Stanisław Leśniewski

Collected Works

Volume II

S.J. Surma, J.T. Srzednicki,
D.I. Barnett and V.F. Rickey (editors)

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Collected works

Volume II

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TABLE OF CONTENTS

ON FUNCTIONS WHOSE FIELDS, WITH RESPECT TO THESE FUNCTIONS ARE GROUPS[1]

The objects which satisfy a given function f form a group with respect to a given function φ if and only if the following conditions are fulfilled:[2]

(a) $[A, B] : f(A) . f(B) . \supset . [\exists C] . f(C) . \varphi(A, B, C)$

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[1] [By 'group' Leśniewski means a set that with respect to a given function, satisfies certain conditions (axioms). Modern algebraic convention would call the ordered pair of them a group. By 'field' Leśniewski means the set-theoretic concept of the union of the domain and range of a function. Strictly speaking, the field of φ (which Leśniewski refers to immediately below) is the union of the set $\{T, F\}$ of truth-values and the set of objects on which φ is defined, which is stipulated by formula [A] to be the set f (i.e., the set of objects which satisfy f). Leśniewski seems to regard (f) as the field of φ, and this is perhaps understandable considering the fact that there must exist a two-place operation $\circ$ defined on f such that for any $< A, B, C >$ where $A, B, C \in f : \varphi(A, B, C)$ iff $A \circ B = C$, and $< f, \circ >$ is a group with essentially the same axioms. When Leśniewski refers to f as the field of φ, he apparently means the domain of φ, or perhaps the field of such a two-place operation as just described. – *tr.*]

[2] In connection with the contents of these conditions, cf. (1) H. Weber, 'Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie', *Mathematische Annalen* 43 (1893), pp. 522, 523. (2) Edward V. Huntington, 'Note on the Definitions of Abstract Groups and Fields by Sets of Independent Postulates', *Transactions of the American Mathematical Society* 6 (1905), p. 192. In connection with the expressions of type '$\varphi(A, B, C)$' which appear below, cf. Maxime Bocher, 'The Fundamental Conceptions and Methods of Mathematics': Address delivered before the Department of Mathematics of the International Congress of Arts and Science, St. Louis, Sept. 20, 1904, *Bulletin of the American Mathematical Society* XI, (Oct. 1904 to July 1905), 1905, p. 126. In connection with the logical symbolism used in my paper, cf. Alfred North Whitehead and Bertrand Russell, *Principia Mathematica* I (2nd ed.), Cambridge 1925, pp. 6, 7, 9–11, 15.

(b) $[A, B, C, D] : f(A) . f(B) . f(C) . f(D) . \varphi(A, B, C) . \varphi(A, B, D) . \supset . C = D$

(c) $[A, B] : f(A) . f(B) . \supset . [\exists C] . f(C) . \varphi(A, C, B)$

(d) $[A, B, C, D] : f(A) . f(B) . f(C) . f(D) . \varphi(A, C, B) . \varphi(A, D, B) . \supset . C = D$

(e) $[A, B] : f(A) . f(B) . \supset . [\exists C] . f(C) . \varphi(C, A, B)$

(f) $[A, B, C, D] : f(A) . f(B) . f(C) . f(D) . \varphi(C, A, B) . \varphi(D, A, B) . \supset . C = D$

(g) $[A, B, C, D, E, F, G] : f(A) . f(B) . f(C) . f(D) . f(E) . f(F) . f(G) . \varphi(A, B, C) . \varphi(C, D, E) . \varphi(B, D, F) . \varphi(A, F, G) . \supset . E = G.$

As is well known, Huntington proved in 1904 that conditions (b), (d), and (f) follow from the other four together.[3] In this paper, therefore, I shall have already taken into account the fact that the conditions (a)–(g) above can be replaced by (a), (c), (e), and (g).

While investigating various well-known systems of arithmetic from the aspect of different possible methods of simplifying their fundamental axioms, I noticed that from this point of view it is important to obtain the simplest possible system of conditions which would (with the aid of any trustworthy sketch of the theory of deduction, universal and particular quantification, the identity sign, and appropriate expressions of the type '$\varphi(A, B, C)$') unequivocally characterize a certain special situation in which a group is formed, with respect to a given function φ, by the objects which, for some specific function f, satisfy the following formula:

$$[A] \therefore f(A) . \equiv : [\exists B, C] : \varphi(A, B, C) . \vee . \varphi(B, A, C) . \vee . \varphi(B, C, A).$$

(Loosely speaking, I could characterize this situation by explaining that here the function φ is such that with respect to itself, its whole field is a group.) When applied to this situation, the conditions (a), (c), (e), and (g) given above correspondingly take,

[3] Cf. Huntington, *op. cit.*, pp. 181, 192, 196.

on removal of the obvious redundancies, the form of the following four conditions:

1. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \lor . \varphi(D, A, E) . \lor . \varphi(D, E, A) : \varphi(B, F, G) . \lor . \varphi(F, B, G) . \lor . \varphi(F, G, B) : \supset . [\exists C] . \varphi(A, B, C)$
2. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \lor . \varphi(D, A, E) . \lor . \varphi(D, E, A) : \varphi(B, F, G) . \lor . \varphi(F, B, G) . \lor . \varphi(F, G, B) : \supset . [\exists C] . \varphi(A, C, B)$
3. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \lor . \varphi(D, A, E) . \lor . \varphi(D, E, A) : \varphi(B, F, G) . \lor . \varphi(F, B, G) . \lor . \varphi(F, G, B) : \supset . [\exists C] . \varphi(C, A, B)$
4. $[A, B, C, D, E, F, G] : \varphi(A, B, C) . \varphi(C, D, E) . \varphi(B, D, F) . \varphi(A, F, G) . \supset . E = G.$

In this article I wish to show that the system consisting of these four conditions is equivalent to a single condition, which has the form of the following equivalence:[4]

I. $[A, B, C] :: \varphi(A, B, C) . \equiv \because [\exists D, E, F, G] . \varphi(A, D, E) . \varphi(C, F, G) \because [H, I] :: \varphi(H, B, I) . \equiv \therefore [\exists K, L, M, N] . \varphi(K, H, L) . \varphi(M, N, I) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P$

The derivation of thesis I from theses 1–4 proceeds without difficulty on consideration of well-known elementary results of group theory. I shall give the relevant deductions explicitly only half-way through.

5. $[B, H, I] : \varphi(H, B, I) . \supset . [\exists K, L] . \varphi(K, H, L)$ (from 1)
6. $[A, B, C] : \varphi(A, B, C) . \supset . [\exists F, G] . \varphi(C, F, G)$ (from 2)
7. $[A, B, C, H, I, K, L, M, N] :: \varphi(A, B, C) . \varphi(K, H, L) . \varphi(M, N, I) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P \therefore \supset . \varphi(H, B, I)$

Proof:

$[A, B, C, H, I, K, L, M, N] \because$

(α) $\varphi(A, B, C)$.

[4] These results date from 1926.

(β) $\varphi(K,H,L)$.
(γ) $\varphi(M,N,I)$.·.
(δ) $[O,P]:\varphi(O,C,I).\varphi(P,A,H).\supset.O=P$.·. $\supset$: :
$[\exists D]$.·.
(ϵ) $\varphi(H,B,D)$.·. (from 1,(β),(α))
$[\exists P]$:
(ζ) $\varphi(P,A,H)$: (3,(α),(β))
$[\exists O]$.
(η) $\varphi(O,C,I)$. (3,(α),(γ))
(ϑ) $O=P$: ((δ),(η),(ζ))
(ι) $\varphi(P,C,I)$.·. ((η),(ϑ))
(κ) $D=I$: : (4, (ζ),(ϵ),(α),(ι))
$\varphi(H,B,I)$ ((ϵ),(κ))

8. $[A,B,C,D]:\varphi(A,B,C).\varphi(D,B,C).\supset.A=D$

Proof:[5]

$[A,B,C,D]$: :
(α) $\varphi(A,B,C)$.
(β) $\varphi(D,B,C).\supset$.·.
$[\exists E]$:
(γ) $\varphi(E,A,D)$: (3,(α),(β))
$[\exists F]$.
(δ) $\varphi(E,C,F)$. (1,(γ),(α))
(ϵ) $C=F$: (4,(γ),(β),(α),(δ))
(ζ) $\varphi(E,C,C)$: ((δ),(ϵ))
$[\exists G]$.
(η) $\varphi(C,G,A)$.·. (2,(α))
$A=D$ (4,(ζ),(η),(γ))

9. $[A,C,O,P]:\varphi(O,C,C).\varphi(P,A,A).\supset.O=P$

[5] This proof is essentially only a repetition of the relevant proofs in Huntington. Cf. *op. cit.*, p. 196.

Proof:

$[A, C, O, P] ::$

(α) $\varphi(O, C, C)$.

(β) $\varphi(P, A, A) . \supset \therefore$

$[\exists B] :$

(γ) $\varphi(O, A, B) :$ (1,(α), (β)

$[\exists D]$.

(δ) $\varphi(C, D, A) :$ (2,(α), (β))

(ε) $A = B \therefore$ (4,(α), (δ), (γ))

(ζ) $\varphi(O, A, A)$. ((γ), (ε))

$O = P$ (8,(ζ), (β))

10. $[A, B, C, D, E, F, G] \mathbin{:\,:} \varphi(A, D, E) . \varphi(C, F, G) \mathbin{\because\!\cdot} [H, I] :: \varphi(H, B, I) . \equiv \therefore [\exists K, L, M, N] . \varphi(K, H, L) . \varphi(M, N, I) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P \mathbin{\because\!\cdot} \supset . \varphi(A, B, C)$

Proof:

$[A, B, C, D, E, F, G] \mathbin{:\,:}$

(α) $\varphi(A, D, E)$.

(β) $\varphi(C, F, G) \mathbin{\because\!\cdot}$

(γ) $[H, I] :: \varphi(H, B, I) . \equiv \therefore [\exists K, L, M, N] . \varphi(K, H, L) . \varphi(M, N, I) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P \mathbin{\because\!\cdot} \supset \therefore$

(δ) $[O, P] : \varphi(O, C, C) . \varphi(P, A, A) . \supset . O = P \therefore$ (9)

$[\exists K, L] :$

(ε) $\varphi(K, A, L) :$ (5,(α))

$[\exists N]$.

(ζ) $\varphi(C, N, C) \therefore$ (2,(β))

$\varphi(A, B, C)$ ((γ), (ε), (ζ), (δ))

11. $[A, B, C, H, I, O, P] : \varphi(A, B, C) . \varphi(H, B, I) . \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P$

Proof:

$[A, B, C, H, I, O, P] ::$

$(\alpha)\ \varphi(A,B,C)\,.$
$(\beta)\ \varphi(H,B,I)\,.$
$(\gamma)\ \varphi(O,C,I)\,.$
$(\delta)\ \varphi(P,A,H)\,.\supset\,.\!\cdot\!.$
$[\exists D]:$
$(\epsilon)\quad \varphi(O,A,D):$ $(1,(\gamma),(\alpha)$
$[\exists E]\,.$
$(\zeta)\qquad \varphi(D,B,E)\,.$ $(1,(\epsilon),(\alpha))$
$(\eta)\qquad E=I:$ $(4,(\epsilon),(\zeta),(\alpha),(\gamma))$
$(\vartheta)\quad \varphi(D,B,I)\,.$ $((\zeta),(\eta))$
$(\iota)\quad D=H\,.\!\cdot\!.$ $(8,(\vartheta),(\beta))$
$(\kappa)\ \varphi(O,A,H)\,.$ $((\epsilon),(\iota))$
$O=P$ $(8,(\kappa),(\delta))$

12. $[A,B,C,H,I]\therefore\varphi(A,B,C)\,.\supset::\varphi(H,B,I)\,.\supset\therefore$
$[\exists K,L,M,N]\,.\,\varphi(K,H,L)\,.\,\varphi(M,N,I)\therefore[O,P]:\varphi(O,C,I)\,.$
$\varphi(P,A,H)\,.\supset.\,O=P$ (from 5, 11, 7)

Thesis I follows from 6, 12, and 10.

I now come to the derivation of theses 1–4 from thesis I:

II. $[A,B,C]:\varphi(A,B,C)\,.\supset.\,[\exists F,G]\,.\,\varphi(C,F,G)$ (from I)

III. $[B,H,I]:\varphi(H,B,I)\,.\supset.\,[\exists K,L]\,.\,\varphi(K,H,L)$ (from I)

IV. $[A,B,C,H,I,O,P]:\varphi(A,B,C)\,.\,\varphi(H,B,I)\,.\,\varphi(O,C,I)\,.$
$\varphi(P,A,H)\,.\supset.\,O=P$ (from I)

V. $[A,B,C,H,I,K,L,M,N]::\varphi(A,B,C)\,.\,\varphi(K,H,L)\,.$
$\varphi(M,N,I)\therefore[O,P]:\varphi(O,C,I)\,.\,\varphi(P,A,H)\,.\supset.\,O=P\therefore\supset$
$.\,\varphi(H,B,I)$ (from I)

VI. $[A,B,C,D,E,F,G]::\varphi(A,D,E)\,.\,\varphi(C,F,G)\therefore[H,I]::$
$\varphi(H,B,I)\,.\equiv\therefore[\exists K,L,M,N]\,.\,\varphi(K,H,L)\,.\,\varphi(M,N,I)\therefore$
$[O,P]:\varphi(O,C,I)\,.\,\varphi(P,A,H)\,.\supset.\,O=P\therefore\supset.\,\varphi(A,B,C)$
(from I)

VII. $[A,B,C]:\varphi(A,B,C)\,.\supset.\,[\exists K,L]\,.\,\varphi(K,C,L)$
(from II, III)

VIII. $[E,Q]:\varphi(Q,Q,Q)\,.\,\varphi(E,Q,Q)\,.\supset.\,\varphi(Q,Q,E)$

Proof:

$[E, Q]$:

(α) $\varphi(Q, Q, Q)$.

(β) $\varphi(E, Q, Q) . \supset .$

(γ) $Q = E$. (IV,(α), (β))

$\varphi(Q, Q, E)$ ((α), (γ))

IX. $[A, B, C, E, H] : \varphi(E, E, B) . \varphi(A, E, B) . \varphi(C, B, H) .$
$\varphi(B, E, H) . \supset . \varphi(E, A, B)$

Proof:

$[A, B, C, E, H]$:

(α) $\varphi(E, E, B)$.

(β) $\varphi(A, E, B)$.

(γ) $\varphi(C, B, H)$.

(δ) $\varphi(B, E, H) . \supset .$

(ε) $C = E$. (IV,(α), (δ), (γ))

(ζ) $C = A$. (IV,(α), (δ), (γ), (β))

(η) $E = A$. ((ε), (ζ))

$\varphi(E, A, B)$ ((α), (η))

X. $[A, B, C] : \varphi(A, B, C) . \supset . [\exists F, G] . \varphi(B, F, G)$

Proof:

$[A, B, C] \therefore$

(α) $\varphi(A, B, C) . \supset :$

(β) $[\exists P] . \varphi(P, A, B) . \vee . \varphi(B, B, C)$: (V,(α))

$[\exists F, G] . \varphi(B, F, G)$ ((β), II)

XI. $[A, B, D, E, F, G] : \varphi(D, E, A) . \varphi(F, G, B) . \supset . [\exists O] .$
$\varphi(O, A, B)$

Proof:

$[A, B, D, E, F, G]$:

(α) $\varphi(D, E, A)$.

(β) $\varphi(F, G, B) . \supset \therefore$

(γ) $[\exists O] . \varphi(O, A, B) . \vee . \varphi(E, E, B)$: (V,(α), (β))

$[\exists K, L]$.

(δ) $\varphi(K, A, L)$: (VII,(α))
(ϵ) $[\exists O] . \varphi(O, A, B) . \vee . \varphi(A, E, B) .\!\cdot\!.$ (V,$(\alpha), (\delta), (\beta)$)
$[\exists C, H] .\!\cdot\!.$
(ζ) $\varphi(C, B, H)$: (VII,(β))
(η) $[\exists O] . \varphi(O, A, B) . \vee . \varphi(B, E, H)$: $((\epsilon), \text{V}, (\zeta))$
(ϑ) $[\exists O] . \varphi(O, A, B) . \vee . \varphi(E, E, B) . \varphi(A, E, B) .$
$\varphi(B, E, H) .\!\cdot\!.$ $((\gamma), (\epsilon), (\eta))$
$[\exists O] . \varphi(O, A, B)$ $((\gamma), \text{IX}, (\zeta))$

XII. $[A, K, P, Q] : \varphi(Q, A, A) . \varphi(P, K, Q) . \supset . \varphi(Q, Q, Q)$

Proof:

$[A, K, P, Q] .\!\cdot\!.$
(α) $\varphi(Q, A, A) .$
(β) $\varphi(P, K, Q) . \supset :$
$[\exists O] .$
(γ) $\varphi(O, Q, Q) .$ (XI,(β))
(δ) $Q = O$: (IV,$(\alpha), (\gamma)$)
$\varphi(Q, Q, Q)$ $((\gamma), (\delta))$

XIII. $[A, Q] : \varphi(Q, A, A) . \sim \{\varphi(Q, Q, A)\} . \supset . \varphi(Q, Q, Q)$

Proof:

$[A, Q] : :$
(α) $\varphi(Q, A, A) .$
(β) $\sim \{\varphi(Q, Q, A)\} . \supset .\!\cdot\!.$
$[\exists K, L]$:
(γ) $\varphi(K, Q, L)$: (III,(α))
$[\exists P] .$
(δ) $\varphi(P, K, Q) .\!\cdot\!.$ (V,$(\gamma), (\alpha), (\beta)$)
$\varphi(Q, Q, Q)$ (XII,$(\alpha), (\beta)$)

XIV. $[A, Q] : \varphi(Q, A, A) . \varphi(Q, Q, A) . \supset . \varphi(Q, Q, Q)$

Proof:

$[A, Q] \vdots \vdots$
(α) $\varphi(Q, A, A) .$
(β) $\varphi(Q, Q, A) . \supset \because$

(γ) $[H,I] :: \varphi(H,Q,I) . \equiv .\because [\exists K,L,M,N] . \varphi(K,H,L) .$
$\varphi(M,N,I) \because [O,P] : \varphi(O,A,I) . \varphi(P,Q,H) . \supset . O = P ::$
(III, IV, V, (β)[6])

(δ) $[H,I] :: \varphi(H,A,I) . \equiv .\because [\exists K,L,M,N] . \varphi(K,H,L) .$
$\varphi(M,N,I) \because [O,P] : \varphi(O,A,I) . \varphi(P,Q,H) . \supset . O = P ::$
(III, IV, V, (α)[7])

(ϵ) $[H,I] : \varphi(H,A,I) . \equiv . \varphi(H,Q,I) ::$ $((\delta),(\gamma))$

(ζ) $[H,I] :: \varphi(H,Q,I) . \equiv .\because [\exists K,L,M,N] . \varphi(K,H,L) .$
$\varphi(M,N,I) \because [O,P] : \varphi(O,Q,I) . \varphi(P,Q,H) . \supset . O = P ::$
$((\gamma),(\epsilon))$

$\varphi(Q,Q,Q)$ (VI, (α), (ζ))

XV. $[A,Q] : \varphi(Q,A,A) . \supset . \varphi(Q,Q,Q)$ (from XIV, XIII)

XVI. $[A,D,E] : \varphi(D,E,A) . \supset . [\exists P,Q] . \varphi(P,Q,E)$

Proof:

$[A,D,E] ::$

(α) $\varphi(D,E,A) . \supset \therefore$

$[\exists Q] :$

(β) $\varphi(Q,A,A) .$ (XI, (α)

(γ) $\varphi(Q,Q,Q) :$ (XV, (β))

(δ) $\sim \{\varphi(E,Q,Q)\} . \vee . \varphi(Q,Q,E) \therefore$ (VIII, (γ))

$[\exists P,Q] . \varphi(P,Q,E)$ $((\delta)$, V, (γ), $(\alpha))$

XVII. $[B,H,I] : \varphi(H,B,I) . \supset . [\exists P,Q] . \varphi(P,Q,H)$
(from III, XVI)

XVIII. $[A,D,E] \therefore \varphi(A,D,E) . \vee . \varphi(D,A,E) . \vee . \varphi(D,E,A) :$
$\supset . [\exists F,G,K,L,P,Q] . \varphi(A,F,G) . \varphi(K,A,L) . \varphi(P,Q,A)$
(from III, XVII, X, XVI, II, VII)

3. $[A,B,D,E,F,G] \therefore \varphi(A,D,E) . \vee . \varphi(D,A,E) . \vee .$
$\varphi(D,E,A) : \varphi(B,F,G) . \vee . \varphi(F,B,G) . \vee . \varphi(F,G,B) :$
$\supset . [\exists C] . \varphi(C,A,B)$

---

[6] It would be simpler here to appeal to I and (β); I do not do this for reasons which will be clear further below.

[7] It would be simpler here to appeal to I and (α).

Proof:

$[A, B, D, E, F, G] :\, :$

(α) $\varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee . \varphi(D, E, A) :$

(β) $\varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) : \supset \therefore$

$[\exists P, Q] :$

(γ) $\varphi(P, Q, A) :$ (XVIII,(α))

$[\exists H, I] .$

(δ) $\varphi(H, I, B) \therefore$ (XVIII,(β))

$[\exists C] . \varphi(C, A, B)$ (XI,$(\gamma), (\delta)$)

XIX. $[B, C, O, P] : \varphi(O, B, C) . \varphi(P, B, C) . \supset . O = P$

Proof:

$[B, C, O, P] ::$

(α) $\varphi(O, B, C) .$

(β) $\varphi(P, B, C) . \supset :\, :$

$[\exists A, Q] \therefore$

(γ) $\varphi(A, Q, B) \therefore$ (XVI,(β))

$[\exists D] :$

(δ) $\varphi(D, Q, C) :$ (3,$(\gamma), (\alpha)$)

$[\exists E] .$

(ϵ) $\varphi(E, A, D) .$ (3,$(\gamma), (\delta)$)

(ζ) $O = E .$ (IV, $(\gamma), (\delta), (\alpha), (\epsilon)$)

(η) $P = E :\, :$ (IV, $(\gamma), (\delta), (\beta), (\epsilon)$)

$O = P$ ($(\zeta), (\eta)$)

XX. $[A, B, C, H, I] \therefore \varphi(C, B, B) : \varphi(I, H, A) . \vee . \varphi(H, I, A) . \vee . \varphi(H, A, I) : \supset . \varphi(I, C, I)$

Proof:

$[A, B, C, H, I] :\, :$

(α) $\varphi(C, B, B) :$

(β) $\varphi(I, H, A) . \vee . \varphi(H, I, A) . \vee . \varphi(H, A, I) : \supset \therefore$

$[\exists P, Q] .$

(γ) $\varphi(P, Q, C) :$ (XVII,(α))

(δ) $\varphi(C, C, C) \therefore$ (XII,$(\alpha), (\gamma)$)

(ϵ) $[O,P]:\varphi(O,C,I).\varphi(P,C,I).\supset.O=P\therefore$ (XIX)
$[\exists K,L,M,N]$.
(ζ) $\varphi(K,I,L).\varphi(M,N,I):$ (XVIII,(β))
$\varphi(I,C,I)$ (V,$(\delta),(\zeta),(\epsilon)$)

XXI. $[A,B,F,G,H,I,K,L,M,N,Q]::\varphi(B,F,G).\vee.\varphi(F,B,G).\vee.\varphi(F,G,B):\varphi(Q,A,A).\varphi(K,H,L).\varphi(M,N,I)\therefore[O,P]:\varphi(O,B,I).\varphi(P,Q,H).\supset.O=P\therefore\supset.\varphi(H,B,I)$

Proof:

$[A,B,F,G,H,I,K,L,M,N,Q]::$
(α) $\varphi(B,F,G).\vee.\varphi(F,B,G).\vee.\varphi(F,G,B):$
(β) $\varphi(Q,A,A)$.
(γ) $\varphi(K,H,L)$.
(δ) $\varphi(M,N,I)\therefore$
(ϵ) $[O,P]:\varphi(O,B,I).\varphi(P,Q,H).\supset.O=P\therefore\supset:$
(ζ) $\varphi(H,Q,H):$ (XX,$(\beta),(\gamma)$)
$[\exists C]$.
(η) $\varphi(C,B,I)$. (3,$(\alpha),(\delta)$)
(ϑ) $C=H:$ ($(\epsilon),(\eta),(\zeta)$)
$\varphi(H,B,I)$ $(\eta),(\vartheta)$)

XXII. $[A,B,C,H,I,O,P]:\varphi(C,B,B).\varphi(I,H,A).\varphi(O,C,I).\varphi(P,H,A).\supset.O=P$

Proof:

$[A,B,C,H,I,O,P]:$
(α) $\varphi(C,B,B)$.
(β) $\varphi(I,H,A)$.
(γ) $\varphi(O,C,I)$.
(δ) $\varphi(P,H,A).\supset$.
(ϵ) $\varphi(I,C,I)$. (XX,$(\alpha),(\beta)$)
(ζ) $I=O$. (XIX,$(\epsilon),(\gamma)$)
(η) $I=P$. (XIX,$(\beta),(\delta)$)
$O=P$ ($(\zeta),(\eta)$)

XXIII. $[A, B, C, H, I] \colon \varphi(C, B, B) . \varphi(H, B, C) . \varphi(I, H, A) . \supset .$
$\varphi(A, B, I)$
Proof:
$[A, B, C, H, I] \colon\colon$
(α) $\varphi(C, B, B) .$
(β) $\varphi(H, B, C) .$
(γ) $\varphi(I, H, A) . \supset .\therefore$
(δ) $[O, P] \colon \varphi(O, C, I) . \varphi(P, H, A) . \supset . O = P \therefore$ (XXII,(α), (γ))
$[\exists D] .$
(ϵ) $\varphi(D, A, I) \colon$ (3, (γ))
$\varphi(A, B, I)$ (V,(β), (ϵ), (δ))
1. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee .$
$\varphi(D, E, A) \colon \varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) .$
$\supset . [\exists C] . \varphi(A, B, C)$
Proof:
$[A, B, D, E, F, G] \because$
(α) $\varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee . \varphi(D, E, A) \colon$
(β) $\varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) \colon \supset \colon\colon$
$[\exists C] \therefore$
(γ) $\varphi(C, B, B) \therefore$ (3,(β))
$[\exists H] \colon$
(δ) $\varphi(H, B, C) \colon$ (3,(γ))
$[\exists I] .$
(ϵ) $\varphi(I, H, A) \colon\colon$ (3,(δ), (α))
$[\exists C] . \varphi(A, B, C)$ (XXIII,(γ), (δ), (ϵ))
XXIV. $[B, H, I] \colon \varphi(H, B, B) . \varphi(I, B, H) . \supset . \varphi(B, I, H)$
Proof:
$[B, H, I] \therefore$
(α) $\varphi(H, B, B) .$
(β) $\varphi(I, B, H) . \supset \colon$
$[\exists C] .$

(γ) $\varphi(B, I, C)$. $(1, (\beta))$

(δ) $\varphi(C, B, B)$. $(\text{XXIII}, \alpha), (\beta), (\gamma))$

(ϵ) $C = H$: $(\text{XIX}, (\delta), (\alpha))$

$\varphi(B, I, H)$ $((\gamma), (\epsilon))$

XXV. $[A, B, C, H, I] : \varphi(H, B, B) . \varphi(I, B, H) . \varphi(A, B, C) . \supset .$
$\varphi(C, I, A)$

Proof:

$[A, B, C, H, I] \therefore$

(α) $\varphi(H, B, B)$.

(β) $\varphi(I, B, H)$.

(γ) $\varphi(A, B, C) . \supset :$

(δ) $\varphi(H, H, H)$: $(\text{XII}, (\alpha), (\beta))$

$[\exists P]$.

(ϵ) $\varphi(P, I, I)$. $(3, (\beta))$

(ζ) $H = P$: $(\text{IV}, (\beta), (\delta), (\epsilon))$

(η) $\varphi(H, I, I)$. $((\epsilon), (\zeta))$

(ϑ) $\varphi(B, I, H)$. $(\text{XXIV}, (\alpha), (\beta))$

$\varphi(C, I, A)$ $(\text{XXIII}, (\eta), (\vartheta), (\gamma))$

4. $[A, B, C, D, E, F, G] : \varphi(A, B, C) . \varphi(C, D, E) . \varphi(B, D, F)$.
$\varphi(A, F, G) . \supset . E = G$

Proof:

$[A, B, C, D, E, F, G] ::::$

(α) $\varphi(A, B, C)$.

(β) $\varphi(C, D, E)$.

(γ) $\varphi(B, D, F)$.

(δ) $\varphi(A, F, G) . \supset ::::$

$[\exists H] ::.$

(ϵ) $\varphi(H, B, B) ::.$ $(3, (\alpha))$

$[\exists I] ::$

(ζ) $\varphi(I, B, H)$. $(3, (\epsilon))$

(η) $\varphi(C, I, A) ::$ $(\text{XXV}, (\epsilon), (\zeta), (\alpha))$

$[\exists Q] \therefore$

(ϑ) $\varphi(Q,D,D)\,.$ $(3,(\beta))$

(ι) $\varphi(G,Q,G)\,:\cdot:$ $(\mathrm{XX},(\vartheta),(\delta))$

$[\exists R]\,::$

(κ) $\varphi(R,D,Q)\,.$ $((\beta),(\vartheta))$

(λ) $\varphi(E,R,C)\,::$ $(\mathrm{XXV},(\vartheta),(\kappa),(\beta))$

$[\exists S]\,.\cdot.$

(μ) $\varphi(R,I,S)\,.$ $(1,(\kappa),(\zeta))$

(ν) $\varphi(S,B,R)\,.\cdot.$ $(\mathrm{XXIII},(\epsilon),(\zeta),(\mu))$

$[\exists T]\,:$

(ξ) $\varphi(T,S,A)\,.$ $(3,(\mu),(\alpha))$

(o) $T=E\,:$ $(\mathrm{IV},(\mu),(\eta),(\xi),(\lambda))$

$[\exists O]\,.$

(π) $\varphi(O,F,Q)\,.$ $(\mathrm{XI},\gamma),(\kappa))$

(ρ) $O=S\,.$ $(\mathrm{IV},\gamma),(\kappa),(\pi),(\nu))$

(σ) $\varphi(T,O,A)\,:$ $((\xi),(\rho))$

(τ) $G=T\,\vdots\!\cdot\!\vdots$ $(\mathrm{IV},(\pi),(\delta),(\iota),(\sigma))$

$E=G$ $((\sigma),(\tau))$

2. $[A,B,D,E,F,G]\,.\cdot.\,\varphi(A,D,E)\,.\vee.\,\varphi(D,A,E)\,.\vee.$
$\varphi(D,E,A)\,:\,\varphi(B,F,G)\,.\vee.\,\varphi(F,B,G)\,.\vee.\,\varphi(F,G,B)\,:$
$\supset.\,[\exists C]\,.\,\varphi(A,C,B)$

Proof:

$[A,B,D,E,F,G]\,\vdots\!\vdots$

(α) $\varphi(A,D,E)\,.\vee.\,\varphi(D,A,E)\,.\vee.\,\varphi(D,E,A)\,:$

(β) $\varphi(B,F,G)\,.\vee.\,\varphi(F,B,G)\,.\vee.\,\varphi(F,G,B)\,:\supset\,\vdots$

(γ) $[H,I]\,:\,\varphi(H,B,I)\,.\supset.\,[\exists K,L,M,N]\,.\,\varphi(K,H,L)\,.$
$\varphi(M,N,I)\,.\cdot.$ (III)

$[\exists Q]\,:\cdot:$

(δ) $\varphi(Q,A,A)\,.\cdot.$ $(3,(\alpha))$

(ϵ) $[H,I,O,P]\,:\,\varphi(H,B,I)\,.\,\varphi(O,B,I)\,.\,\varphi(P,Q,H)\,.\supset.$
$P=O\,:\cdot:$ $(\mathrm{XXII},(\delta))$

(ζ) $[H,I,K,L,M,N]::\varphi(K,H,L).\varphi(M,N,I)\therefore[O,P]:$
$\varphi(O,B,I).\varphi(P,Q,H).\supset.O=P\therefore\supset.\varphi(H,B,I)::$ (XXI,$(\beta),(\delta)$)

$[\exists R,S].$

(η) $\varphi(B,R,S):$ (XVIII,(β))

(ϑ) $\varphi(Q,B,B)::$ (VI,$(\delta),(\eta),(\gamma),(\epsilon),(\zeta)$)

$[\exists T]\therefore$

(ι) $\varphi(T,A,Q).$ (3,$(\alpha),(\delta)$)

(κ) $\varphi(A,T,Q)\therefore$ (XXIV,$(\delta),(\iota)$)

$[\exists C]:$

(λ) $\varphi(T,B,C):$ (1,$(\iota),(\vartheta)$)

$[\exists U].$

(μ) $\varphi(A,C,U).$ (1,$(\delta),(\lambda)$)

(ν) $B=U::$ (1,$(\kappa),(\vartheta),(\lambda),(\iota)$)

$[\exists C].\varphi(A,C,B)$ ($(\mu),(\nu)$)

The deduction gone through here show that the system of theses 1–4 is equivalent to thesis I, as I claimed above. (We could also express this result by pointing out that a necessary and sufficient conditions for any function φ satisfying thesis I is that the field of φ should be a group with respect to φ.)

I might also point out that I had derived from thesis I its 'natural' factors (theses II–VI) prior to my having ever appealed to thesis I in the derivations of theses 1–4.[8] It follows that the system of theses II–VI forms by itself a rather remarkable (from the point of view of the prevailing traditions in group theory) postulate system, which is equivalent to the system of postulates 1–4. It is easy to convince oneself that each of the five theses belonging to this system (i. e., II–VI) is independent of the other four: all of them except one (which is different for the five different cases) satisfy the functions φ, which are defined in turn by the following five formulae:

[8] Cf. the comments above on (γ) and (δ) in the proof of thesis XIV.

$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . B = 1 : C = 1 . \vee . C = 2$

$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . \vee . A = 2 : B = 2 . C = 1$

$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = B . B = C . \supset . A = C$

$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . B = 1 . C = 1 . \vee . A = 1 . B = 2 . C = 2 . \vee . A = 2 . B = 1 . C = 2$

$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . C = 1 . \vee . A = 2 . C = 2 : B = 1 . \vee . B = 2.$

The fact that the functions whose fields, with respect to these functions, are groups can be characterized by a single equivalence which can be broken up into mutually independent postulates II–VI, and of which one side is an appropriate expression of the type '$\varphi(A, B, C)$',[9] possesses for me personally a universal importance: I am inclined to assume that choosing single equivalences of this kind, which, in addition to characterizing these functions, would be satisfied by functions appearing as primitives in various deductive theories, casts much light on the axiom systems of those theories, and can contribute towards an important simplification of those axiom systems. (I have already succeeded in establishing such simplifications in many quite independent cases.) I might add at this point that, for me, the greatest difficulties with constructions from equivalences of the type mentioned have so far always been inflicted by those theories whose axioms are concerned with the question of how many elements belong to the field of the primitive functions of the theories. I suspect that there is material in the facts I have here so very generally touched upon for a future precise synthesis in the area of the theory of deductive systems.

[9] I shall present an analogous result for Abelian groups in a separate paper.

ON FUNCTIONS WHOSE FIELDS, WITH RESPECT TO THESE FUNCTIONS, ARE ABELIAN GROUPS[1]

The objects which satisfy a given function f form an Abelian group with respect to a given function φ if and only if the following conditions are fulfilled:[2]

(a) $[A, B] : f(A) . f(B) . \supset . [\exists C] . f(C) . \varphi(A, B, C)$

(b) $[A, B] : f(A) . f(B) . \supset . [\exists C] . f(C) . \varphi(A, C, B)$

(c) $[A, B] : f(A) . f(B) . \supset . [\exists C] . f(C) . \varphi(C, A, B)$

(d) $[A, B, C, D, E, F, G] : f(A) . f(B) . f(C) . f(D) . f(E) . f(F) . f(G) . \varphi(A, B, C) . \varphi(C, D, E) . \varphi(B, D, F) . \varphi(A, F, G) . \supset . E = G$

(e) $[A, B, C] : f(A) . f(B) . f(C) . \varphi(A, B, C) . \supset . \varphi(B, A, C)$.

In this paper I shall deal with a certain special situation in which an Abelian group is formed, with respect to a given function φ, by the objects which, for some specific function f, satisfy the following formula:[3]

$$[A] \therefore f(A) . \equiv : [\exists B, C] : \varphi(A, B, C) . \vee . \varphi(B, A, C) . \vee . \varphi(B, C, A).$$

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[1] (*Translator's note*): see footnote 1 in my translation of Leśniewski's paper 'On Functions Whose Fields, with Respect to These Functions, are Groups'.

[2] In connection with these conditions, cf. (1) H. Weber, 'Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie', *Mathematische Annalen* 43 (1893), pp. 522, 523. (2) Edward V. Huntington, 'Note on the Definitions of Abstract Groups and Fields by Sets of Independent Postulates', *Transactions of the American Mathematical Society* 6 (1905), p. 192. (3) Stanisław Leśniewski, 'Über Funktionen, deren Felder Gruppen mit Rücksicht auf diese Funktionen sind', *Fundamenta Mathematicae* XIII (1929), pp. 319, 320.

[3] Cf. Leśniewski, *op. cit.*, p. 320.

(Loosely speaking, I could characterize this situation by explaining that here the function φ is such that with respect to itself, its whole field is an Abelian group.) When applied to this situation, the conditions (a)–(e) given above correspondingly take, on removal of the obvious redundancies, the form of the following five conditions:[4]

1. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee . \varphi(D, E, A) : \varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) : \supset . [\exists C] . \varphi(A, B, C)$
2. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee . \varphi(D, E, A) : \varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) : \supset . [\exists C] . \varphi(A, C, B)$
3. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee . \varphi(D, E, A) : \varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) : \supset . [\exists C] . \varphi(C, A, B)$
4. $[A, B, C, D, E, F, G] : \varphi(A, B, C) . \varphi(C, D, E) . \varphi(B, D, F) . \varphi(A, F, G) . \supset . E = G$
5. $[A, B, C] : \varphi(A, B, C) . \supset . \varphi(B, A, C)$.

At this point I would like to prove that the system of conditions 1–5 is equivalent to a single condition, which has the form of the following equivalence:[5]

I. $[A, B, C] :: \varphi(B, A, C) . \equiv \therefore [\exists D, E, F, G] . \varphi(A, D, E) . \varphi(F, G, C) \therefore [H, I] :: \varphi(H, B, I) . \equiv \therefore [\exists K, L, M, N] . \varphi(H, K, L) . \varphi(M, I, N) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P$.

In my paper 'On Functions whose Fields, with Respect to these Functions, are Groups', I derived from the theses 1–4 given above the thesis which says that:[6]

[4] In connection with the first four of these conditions, cf. *op. cit.*, pp. 320, 321.

[5] This result dates from 1926. Cf. *op. cit.*, p. 332.

[6] Cf. *op. cit.*, pp. 320–323. In connection with the relation of thesis I to thesis 6, cf. Wallie Abraham Hurwitz, 'Postulate-Sets for Abelian Groups

6. $[A, B, C] \because\because \varphi(A, B, C) . \equiv \therefore\because [\exists D, E, F, G] . \varphi(A, D, E) . \varphi(C, F, G) \therefore\because [H, I] :: \varphi(H, B, I) . \equiv \therefore [\exists K, L, M, N] . \varphi(K, H, L) . \varphi(M, N, I) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . 0 = P.$

I now make use of this thesis to derive thesis I from theses 1–5:

7. $[A, C] : [\exists D, E, F, G] . \varphi(A, D, E) . \varphi(C, F, G) . \equiv . [\exists D, E, F, G] . \varphi(A, D, E) . \varphi(F, G, C)$ (from 2)
8. $[A, B, C] : \varphi(A, B, C) . \equiv . \varphi(B, A, C)$ (from 5)
9. $[H, I] : [\exists K, L, M, N] . \varphi(K, H, L) . \varphi(M, N, I) . \equiv . [\exists K, L, M, N] . \varphi(H, K, L) . \varphi(M, I, N)$ (from 5 and 3)

Thesis I follows from 6, 8, 7, and 9.

I now come to the derivation of theses 1–5 from thesis I:

II. $[A, B, C] : \varphi(B, A, C) . \supset . [\exists D, E] . \varphi(A, D, E)$ (from I)

III. $[A, B, C, H, I] : \varphi(B, A, C) . \varphi(H, B, I) . \supset . [\exists M, N] . \varphi(M, I, N)$ (from I)

IV. $[A, B, C, H, I, O, P] : \varphi(B, A, C) . \varphi(H, B, I) . \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P$ (from I)

V. $[A, B, C, H, I, K, L, M, N] :: \varphi(B, A, C) . \varphi(H, K, L) . \varphi(M, I, N) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P \therefore \supset . \varphi(H, B, I)$ (from I)

VI. $[A, B, C, D, E, F, G] \because\because \varphi(A, D, E) . \varphi(F, G, C) \therefore\because [H, I] :: \varphi(H, B, I) . \equiv \therefore [\exists K, L, M, N] . \varphi(H, K, L) . \varphi(M, I, N) \therefore [O, P] : \varphi(O, C, I) . \varphi(P, A, H) . \supset . O = P \therefore\because \supset . \varphi(B, A, C)$ (from I)

VII. $[B, H, I] : \varphi(H, B, I) . \supset . [\exists M, N] . \varphi(M, I, N)$

Proof:

$[B, H, I] \therefore$

(α) $\varphi(H, B, I) . \supset :$

$[\exists D, E] .$

(β) $\varphi(B, D, E) :$ (from II and (α))

and Fields', *Annals of Mathematics*, December 1913, Second Series, vol. 15, no. 2, pp. 93, 94.

$[\exists M, N] . \varphi(M, I, N)$ (III,(β), (α))

VIII. $[B, H, I] : \varphi(H, B, I) . \supset . [\exists D, E] . \varphi(I, D, E)$
(from VII and II)

IX. $[I, M, N] : \varphi(M, I, N) . \supset . [\exists O, P] . \varphi(O, P, I)$

Proof:

$[I, M, N] \therefore$

(α) $\varphi(M, I, N) . \supset :$

(β) $[\exists O] . \varphi(O, N, I) . \vee . \varphi(M, M, I) :$ (V,(α))

$[\exists O, P] . \varphi(O, P, I)$ (β))

X. $[B, H, I] : \varphi(H, B, I) . \supset . [\exists O, P] . \varphi(O, P, H)$

Proof:

$[B, H, I] : :$

(α) $\varphi(H, B, I) . \supset \therefore$

$[\exists M, N] :$

(β) $\varphi(M, I, N) :$ (VII,(α))

(γ) $[\exists P] . \varphi(P, B, H) . \vee . \varphi(H, H, I) \therefore$ (V,(α), (β))

$[\exists O, P] . \varphi(O, P, H)$ ((γ), IX)

XI. $[B, H, I] : \varphi(H, B, I) . \supset . [\exists M, N] . \varphi(M, H, N)$
(from X and VII)

XII. $[B, H, I] \therefore \varphi(H, B, I) . \vee . \varphi(B, H, I) . \vee . \varphi(B, I, H) : \supset .$
$[\exists D, E, M, N, O, P] . \varphi(H, D, E) . \varphi(M, H, N) . \varphi(O, P, H)$
(from XI, X, II, IX, VIII, and VII)

XIII. $[A, B, O] : \varphi(O, O, B) . \varphi(B, O, B) . \varphi(A, O, B) . \supset .$
$\varphi(A, A, B)$

Proof:

$[A, B, O] \therefore$

(α) $\varphi(O, O, B) .$

(β) $\varphi(B, O, B) .$

(γ) $\varphi(A, O, B) . \supset :$

$[\exists M, N] .$

(δ) $\varphi(M, B, N) :$ (VII,(β))

$[\exists C] .$

(ε) $\varphi(C, B, B)$. (V,(β),(δ))

(ζ) $C = O$. (IV,(α),(β),(ε))

(η) $C = A$: (IV,(α),(β),(ε),(γ))

(ϑ) $O = A$. ((ζ),(η))

$\varphi(A, A, B)$ ((γ),(ϑ))

3. $[A, B, D, E, F, G] \therefore \varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee .$
$\varphi(D, E, A) : \varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) :$
$\supset . [\exists C] . \varphi(C, A, B)$

Proof:

$[A, B, D, E, F, G] :\cdot:$

(α) $\varphi(A, D, E) . \vee . \varphi(D, A, E) . \vee . \varphi(D, E, A) :$

(β) $\varphi(B, F, G) . \vee . \varphi(F, B, G) . \vee . \varphi(F, G, B) : \supset ::$

$[\exists H, I, O, P] \therefore$

(γ) $\varphi(A, H, I) . \varphi(O, P, A) \therefore$ (XII,(α))

$[\exists K, L, M, N] :$

(δ) $\varphi(B, K, L) . \varphi(M, B, N) :$ (XII,(β))

(ε) $[\exists C] . \varphi(C, A, B) . \vee . \varphi(O, O, B) :$ (V,(γ),(δ))

(ζ) $[\exists C] . \varphi(C, A, B) . \vee . \varphi(B, O, B) :$ (V,(γ),(δ))

(η) $[\exists C] . \varphi(C, A, B) . \vee . \varphi(A, O, B) \therefore$ (V,(γ),(δ))

(ϑ) $[\exists C] . \varphi(C, A, B) . \vee . \varphi(O, O, B) . \varphi(B, O, B) . \varphi(A, O, B)$
$::$ ((ε),(ζ),(η))

$[\exists C] . \varphi(C, A, B)$ ((ϑ),XIII)

XIV. $[B, C, O, P] : \varphi(O, B, C) . \varphi(P, B, C) . \supset . O = P$

Proof:

$[B, C, O, P] :\cdot:$

(α) $\varphi(O, B, C)$.

(β) $\varphi(P, B, C) . \supset ::$

$[\exists A] \therefore$

(γ) $\varphi(A, B, B) \therefore$ (3,(α))

$[\exists D] :$

(δ) $\varphi(D, A, C) :$ (3,(γ),(α))

$[\exists E]$.

(ϵ) $\varphi(E,B,D)$. (3,(α), (δ))

(ζ) $O=E$. (IV,(γ), (δ), (α), (ϵ))

(η) $P=E$ :: (IV,(γ), (δ), (β), (ϵ))

$O=P$ ((ζ), (η))

XV. $[A,B,C,H,I]$ ∴ $\varphi(H,B,B)$: $\varphi(A,C,I)$. ∨ . $\varphi(A,I,C)$: ⊃ . $\varphi(C,H,C)$

Proof:

$[A,B,C,H,I]$ ::

(α) $\varphi(H,B,B)$:

(β) $\varphi(A,C,I)$. ∨ . $\varphi(A,I,C)$: ⊃ ∴

(γ) $[O,P]$: $\varphi(O,B,C)$. $\varphi(P,B,C)$. ⊃ . $O=P$ ∴ (XIV)

$[\exists D,E,M,N]$.

(δ) $\varphi(C,D,E)$. $\varphi(M,C,N)$: (XII,(β))

$\varphi(C,H,C)$ (V,(α), (δ), (γ))

XVI. $[B,C,H,O]$: $\varphi(H,B,B)$. $\varphi(O,H,C)$. ⊃ . $C=O$

Proof:

$[B,C,H,O]$:

(α) $\varphi(H,B,B)$.

(β) $\varphi(O,H,C)$. ⊃ .

(γ) $\varphi(C,H,C)$. (XV,(α), (β))

$C=O$ (XIV,(γ), (β))

XVII. $[A,B,C,H,O,P]$: $\varphi(H,B,B)$. $\varphi(C,B,A)$. $\varphi(O,H,C)$. $\varphi(P,B,A)$. ⊃ . $O=P$

Proof:

$[A,B,C,H,O,P]$:

(α) $\varphi(H,B,B)$.

(β) $\varphi(C,B,A)$.

(γ) $\varphi(O,H,C)$.

(δ) $\varphi(P,B,A)$. ⊃ .

(ϵ) $C=O$. (XVI,(α), (γ))

(ζ) $C=P$. (XIV,(β), (δ))

$O=P$ ((ϵ), (ζ))

XVIII. $[B, C, E, F, G, H, I] : \varphi(E, C, C) \,.\, \varphi(H, E, I) \,.\, \varphi(F, B, I) \,.$
$\varphi(G, B, H) \,.\, \supset \,.\, F = G$

Proof:

$[B, C, E, F, G, H, I] :$

(α) $\varphi(E, C, C) \,.$

(β) $\varphi(H, E, I) \,.$

(γ) $\varphi(F, B, I) \,.$

(δ) $\varphi(G, B, H) \,.\, \supset \,.$

(ϵ) $I = H \,.$ (XVI,(α), (β))

(ζ) $\varphi(F, B, H) \,.$ ((γ), (ϵ))

$F = G$ (XIV,(ζ), (δ))

XIX. $[A, B, C, H, I] : \varphi(H, B, B) \,.\, \varphi(I, B, H) \,.\, \varphi(C, B, A) \,.\, \supset \,.$
$\varphi(A, I, C)$

Proof:

$[A, B, C, H, I] : :$

(α) $\varphi(H, B, B) \,.$

(β) $\varphi(I, B, H) \,.$

(γ) $\varphi(C, B, A) \,.\, \supset \,.\!\cdot\,.$

(δ) $[O, P] : \varphi(O, H, C) \,.\, \varphi(P, B, A) \,.\, \supset \,.\, O = P \,.\!\cdot\,.$ (XVII,(α), (γ))

$[\exists D, E] :$

(ϵ) $\varphi(A, D, E) :$ (VIII,(γ))

$[\exists M, N] \,.$

(ζ) $\varphi(M, C, N) \,.\!\cdot\,.$ (XI,(γ))

$\varphi(A, I, C)$ (V,(β), (ϵ), (ζ), (δ))

1. $[A, B, D, E, F, G] \,.\!\cdot\,.\, \varphi(A, D, E) \,.\, \vee \,.\, \varphi(D, A, E) \,.\, \vee \,.$
$\varphi(D, E, A) : \varphi(B, F, G) \,.\, \vee \,.\, \varphi(F, B, G) \,.\, \vee \,.\, \varphi(F, G, B) :$
$\supset \,.\, [\exists C] \,.\, \varphi(A, B, C)$

Proof:

$[A, B, D, E, F, G] : \cdot :$

(α) $\varphi(A, D, E) \,.\, \vee \,.\, \varphi(D, A, E) \,.\, \vee \,.\, \varphi(D, E, A) :$

(β) $\varphi(B, F, G) \,.\, \vee \,.\, \varphi(F, B, G) \,.\, \vee \,.\, \varphi(F, G, B) : \supset : :$

$[\exists H] \therefore$

(γ) $\varphi(H, B, B) \therefore$ (3,(β))

$[\exists I]:$

(δ) $\varphi(I, B, H) .$ (3,(β), (γ))

(ϵ) $\varphi(H, I, I) .$ (XIX,(γ), (δ))

(ζ) $\varphi(B, I, H):$ (XIX,(γ), (δ))

$[\exists C] .$

(η) $\varphi(C, I, A) ::$ (3,(δ), (α))

$[\exists C] . \varphi(A, B, C)$ (XIX,(ϵ), (ζ), (η))

XX. $[B, G, H, I] : \varphi(G, B, I) . \varphi(G, B, H) . \supset . I = H$

Proof:

$[B, G, H, I] ::$

(α) $\varphi(G, B, I) .$

(β) $\varphi(G, B, H) . \supset \therefore$

$[\exists C]:$

(γ) $\varphi(C, B, B):$ (3,(α))

$[\exists A] .$

(δ) $\varphi(A, B, C) .$ (3,(γ))

(ϵ) $\varphi(I, A, G) .$ (XIX,(γ), (δ), (α))

(ζ) $\varphi(H, A, G) \therefore$ (XIX,(γ), (δ), (β))

$I = H$ (XIV,(ϵ), (ζ))

XXI. $[A, B, C, D, E, F, G] : \varphi(A, B, C) . \varphi(C, D, E) . \varphi(D, B, F) .$
$\varphi(A, F, G) . \supset . E = G$

Proof:

$[A, B, C, D, E, F, G] \therefore$

(α) $\varphi(A, B, C) .$

(β) $\varphi(C, D, E) .$

(γ) $\varphi(D, B, F) .$

(δ) $\varphi(A, F, G) . \supset :$

$[\exists O] .$

(ϵ) $\varphi(O, F, E) .$ (3,(γ), (β))

(ζ) $O = A .$ (IV,(γ), (β), (ϵ), (α))

$(\eta)\quad \varphi(O,F,G):$ $\qquad ((\delta),(\zeta))$

$E = G$ $\qquad (\mathrm{XX},(\epsilon),(\eta))$

XXII. $[B,C,E,H,I,K,L,M,N,O,P] :: \varphi(O,P,B) . \varphi(E,C,C) . \varphi(H,K,L) . \varphi(M,I,N) \therefore [F,G] : \varphi(F,B,I) . \varphi(G,B,H) . \supset . F = G \therefore \supset . \varphi(H,E,I)$

Proof:

$[B,C,E,H,I,K,L,M,N,O,P] ::$

(α) $\varphi(O,P,B)$.

(β) $\varphi(E,C,C)$.

(γ) $\varphi(H,K,L)$.

(δ) $\varphi(M,I,N) \therefore$

(ϵ) $[F,G] : \varphi(F,B,I) . \varphi(G,B,H) . \supset . F = G \therefore \supset \therefore$

(ζ) $\varphi(I,E,I) \therefore$ $\qquad (\mathrm{XV},(\beta),(\delta))$

$[\exists F]:$

(η) $\quad \varphi(F,B,I):$ $\qquad (3,(\alpha),(\delta))$

$\quad [\exists G]$.

(ϑ) $\qquad \varphi(G,B,H)$. $\qquad (3,(\alpha),(\gamma))$

(ι) $\qquad F = G$. $\qquad ((\epsilon),(\eta),(\vartheta))$

(κ) $\qquad \varphi(G,B,I) \therefore$ $\qquad ((\eta),(\iota))$

(λ) $I = H$. $\qquad (\mathrm{XX},(\kappa),(\vartheta))$

$\varphi(H,E,I)$ $\qquad ((\zeta),(\lambda))$

5. $[A,B,C] : \varphi(A,B,C) . \supset . \varphi(B,A,C)$

Proof:

$[A,B,C] ::::$

(α) $\varphi(A,B,C) . \supset ::::$

$[\exists D] :\cdot:$

(β) $\quad \varphi(B,A,D) :\cdot:$ $\qquad (1,(\alpha))$

$\quad [\exists O,P] ::$

(γ) $\qquad \varphi(O,P,B) ::$ $\qquad (\mathrm{IX},(\alpha))$

$\qquad [\exists E] :\cdot.$

(δ) $\qquad\quad \varphi(E,C,C) \therefore$ $\qquad (3,(\alpha))$

(ε) $[H,I]:\varphi(H,E,I).\supset.[\exists M,N].\varphi(M,I,N)\therefore$ (VII)

(ζ) $[F,G,H,I]:\varphi(H,E,I).\varphi(F,B,I).\varphi(G,B,H).$
$\supset.F=G::\cdot$ (XVIII,(δ))

(η) $[H,I,K,L,M,N]::\varphi(H,K,L).\varphi(M,I,N)\therefore$
$[F,G]:\varphi(F,B,I).\varphi(G,B,H).\supset.F=G\therefore\supset.$
$\varphi(H,E,I)::\cdot$ (XXII,(γ),(δ))

(ϑ) $\varphi(E,B,B)::\cdot$ (VI,(β),(γ),(ε),(ζ),(η))

(ι) $D=C:::$ (XXI,(ϑ),(β),(α),(δ))

$\varphi(B,A,C)$ ((β),(ι))

2. $[A,B,D,E,F,G]\therefore\varphi(A,D,E).\vee.\varphi(D,A,E).\vee.$
$\varphi(D,E,A):\varphi(B,F,G).\vee.\varphi(F,B,G).\vee.\varphi(F,G,B).$
$\supset.[\exists C].\varphi(A,C,B)$ (from 3 and 5)

4. $[A,B,C,D,E,F,G]:\varphi(A,B,C).\varphi(C,D,E).\varphi(B,D,F).$
$\varphi(A,F,G).\supset.E=G$

Proof:

$[A,B,C,D,E,F,G]:$

(α) $\varphi(A,B,C).$

(β) $\varphi(C,D,E).$

(γ) $\varphi(B,D,F).$

(δ) $\varphi(A,F,G).\supset.$

(ε) $\varphi(D,B,F).$ (5,(γ)

$E=G$ (XXI,(α),(β),(ε),(δ))

The deductions gone through here show that the system of theses 1–5 is equivalent to thesis I, as I claimed above. (We could also get this result from the statement expressing the fact that a necessary and sufficient conditions for any function φ satisfying thesis I is that the field of φ should be an Abelian group with respect to φ.)

I should further point out that I had derived theses II–VI from thesis I prior to my having ever appealed to it in the derivation of theses 1–5. It follows that the system of theses II–VI forms a postulate system which is equivalent to the system of postulates 1–5.

It is easy to convince oneself that each of the five theses belonging to this system (i.e., II–VI) is independent of the other four: all of them except one (which is different in the five different cases) satisfy the functions φ, which are defined in turn by the following five formulae:

$$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . B = 1 . C = 1 . \vee . A = 1 . B = 1 . C = 2 . \vee . A = 1 . B = 1 . C = 3 . \vee . A = 1 . B = 2 . C = 3 . \vee . A = 1 . B = 3 . C = 3$$

$$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . B = 1 . C = 1 . \vee . A = 1 . B = 1 . C = 2$$

$$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = B . B = C . \supset . A = C$$

$$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . B = 2 . C = 2 . \vee . A = 2 . B = 1 . C = 2$$

$$[A, B, C] \therefore \varphi(A, B, C) . \equiv : A = 1 . B = 1 . C = 1 . \vee . A = 1 . B = 2 . C = 1 . \vee . A = 2 . B = 1 . C = 2 . \vee . A = 2 . B = 2 . C = 2$$

FUNDAMENTALS OF A NEW SYSTEM OF THE FOUNDATIONS OF MATHEMATICS[0]

INTRODUCTION

In 1927 I began in *Przegląd Filozoficzny* the publication of a larger work entitled 'On the Foundations of Mathematics.'[1] In the introduction to this work I wrote:[2]

"The purpose of this work is to deal with an awkward state of affairs in which I have found myself for a number of years. The situation is that I possess a great deal of unpublished scientific results in various areas of the foundations of mathematics. The number of these unpublished results continually grows, and because they interrelate with one another and with the results

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[0] *Translator's Acknowledgements*: I would like to personally thank a number of people who contributed in various ways towards the final realization of this translation. Marie Long produced the first typescript, and Karen Rhodes produced the revised version. They both labored long and corrected many errors and infelicities. Work on these manuscripts was supported by grants from Dean Tillman of North Carolina State University, and Professor Tilghman, Chairman of the Philosophy Department of Kansas State University. I wish to express special thanks to Dr. Janet Gardiner and also to Professor Jan Srzednicki. Their comments on various difficult passages were excellent and always led to significant improvements in the translation. But my deepest thanks are due to my wife Sandy and our children, for their accommodating tolerance of the many odd hours I have spent alone with these writings. *(Michael P. O'Neil)*

[1] Leśniewski [1].

[2] I give the footnotes which belong to the quoted sections in appropriately inserted parentheses. [They will here be collected at the end of the quotation, on pp. 415–416. Footnote numbers which occur in quoted passages will be contained in parentheses throughout. – *Tr.*]

of others who are investigating this area, the technical difficulties connected with editorial preparation for the press continually multiply.

"While experimenting with various ways of organizing the results I had reached, I considered, among other things, setting them out systematically after the example of Whitehead and Russell.(1) Again, however, the task is spreading itself over a number of years, and it is very difficult to estimate how long it would still take me to use this method of submitting my results for a wider technical discussion, results which already stem from over ten years of reflection on the foundations of mathematics.

"This difficult situation is also complicated by the fact that I reached my views under the influence of conversations with my colleagues and in connection with their still unpublished scientific results, and have achieved similar results which agree with theirs. Likewise, my views and observations, which I had been formulating for several years while lecturing at the university and through participating in numerous scientific discussions, have contributed towards the growth of similar views and results on the part of my colleagues. Out of an estimable loyalty to me they have until now withheld the publication of a number of their scientific results, until my own results relevant to these are published.

"Since I wished to hasten the publication of the results of my investigations into the area of the foundations of mathematics, I felt obliged to change my approach once more. I decided this time to use a method of presentation which could be called an autobiographical sketch, in contrast to the method of systematic compendium. I decided for the time being to pass over in silence the majority of the consequences which I previously had intended to derive explicitly from my various assumptions, and to concentrate on the most lucid presentation possible of the foundations and basic shape of the particular theories I have been constructing. In my presentation I shall take care that from the chronological order and from the interdependence of certain scientific facts, the reader himself will be able to give an account of and, especially, to learn of the still unpublished results of the investigations of other scholars upon which I have relied for some of my assertions and constructions.

"The system of foundations of mathematics whose basic plan I intend to present in this work is, in certain respects, both objectively and methodologically new. It consists of three deductive theories, the union of which I consider to be but one of the possible foundations of the whole of the systems of mathematical science. These theories are:

1) the theory I call Protothetic;(2) as regards its content, it corresponds in very rough outline, no doubt, to the theories known in the literature as 'calculus of equivalent statements',(3) 'propositional calculus',(4) 'theory of deduction',(5) in connection with the 'theory of apparent variables',(6) etc.

2) the theory I have designated Ontology, which forms a modernized traditional logic of a certain kind, and which – in its content and power – most approaches Schröder's 'class calculus'(7) when this is considered to include the theory of 'individuals',(8)

3) the theory I call Mereology, of whose primary and, in many respects, imperfect characteristics I published an account in a work entitled 'The Foundations of General Set Theory. I'.(9)

"To me *The Basic Laws of Arithmetic* by Gottlob Frege(10) is still the most impressive embodiment of the solid deductive results won during the historical establishment of mathematics, and the most valuable authority on them since the time of the Greeks. Frege's system, however, is not free of contradiction, as Russell proved when he constructed his famous antinomy concerning 'the class of classes which are not their own elements'.(11)

"Under the overwhelming influence of Russell's findings, the problem of the antinomies has become the central concern in the work of a number of prominent mathematicians. Their efforts, however, sometimes led them far away from the historical and intuitive basis out of which the antinomies grew. This favored the disappearance of a feeling for the distinction between mathematical science – understood as deductive theories serving to inscribe various realities of the world in the most exact laws possible – and certain consistent systems which do indeed ensure the possibility of obtaining an abundance of new theorems, yet which at the same time distinguish themselves by their lack whatsoever of any reality-linked intuitive scientific merit.

"Frege gives in the appendix to the second volume of *The Basic Laws of Arithmetic* a way of modifying his system so that Russell's antinomy cannot be constructed: it consists in replacing one axiom of the system with a different, obviously true axiom.(12) Yet from the general tone of the appendix, it could be surmised that the new axiom was not supported even by the author's own intuitions. Again, in order to eliminate the antinomies Zermelo(13) has introduced into his subtly constructed set theory a number of restrictions that have no intuitive basis. From a deeper point of view, however, one that issues from an irresistable intuitive necessity to believe in the truth of various assumptions and in the correctness of various inferences that lead from them to contradictions, it is quite immaterial whether Frege's system is changed in the way indicated, or whether Zermelo's set theory will ever lead to contradictions. The only way to a real solution of the antinomies from this point of view is through an intuitive undermining of the inferences or assumptions which contribute to the contradiction.(14) An unintuitive mathematics contains no effective remedy for any malady of the intuition.

"In creating his theory of types to do away with the antinomies, Russell appealed to, among other things, considerations of an intuitive nature.(15) As is well known, the theory of types is one of the cardinal elements of Whitehead and Russell's work mentioned above.(16) It stands as the most representative synthesis yet in the struggle against the antinomies. However, not even Whitehead and Russell are satisfied with it in its present form.(17) "Both versions of Whitehead and Russell's system possess striking deficiencies.(18) In particular, the matter of determining the conditions which any expression ought to satisfy in order to be accepted as a definition or added to the system as a new theorem has been an embarrassment in the foundations of these systems.(19)

"Leon Chwistek, in his system of the foundations of mathematics,(20) took care to formulate the directives concerning the assertion of definitions and addition of new theorems to the system more scrupulously than Whitehead and Russell did.(21) In this work I shall subject Chwistek's system to a critique.

"I have not encountered in the scientific literature any theoretical concept which would meet the requirements I place upon deductive theories, and which would also eliminate the existing antinomies in a way I would consider adequate. In both respects I am satisfied for now with the concept I shall develop below.

"This system of the foundations of mathematics I have constructed is indebted for a number of important improvements to Alfred Tarski, a lecturer in philosophy of mathematics at the University of Warsaw, who was my pupil there from 1919 to 1923, and who earned his doctorate under me in 1924. With regard to the concrete results of the considerations which Tarski has carried out in connection with my system, I shall try to illustrate them explicitly. But because of the nature of things, I cannot illustrate all of Tarski's occasional critical observations which have undermined one or another link in the chain of my theoretical conceptions through the various stages of the system's coming into being, such as, (e.g.,) all the subtle and well-intentioned advice and sometimes impalpable suggestions from which I had occasion to profit in numerous conversations with him.

"The present form of a single axiom of the theory I call Protothetic is the result of successive simplifications due in important respects to the results of Mordechaj Wajsberg's investigations, who was then a student at the University of Warsaw."[3]

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[3] Cf. Leśniewski [1], pp. 164–169.

[Footnotes for the quotation:]

(1) Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, Cambridge, Vol. I, 1910; Vol. II, 1912; Vol. III, 1913; Vol. I, second edition, 1925.

(2) Much earlier I employed for its designation the word 'logistic'; cf. Adolf Lindenbaum and Alfred Tarski, [1926], p. 322. Article on investigations in the area of the theory of manifolds. Proposed by W. Sierpiński.

(3) Cf. Ernst Schröder, *Vorlesungen über die Algebra der Logik (exacte Logik)*, First vol., Leipzig 1890, p. 161.

(4) Cf. Schröder, *op. cit.*, second vol., first section, Leipzig 1891, pp. 1–84 and 256–276.

(5) Cf. Whitehead and Russell, *op. cit.*, Vol. I, second ed., pp. 90–126.

(6) Cf. *l. c.*, pp. 127–186.

(7) Cf. Schröder, *op. cit.*, first vol., pp. 160 and 161.

(8) Cf. *op. cit.*, second vol., first section, pp. 318–349.

(9) Stanisław Leśniewski, [1916].

(10) *Grundgesetze der Arithmetik*, Ideographically derived by G. Frege, Jena, first vol. 1893; second vol. 1903.

(11) Cf. Frege, *op. cit.*, second vol., pp. 253 and 254.

(12) Cf. *l. c.*, pp. 262–265.

(13) E. Zermelo, 'Untersuchungen über die Grundlagen der Mengenlehre, I', *Mathematische Annalen 65* (1908).

(14) Cf. K. Grelling and L. Nelson, 'Remarks on the Paradoxes of Russell and Burali Forti', *Abhandlungen der Fries'schen Schule*, Neue Folge, 2, no. 3(1908), VIII, p. 314.

(15) Cf. Bertrand Russel, 'Mathematical Logic as based on the Theory of Types', *American Journal of Mathematics*, XXX, no. 3, July, 1908. p. 222. Cf. also Whitehead and Russell, *loc. cit.*, p. 37.

(16) Cf. *l. c.*, sec. VII.

(17) Cf. *l. c.*, sec. XIV.

(18) In connection with the first edition of the system, Cf. e.g., Leon Chwistek, 'The Theory of Constructive Types. (Principles of Logic and Mathematics). Part I. General Principles of Logic. Theory of Classes and Relations', Extracted from the *Annales de la Société Polonaise de Mathématique* Cracow 1923, p. 22, n. 3. The work of Chwistek contains a number of interesting and appropriately severe critical observations on the 1st edition of Whitehead and Russell's system.

(19) Cf. Frege's 'principles' which, concerning the assertion of definitions, and 'rules' which, concerning the proofs of theorems, have binding force (Frege, *op. cit.*, first vol., pp. 51, 52, and 61–64). Cf. as well, *l. c.*, Sec. VI and VII. Cf. also (1) Bertrand Russell, *Introduction to Mathematical*

Philosophy, London, New York, second ed., April, 1920, p. 151. (2) Cf. *l. c.*, p. 21.

(20) Cf. *op. cit.* Also, the continuation of this work has already appeared in print: Leon Chwistek. 'The Theory of Constructive Types. (Principles of Logic and Mathematics). Part II. Cardinal Arithmetic.' *Rocznik Polskiego Tow. Matematycznego. Annales de la Société Polonaise de Mathematique* III (1924, 1925).

It consists of a printing of Part II with a separate pagination which forms the continuation of the pagination of the printing of Part I. In referring to the 'Theory of Constructive Types' I shall give the page numbers in accordance with the pagination of this printing.

(21) Cf. Chwistek, *op. cit.*, pp. 20–33.

To this day only the introduction and first three sections of my work just quoted, 'On the Foundations of Mathematics', have appeared in print; they are entitled 'On Certain Questions Concerning the Meaning of Logistic Theses',[4] 'On Russell's Antinomy Concerning the Class of Classes Which are Not Their Own Elements',[5] 'On Various Ways of Understanding the Words 'Class' and 'Set''.[6] Because the editorial staff of *Przegląd Filozoficzny*, in view of the current abundance of 'philosophical' material, is able to devote only a few pages at a time in their columns for my work, which still has a mathematical character, the printing of this work will certainly take several more years. This circumstance has moved me to publish at the present time a briefer communication in the same area, in which I intend only to indicate the theoretical positions of my system of the foundations of mathematics. For questions concerning numerous details, I refer the reader, if I may, to the work printed in *Przegląd Filozoficzny*.

SECTION I. THE FOUNDATIONS OF PROTOTHETIC

§1. In 1912 Henry Maurice Sheffer proved that in Whitehead and Russell's theory of deduction one may define functions of two

4 *Op. cit.*, (Leśniewski [1], pp. 169–181.

5 *Op. cit.*, pp. 182–189.

6 *Op. cit.*, pp. 190–206.

propositional variables having the following property: each one can be used in the theory of deduction to define both alternation and negation (the two primitive functions in Whitehead and Russell's system[7]) if instead of them it is adopted as the primitive function. According to Sheffer's discovery, one of the functions with this property is that which for all values of the variables is equivalent to the function '$\sim (p \vee q)$'; the second is that function which for all values of the variables is equivalent to the function '$\sim p \vee \sim q$'.[8]

In 1916 J. G. P. Nicod used the second Sheffer function to base the theory of deduction on a single axiom.[9] If in accordance with Nicod the vertical stroke '|' is used to express that Sheffer function which is primitive in Nicod's system, and if dots are used as in Whitehead and Russell's system[10], then Nicod's single axiom for the theory of deduction can be written as follows:

$$p \mid . q \mid r : \mid :: t \mid . t \mid t : \mid .\because s \mid q . \mid : p \mid s . \mid . p \mid s .$$ [11]

(I might add that this axiom of Nicod's was simplified in 1925 by Jan Łukasiewicz, Professor of Philosophy at the University of Warsaw. Łukasiewicz, without changing Nicod's final directives, reduced the number of different variables in Nicod's axiom from five to four, having replaced in this axiom the variable 't' by the variable 's'. The simplification in question is by no means trivial: the complete proof of Nicod's axiom on the basis of Łukasiewicz's axiom rests upon twenty-four successive applications of the directives of Nicod's system, including the assumed insertion directive.[12]

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7 Cf. Whitehead–Russell [1], pp. 91–93.

8 Cf. Sheffer [1], pp. 487 and 488; cf. also Żyliński [1]; Żyliński[2], p. 208.

9 Cf. Nicod [1].

10 Cf. Whitehead–Russell [1], pp. 9–11.

11 *Ibid.*, Sect. XIX.

12 Cf. Russell [1], p. 151.

In defining the functions of the theory of deduction in terms of other such functions, both Sheffer and Nicod use a special equal-sign for definitions which they do not define in terms of the primitive functions of the system. The definitions of Sheffer have the form of expressions of the type '$p = q$'. Nicod's definitions have the form of expressions of the type '$p = q$ Df', used also by Whitehead and Russell.[13] This circumstance makes it difficult to say whether Nicod's theory of deduction is in fact constructed out of the single primitive term '$|$'.

In 1921 I realized that a system of the theory of deduction containing definitions would actually be constructed from a single term only if the definitions were written down with just that primitive term and without recourse to a special equal-sign for definitions. In particular, if this reform were introduced into Nicod's system, its definitions could be written out using some selected function constructed only with the primitive function '$p \mid q$' and which is equivalent for all its values to the ordinary equivalence function of the type '$r \equiv s$'. For example, in accordance with Nicod's definitions relevant to

$$p \equiv q \,.\, \equiv \,::\, p \mid .\, q \mid q : \mid : q \mid .\, p \mid p \therefore \mid \therefore p \mid .\, q \mid q : \mid : q \mid .\, p \mid p,$$

appropriate expressions of type

$$p \mid .\, q \mid q : \mid : q \mid .\, p \mid p \therefore \mid \therefore p \mid .\, q \mid q : \mid : q \mid .\, p \mid p$$

could be formulated as definitions instead of expressions of type '$p = q$ Df', which Nicod used. The definition of negation in circumstances thus turned around could have the form of the proposition

$$\sim p \,.\, \mid : p \mid p \,.\, \mid .\, p \mid p \therefore \mid \therefore p \mid p \,.\, \mid : \sim p \,.\, \mid .\, \sim p :: \mid :: \sim p \,.\, \mid : p \mid p \,.\, \mid .\, p \mid p \therefore \mid \therefore p \mid p \,.\, \mid : \sim p \,.\, \mid .\, \sim p;$$

the definition of alternation could have the form of the proposition

[13] Cf. Whitehead–Russell [1], p. 94.

$$p \vee q \,.\, \mid \therefore p \mid p \,.\, \mid .\, q \mid q : \mid : p \mid p \,.\, \mid .\, q \mid q :: \mid \therefore p \mid p \,.\, \mid .\, q \mid q : \mid$$
$$: p \vee q \,.\, \mid .\, p \vee q :\cdot: p :\cdot: p \vee q \,.\, \mid \therefore p \mid p \,.\, \mid .\, q \mid q : \mid : p \mid p \,.\, \mid .\, q \mid q$$
$$:: \mid \therefore p \mid p \,.\, \mid .\, q \mid q : \mid : p \vee q \,.\, \mid .\, p \vee q;$$

etc.

In 1922 Tarski made a paradoxical discovery. He established that with an appropriate use of function variables and quantifiers, all known functions of the theory of deduction can be defined with the equivalence function as the only primitive function. In his article 'On the Primitive Form of Logistic' Tarski wrote:

"I propose in this note to establish a theorem of logistic concerning the connections, unknown until now, which exist between the terms of this discipline. My reasonings are based upon propositions generally recognized by logisticans. However, I do not make them depend upon one or other theory of logical types; on the other hand, among all the theories of logical types that can be constructed(1), there exist some according to which my arguments are perfectly legitimate in their present form.(2)

"The problem for which I present the solution is the following: is it possible to construct a system of logistic recognizing the equivalence sign as the only primitive term (in addition, of course, to the quantifiers(3))?"[14]

And further:

"The theorem which will be demonstrated (Th. 10) represents a positive solution of the problem considered. It could serve, in fact, as the definition of the sign of logical product by means of the equivalence sign and the universal quantifier. But when one is already using the sign of logical product, the other terms of logistic can be easily defined with the aid of the following propositions(4):

$$[p] \therefore \sim (p) \,.\, \equiv : p \equiv .\, [q] \,.\, q$$

$$[p, q] \therefore p \supset q \,.\, \equiv : p \equiv .\, p \,.\, q$$

[14] Tarski [1], p. 196. Cf. Tarski [2], pp. 4 and 5.

$[p, q] : p \vee q . \equiv . \sim (p) \supset q^{(5)}$".[15]

[Footnotes for the quotation]:

(1) The possibility of constructing different theories of logical types is also recognized by the inventor of the best known of them. Cf. A. N. Whitehead and B. Russell, *Principia Mathematica*, Cambridge 1910, Sec. VII.

(2) One such theory was developed in 1920 by Professor S. Leśniewski in his course on the principles of arithmetic at the University of Warsaw.

(3) According to Peirce ('On the Algebra of Logic', *American Journal of Mathematics* VII (1885), p. 197.), who thus calls them, the symbols '$\prod$' (universal quantifier) and '$\sum$' (particular quantifier) represent abbreviations of the expressions 'for all significations of the terms...' and 'for some significations of the terms...'

(4) I adopt in this note the notations of Whitehead and Russell with some slight modifications; in particular, instead of expressions of the form 'φx' I write '$\varphi(x)$'.

(5) The terms '0' and '1' which figure, e.g., in Couturat, *The Algebra of Logic*, Paris 1905, can be defined as follow:

$$0 \equiv . [q] . q$$
$$1 \equiv . [q] . q \equiv q .$$

The Theorem 10 discussed in the quoted paragraph runs as follows:

$$[p, q] :: p . q . \equiv \therefore [f] \therefore p \equiv : [r] . p \equiv f(r) . \equiv . [r] . q \equiv f(r).$$[16]

In the same paper Tarski proved that in logistic systems, under his axioms or theorems, the thesis holds which says:

$$[p, q, f] : p \equiv q . f(p) . \supset f(q).$$

Also the thesis holds according to which the following must be true:

$$[p, q] \therefore p . q . \equiv : [f] : p \equiv . f(p) \equiv f(q).$$[17]

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15 Tarski [1], p. 197. Cf. Tarski [2], p. 6.

16 Tarski [1], p. 199, Cf. Tarski [2], p. 9.

17 Incidentally, I might point out that the method used by Tarski in his logistical work for writing down proofs with the aid of a single conditional

§ 2. In 1921 I constructed my 'theory of types', which Tarski mentioned in one of the footnotes to his work quoted above.[18] It was something like Whitehead and Russell's theory of types,[19] which I had generalized and simplified in a certain way. But even as I was constructing my theory of types, I considered it to be only an inadequate stop-gap that would at least avoid the threat of the antinomies, and which would for the time being enable me to operate with each kind of function variable I wanted to use for setting out fundamental mathematical theories (in particular, Ontology and Mereology, both having already been completed by then). To a certain extent my theory of types also filled up conspicuous gaps in the area of definition directives, which are formulated in an insufficient manner or not even formulated at all in the various systems of mathematical logic known to me.

In 1922 I outlined a concept of semantical categories as a replacement for the hierarchy of types, which is quite unintuitive to me. Frankly, I would still today feel obliged to accept this concept even if there were no antinomies at all. From a formal point of view my concept of semantical categories is closely related to the well-known type theories,[20] especially with regard to their theoretical consequences. Intuitively, however, the concept is more easily related to the thread of tradition running through Aristotle's categories, the parts of speech of traditional grammar, and

proposition is a copy of the method which I have used since 1921 in the daily practice of my university lectures.

[18] The date Tarski specified in the footnote quoted is a slight error in my favor. Cf. Tarski [2], p. 4. In addition to the university lectures, I have put forth the mentioned theory of types in the report entitled 'On the Grades of Grammatical Functions' (in Polish) which I gave at the scientific session of the Logical Section of the Warsaw Philosophical Institute on March 10, 1921. Cf. WPI [1], p. 248.

[19] Cf. Whitehead–Russell, pp. 161–167.

[20] Besides the footnote to *Principia Mathematica* mentioned in the last paragraph, cf. further in connection with these theories Chwistek [1], pp. 12–14 and 26–33; Chwistek [2], pp. 49, 50, and 54–56.

Husserl's meaning categories.[21] This concept is used quite generally in mathematics, particularly in mathematical logic, and I did not need to make any sacrifice in the generality of my intuitions concerning various theoretical topics.

In the same year I began the task of formulating, from the perspective of semantical categories, both the definition directives and the final directives for the fundamental mathematical theories (in particular, for Protothetic and Ontology). Testing and modifying various details of these directives lasted several years. But by 1922 the directives were precise enough to form the basis of a considerable amount of axiomatic research. At the time Tarski and I were cooperating closely in the investigation of the axioms and directives of Protothetic and Ontology. His contributions which have particular importance in this area will be illustrated below.

I have been careful to formulate the definition directives and final directives in such a way that they could easily be adapted to the various systems of Protothetic, depending upon which primitive terms are used in their construction. While investigating axioms, however, I concentrated on the problem of constructing for Protothetic the simplest axiom system possible based just on the equivalence sign as the only primitive term. The discovery by Tarski discussed above showed that such systems were indeed possible for Protothetic, even though none had actually yet been realized.

§ 3. I began the construction of the axiom system of Protothetic based upon the equivalence sign by selecting a combination of equivalence propositions demonstrable in the ordinary theory of deduction which could form an adequate axiomatic foundation for the system consisting of all equivalence propositions demonstrable in the ordinary theory of deduction. (For the sake of brevity I say of an expression X that it is an equivalence proposition if the following conditions are fulfilled:

[21] Cf. Husserl [1], pp. 294, 295, 305–312, 316–321, and 326–342.

(a) X is either a propositional variable or an equivalence,

(b) if any Y is an equivalence which forms a proper or improper part of the expression X, then the right and left sides of the equivalence Y are either propositional variables or equivalences.)

I obtained new propositions in this system on the basis of propositions which already belong to the system, by employing the 'substitution' directive, which substitutes equivalence propositions for variables in propositions already belonging to the system, and using the 'detachment' directive, which permits adding a proposition S to the system when both an equivalence A whose right side is equiform with S[22] and a proposition equiform with the left side of equivalence A belong to the system. (I limit myself here to only a very general characterization of these directives. I intend to formulate the directives of Protothetic as precisely as I can later in this article.)

Assuming the consistency of the ordinary theory of deduction,[23] I have convinced myself that a system could be constructed exactly as represented, which contains all equivalence propositions demonstrable in the ordinary theory of deduction and no others, were the following combination taken as a starting point:

A1. $p \equiv r \,.\, \equiv .\, q \equiv p \,\colon \equiv .\, r \equiv q$

A2. $p \equiv .\, q \equiv r \,\colon \equiv \colon p \equiv q \,.\, \equiv r \,.$

(I shall call this system in what follows the system SS; the proposition which appears here as axiom A2 had already been proved in the ordinary theory of deduction by Łukasiewicz prior to 1922.)[24] The method I used to assure myself that this was actually so was based upon my realizing the following facts, which I shall outline here with no pretension to exactness:

22 Cf. Frege [1], p. 107.

23 Cf. Post [1], p. 172.

24 Cf. Tarski [1], p. 199. Tarski [2], p. 8.

1) In SS the following propositions, among others, are demonstrable:

T1. $q \equiv r \,.\, \equiv \,.\, r \equiv q : \equiv \,.\, r \equiv r$ [A1, $q/p, r/q$][25]

T2. $r \equiv \,.\, q \equiv r : \equiv : r \equiv q \,.\, \equiv r \therefore \equiv : q \equiv r \,.\, \equiv \,.\, r \equiv q$ [A1, $r/p, (r \equiv q)/q, (q \equiv r)/r$]

T3. $p \equiv \,.\, r \equiv q : \equiv : p \equiv r \,.\, \equiv q$ [A2, $r/q, q/r$]

T4. $r \equiv \,.\, q \equiv r : \equiv : r \equiv q \,.\, \equiv r$ [A2, r/p]

T5. $s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv \therefore s \equiv \,.\, q \equiv p : \equiv r$ [A2, $s/p, (q \equiv p)/q$]

T6. $r \equiv t \,.\, \equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t :: \equiv \therefore r \equiv t \,.\, \equiv p : \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t$ [A2, $(r \equiv t)/p, p/q, (q \equiv r \,.\, \equiv \,.\, s \equiv t)/r$]

T7. $q \equiv r \,.\, \equiv \,.\, r \equiv q$ [T2, T4]

T8. $p \equiv q \,.\, \equiv \,.\, q \equiv p$ [T7, $p/q, q/r$]

T9. $p \equiv \,.\, q \equiv r : \equiv : p \equiv q \,.\, \equiv r \therefore \equiv \therefore p \equiv q \,.\, \equiv r : \equiv : p \equiv \,.\, q \equiv r$ [T7, $(p \equiv \,.\, q \equiv r)/q, (p \equiv q \,.\, \equiv r)/r$]

T10. $p \equiv \,.\, q \equiv r : \equiv : q \equiv p \,.\, \equiv r \therefore \equiv \therefore q \equiv p \,.\, \equiv r : \equiv : p \equiv \,.\, q \equiv r$ [T7, $(p \equiv \,.\, q \equiv r)/q, (q \equiv p \,.\, \equiv r)/r$]

T11. $p \equiv r \,.\, \equiv : q \equiv p \,.\, \equiv \,.\, q \equiv r \therefore \equiv \therefore q \equiv p \,.\, \equiv \,.\, q \equiv r : \equiv \,.\, p \equiv r$ [T7, $(p \equiv r)/q, (q \equiv p \,.\, \equiv \,.\, q \equiv r)/r$]

T12. $p \equiv r \,.\, \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv q \therefore \equiv \therefore r \equiv q \,.\, \equiv : p \equiv r \,.\, \equiv \,.\, q \equiv p$ [T7, $(p \equiv r \,.\, \equiv \,.\, q \equiv p)/q, (r \equiv q)/r$]

T13. $p \equiv q \,.\, \equiv r : \equiv : p \equiv \,.\, q \equiv r$ [T9, A2]

T14. $p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t$ [T13, $(q \equiv r)/q, (s \equiv t)/r$]

T15. $s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t :: \equiv \therefore s \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv t$ [T13, $(s \equiv \,.\, q \equiv p)/p, r/q, t/r$]

T16. $p \equiv r \,.\, \equiv \,.\, q \equiv p : \equiv \,.\, q \equiv r \therefore \equiv \therefore p \equiv r \,.\, \equiv : q \equiv p \,.\, \equiv \,.\, q \equiv r$ [T13, $(p \equiv r)/p, (q \equiv p)/q, (q \equiv r)/r$]

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[25] No doubt will appear about the way of referring to earlier theses of the system which is assumed here to the reader who knows *Principia Mathematica*. Cf. Whitehead–Russell [1], p. 98.

T17. $q \equiv p . \equiv . q \equiv r : \equiv . p \equiv r \therefore \equiv \therefore q \equiv p . \equiv : q \equiv r . \equiv . p \equiv r$ [T13, $(q \equiv p)/p, (q \equiv r)/q, (p \equiv r)/r$]

T18. $s \equiv . q \equiv p : \equiv . r \equiv t \therefore \equiv \therefore p \equiv : q \equiv r . \equiv . s \equiv t :: \equiv \because s \equiv . q \equiv p : \equiv :: r \equiv t . \equiv \therefore p \equiv : q \equiv r . \equiv . s \equiv t$ [T13, $(s \equiv . q \equiv p)/p, (r \equiv t)/q, (p \equiv : q \equiv r . \equiv . s \equiv t)/r$]

T19. $r \equiv r$ [T1, T7]

T20. $q \equiv r . \equiv . q \equiv r$ [T19, $(q \equiv r)/r$]

T21. $r \equiv r . \equiv . r \equiv r$ [T19, $(r \equiv r)/r$]

T22. $r \equiv q . \equiv : p \equiv r . \equiv . q \equiv p$ [T12, A1]

T23. $s \equiv . p \equiv r : \equiv \therefore q \equiv s . \equiv : p \equiv r . \equiv q$ [T22, $q/p, (p \equiv r)/q, s/r$]

T24. $r \equiv q . \equiv : p \equiv r . \equiv . q \equiv p \therefore \equiv :: q \equiv r . \equiv . r \equiv q : \equiv \therefore p \equiv r . \equiv . q \equiv p : \equiv . q \equiv r$ [T22, $(q \equiv r)/p, (p \equiv r . \equiv . q \equiv p)/q, (r \equiv q)/r$]

T25. $p \equiv . q \equiv r : \equiv : p \equiv r . \equiv q \therefore \equiv :: p \equiv q . \equiv r : \equiv : p \equiv . q \equiv r \therefore \equiv \therefore p \equiv r . \equiv q : \equiv : p \equiv q . \equiv r$ [T22, $(p \equiv q . \equiv r)/p, p \equiv r . \equiv q/q, p \equiv . q \equiv r/r$]

T26. $q \equiv r . \equiv . r \equiv q : \equiv \therefore p \equiv r . \equiv . q \equiv p : \equiv . q \equiv r$ [T24, 22]

T27. $p \equiv r . \equiv . q \equiv p : \equiv . q \equiv r$ [T26, 7]

T28. $p \equiv r . \equiv : q \equiv p . \equiv . q \equiv r$ [T16, T27]

T29. $q \equiv r . \equiv . r \equiv q : \equiv \therefore p \equiv . q \equiv r : \equiv : p \equiv . r \equiv q$ [T28, $(q \equiv r)/p, p/q, (r \equiv q)/r$]

T30. $p \equiv r . \equiv q : \equiv : p \equiv q . \equiv r \therefore \equiv :: q \equiv s . \equiv : p \equiv r . \equiv q \therefore \equiv \therefore q \equiv s . \equiv : p \equiv q . \equiv r$ [T28, $(p \equiv r . \equiv q)/p, (q \equiv s)/q, (p \equiv q . \equiv r)/r$]

T31. $p \equiv q . \equiv r : \equiv : q \equiv p . \equiv r \therefore \equiv :: p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r \therefore \equiv \therefore p \equiv . q \equiv r : \equiv : q \equiv p . \equiv r$ [T28, $(p \equiv q . \equiv r)/p, (p \equiv . q \equiv r)/q, (q \equiv p . \equiv r)/r$]

T32. $p \equiv . r \equiv q : \equiv : p \equiv r . \equiv q \therefore \equiv :: p \equiv . q \equiv r : \equiv : p \equiv . r \equiv q \therefore \equiv \therefore p \equiv . q \equiv r : \equiv : p \equiv r . \equiv q$ [T28, $(p \equiv . r \equiv q)/p, (p \equiv . q \equiv r)/q, (p \equiv r . \equiv q)/r$]

T33. $q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r \therefore \equiv \therefore s \equiv q \,.\, \equiv : p \equiv q \,.\, \equiv r ::$
$\equiv :\cdot: s \equiv \,.\, p \equiv r : \equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r :: \equiv :: s \equiv \,.$
$p \equiv r : \equiv \therefore s \equiv q \,.\, \equiv : p \equiv q \,.\, \equiv r$ $\quad$ [T28, $(q \equiv s \,.\, \equiv : p \equiv q \,.$
$\equiv r)/p, (s \equiv \,.\, p \equiv r)/q, (s \equiv q \,.\, \equiv : p \equiv q \,.\, \equiv r)/r$]

T34. $q \equiv s \,.\, \equiv : p \equiv r \,.\, \equiv q \therefore \equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r ::$
$\equiv :\cdot: s \equiv \,.\, p \equiv r : \equiv \therefore q \equiv s \,.\, \equiv : p \equiv r \,.\, \equiv q :: \equiv :: s \equiv \,.$
$p \equiv r : \equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r$ $\quad$ [T28, $(q \equiv s \,.\, \equiv : p \equiv r \,.$
$\equiv q)/p, (s \equiv \,.\, p \equiv r)/q, (q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r)/r$]

T35. $q \equiv p \,.\, \equiv r : \equiv \,.\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r \therefore t :\cdot:$
$\equiv \vdots\vdots p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv \therefore q \equiv p \,.\, \equiv r : \equiv \,.\, s \equiv t$
$:: \equiv :\cdot: p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r$
$\therefore \equiv t$ $\quad$ [T28, $q \equiv p \,.\, \equiv r : \equiv \,.\, s \equiv t/p, p \equiv \,.\, q \equiv r : \equiv \,.$
$s \equiv t/q, s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv t/r$]

T36. $s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t :: \equiv \therefore s \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv t$
$:\cdot: \equiv \vdots\vdots p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv :: s \equiv \,.\, q \equiv p : \equiv r \therefore$
$\equiv t :\cdot: \equiv :: p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv \therefore s \equiv \,.\, q \equiv p : \equiv \,.$
$r \equiv t$ $\quad$ [T28, $(s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t)/p, (p \equiv \,.\, q \equiv r : \equiv \,.$
$s \equiv t)/q, (s \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv t)/r$]

T37. $s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv t :: \equiv :: s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t :\cdot:$
$\equiv \vdots\vdots p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv t$
$:\cdot: \equiv :\cdot: p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv :: s \equiv \,.\, q \equiv p : \equiv r \therefore$
$\equiv t$ $\quad$ [T28, $(s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv t)/p, (p \equiv \,.\, q \equiv r : \equiv \,.$
$s \equiv t)/q, (s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t)/r$]

T38. $r \equiv t \,.\, \equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t :: \equiv \therefore t \equiv \,.\, r \equiv p :$
$\equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t :\cdot: \equiv \vdots\vdots s \equiv \,.\, q \equiv p : \equiv :: r \equiv t \,.\, \equiv \therefore$
$p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t :\cdot: \equiv :: s \equiv \,.\, q \equiv p : \equiv \therefore t \equiv \,.\, r \equiv p$
$: \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t$ $\quad$ [T28, $(r \equiv t \,.\, \equiv \therefore p \equiv : q \equiv r \,.$
$\equiv \,.\, s \equiv t)/p, (s \equiv \,.\, q \equiv p)/q, (t \equiv \,.\, r \equiv p : \equiv : q \equiv r \,.\, \equiv \,.$
$s \equiv t)/r$]

T39. $r \equiv t \,.\, \equiv p : \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t \therefore \equiv \therefore t \equiv \,.\, r \equiv p : \equiv :$
$q \equiv r \,.\, \equiv \,.\, s \equiv t :: \equiv \vdots\vdots r \equiv t \,.\, \equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t$
$:: \equiv \therefore r \equiv t \,.\, \equiv p : \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t :\cdot: \equiv :\cdot: r \equiv t \,.\, \equiv$
$\therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t :: \equiv \therefore t \equiv \,.\, r \equiv p : \equiv : q \equiv r \,.\, \equiv \,.$
$s \equiv t$ $\quad$ [T28, $(r \equiv t \,.\, \equiv p : \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t)/p, (r \equiv t \,.$

$\equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t)/q, (t \equiv \,.\, r \equiv p : \equiv : q \equiv r \,.\, \equiv \,.$
$s \equiv t)/r]$

T40. $q \equiv p \,.\, \equiv \,.\, q \equiv r : \equiv \,.\, p \equiv r$ [T11, T28]

T41. $p \equiv \,.\, q \equiv r : \equiv : p \equiv \,.\, r \equiv q$ [T29, T7]

T42. $p \equiv \,.\, q \equiv r : \equiv : p \equiv \,.\, r \equiv q \therefore \equiv \therefore p \equiv \,.\, q \equiv r : \equiv : p \equiv r \,.$
$\equiv q$ [T32, T3]

T43. $p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv :: s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t$
$:\cdot: \equiv :: p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv \therefore s \equiv \,.\, q \equiv p : \equiv \,.$
$r \equiv t$ [T36, T15]

T44. $q \equiv p \,.\, \equiv : q \equiv r \,.\, \equiv \,.\, p \equiv r$ [T17, T40]

T45. $p \equiv q \,.\, \equiv \,.\, q \equiv p : \equiv \therefore p \equiv q \,.\, \equiv r : \equiv : q \equiv p \,.$
$\equiv r$ [T44, $(q \equiv p)/p, (p \equiv q)/q$]

T46. $p \equiv \,.\, q \equiv r : \equiv : q \equiv p \,.\, \equiv r \therefore \equiv :: p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t$
$\therefore \equiv \therefore q \equiv p \,.\, \equiv r : \equiv \,.\, s \equiv t$ [T44, $(q \equiv p \,.\, \equiv r)/p, (p \equiv \,.$
$q \equiv r)/q, (s \equiv t)/r$]

T47. $s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv \therefore s \equiv \,.\, q \equiv p : \equiv r :: \equiv :\cdot: s \equiv : q \equiv p \,.$
$\equiv r \therefore \equiv t :: \equiv :: s \equiv \,.\, q \equiv p : \equiv r \therefore \equiv t$ [T44, $(s \equiv \,.\, q \equiv p$
$: \equiv r)/p, (s \equiv : q \equiv p \,.\, \equiv r)/q, t/r$]

T48. $r \equiv t \,.\, \equiv p : \equiv : t \equiv \,.\, r \equiv p \therefore \equiv :: r \equiv t \,.\, \equiv p : \equiv : q \equiv r \,.\, \equiv$
$.\, s \equiv t \therefore \equiv \therefore t \equiv \,.\, r \equiv p : \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t$ [T44, $(t \equiv \,.$
$r \equiv p)/p, (r \equiv t \,.\, \equiv p)/q, (q \equiv r \,.\, \equiv \,.\, s \equiv t)/r$]

T49. $p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv \therefore s \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv t ::$
$\equiv :\cdot: p \equiv \,.\, q \equiv r : \equiv \,.\, s \equiv t \therefore \equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t$
$:: \equiv :: s \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv t \therefore \equiv \therefore p \equiv : q \equiv r \,.\, \equiv \,.$
$s \equiv t$ [T44, $(s \equiv \,.\, q \equiv p : \equiv \,.\, r \equiv t)/p, (p \equiv \,.\, q \equiv r : \equiv \,.$
$s \equiv t)/q, (p \equiv : q \equiv r \,.\, \equiv \,.\, s \equiv t)/r$]

T50. $p \equiv \,.\, q \equiv r : \equiv : p \equiv r \,.\, \equiv q$ [T42, T41]

T51. $p \equiv q \,.\, \equiv r : \equiv : q \equiv p \,.\, \equiv r$ [T45, T8]

T52. $q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r \therefore \equiv \therefore s \equiv q \,.\, \equiv : p \equiv q \,.$
$\equiv r$ [T51, $q/p, s/q, (p \equiv q \,.\, \equiv r)/r$]

T53. $s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv t :: \equiv :: s \equiv \,.\, q \equiv p : \equiv r \therefore$
$\equiv t$ [T47, T5]

T54. $p \equiv q \,.\, \equiv r : \equiv : p \equiv .\, q \equiv r \therefore \equiv \therefore p \equiv r \,.\, \equiv q : \equiv : p \equiv q \,.$
$\equiv r$ [T25, T50]

T55. $p \equiv .\, q \equiv r : \equiv : p \equiv q \,.\, \equiv r \therefore \equiv \therefore p \equiv .\, q \equiv r : \equiv : q \equiv p \,.$
$\equiv r$ [T31, T51]

T56. $s \equiv .\, p \equiv r : \equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r :: \equiv :: s \equiv .\, p \equiv r :$
$\equiv \therefore s \equiv q \,.\, \equiv : p \equiv q \,.\, \equiv r$ [T33, T52]

T57. $p \equiv .\, q \equiv r : \equiv .\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r \therefore \equiv t$
$\because\!: \equiv \because\!: p \equiv .\, q \equiv r : \equiv .\, s \equiv t \therefore \equiv :: s \equiv .\, q \equiv p : \equiv r \therefore$
$\equiv t$ [T37, T53]

T58. $p \equiv r \,.\, \equiv q : \equiv : p \equiv q \,.\, \equiv r$ [T54, T13]

T59. $p \equiv .\, q \equiv r : \equiv : q \equiv p \,.\, \equiv r$ [T55, A2]

T60. $q \equiv p \,.\, \equiv r : \equiv .\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r \therefore$
$\equiv t$ [T59, $(q \equiv p \,.\, \equiv r)/p, s/q, t/r$]

T61. $q \equiv s \,.\, \equiv : p \equiv r \,.\, \equiv q \therefore \equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.$
$\equiv r$ [T30, T58]

T62. $q \equiv p \,.\, \equiv r : \equiv : p \equiv .\, q \equiv r$ [T10, T59]

T63. $r \equiv t \,.\, \equiv p : \equiv : t \equiv .\, r \equiv p$ [T62, $t/p, r/q, p/r$]

T64. $p \equiv .\, q \equiv r : \equiv .\, s \equiv t \therefore \equiv \therefore q \equiv p \,.\, \equiv r : \equiv .$
$s \equiv t$ [T46, T59]

T65. $p \equiv .\, q \equiv r : \equiv .\, s \equiv t \therefore \equiv \therefore q \equiv p \,.\, \equiv r : \equiv .\, s \equiv t ::$
$\equiv \because\!: p \equiv .\, q \equiv r : \equiv .\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r \therefore$
$\equiv t$ [T35, T60]

T66. $s \equiv .\, p \equiv r : \equiv \therefore q \equiv s \,.\, \equiv : p \equiv r \,.\, \equiv q :: \equiv :: s \equiv .\, p \equiv r :$
$\equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r$ [T34, T61]

T67. $r \equiv t \,.\, \equiv p : \equiv : q \equiv r \,.\, \equiv .\, s \equiv t \therefore \equiv \therefore t \equiv .\, r \equiv p : \equiv :$
$q \equiv r \,.\, \equiv .\, s \equiv t$ [T48, T63]

T68. $p \equiv .\, q \equiv r : \equiv .\, s \equiv t \therefore \equiv :: s \equiv : q \equiv p \,.\, \equiv r \therefore$
$\equiv t$ [T65, T64]

T69. $s \equiv .\, p \equiv r : \equiv \therefore q \equiv s \,.\, \equiv : p \equiv q \,.\, \equiv r$ [T66, T23]

T70. $s \equiv .\, p \equiv r : \equiv \therefore s \equiv q \,.\, \equiv : p \equiv q \,.\, \equiv r$ [T56, T69]

T71. $r \equiv t \,.\, \equiv \therefore p \equiv : q \equiv r \,.\, \equiv .\, s \equiv t :: \equiv \therefore r \equiv t \,.\, \equiv p : \equiv :$
$q \equiv r \,.\, \equiv .\, s \equiv t \because\!: \equiv \because\!: r \equiv t \,.\, \equiv \because p \equiv : q \equiv r \,.\, \equiv .\, s \equiv t :: \equiv$
$\therefore t \equiv .\, r \equiv p : \equiv : q \equiv r \,.\, \equiv .\, s \equiv t$ [T39, T67]

T72. $p \equiv . q \equiv r : \equiv . s \equiv t \therefore \equiv :: s \equiv . q \equiv p : \equiv r \therefore$
$\equiv t$ [T57, T68]

T73. $r \equiv t . \equiv \therefore p \equiv : q \equiv r . \equiv . s \equiv t :: \equiv \therefore t \equiv . r \equiv p : \equiv :$
$q \equiv r . \equiv . s \equiv t$ [T71, T6]

T74. $p \equiv . q \equiv r : \equiv . s \equiv t \therefore \equiv \therefore s \equiv . q \equiv p : \equiv .$
$r \equiv t$ [T43, T72]

T75. $s \equiv . q \equiv p : \equiv :: r \equiv t . \equiv \therefore p \equiv : q \equiv r . \equiv . s \equiv t$
$:\!\cdot\!: \equiv :: s \equiv . q \equiv p : \equiv \therefore t \equiv . r \equiv p : \equiv : q \equiv r . \equiv .$
$s \equiv t$ [T38, T73]

T76. $p \equiv . q \equiv r : \equiv . s \equiv t \therefore \equiv \therefore p \equiv : q \equiv r . \equiv . s \equiv t$
$:: \equiv :: s \equiv . q \equiv p : \equiv . r \equiv t \therefore \equiv \therefore p \equiv : q \equiv r . \equiv .$
$s \equiv t$ [T49, T74]

T77. $s \equiv . q \equiv p : \equiv . r \equiv t \therefore \equiv \therefore p \equiv : q \equiv r . \equiv .$
$s \equiv t$ [T76, T14]

T78. $s \equiv . q \equiv p : \equiv :: r \equiv t . \equiv \therefore p \equiv : q \equiv r . \equiv .$
$s \equiv t$ [T18, T77]

T79. $s \equiv . q \equiv p : \equiv \therefore t \equiv . r \equiv p : \equiv : q \equiv r . \equiv . s \equiv$
t [T75, T78]

2) Every equivalence of degree 1 and 2 is demonstrable in SS on the basis of theorems T19, T7, T20, and T21. (I say of an expression X that it is an equivalence of degree n if the following conditions are fulfilled:

(a) X is an equivalence, both sides of which are equivalence propositions that consist of exactly n (not necessarily different) variables respectively,

(b) if one side of the equivalence X contains some variable Y, the second side of the equivalence X contains exactly as many variables equiform with Y as are found in the first side, including the variable Y itself.)

3) If

(a) A is an equivalence proposition which contains exactly n ($n \geq 3$) variables (which need not necessarily be different),

(b) V is a variable belonging to A,

(c) every equivalence of degree lower than n is demonstrable in SS,

then a certain equivalence of degree n is demonstrable in SS whose left side is equiform with V.

We can convince ourselves of this with the help of an inference represented roughly as follows:

Assume that for a certain A, n, and V, conditions (a)–(c) are fulfilled. From (a) it follows that

(d) both sides of the equivalence A contain less than n variables.

From (a) and (b) it follows that

(e) one of the following cases obtains:
   (α) V is the right side of the equivalence A.
   (β) V is the left side of the equivalence A.
   (γ) V belongs to the right side of the equivalence A, but does not exhaust this side.
   (δ) V belongs to the left side of the equivalence A, but does not exhaust this side.

(f) If case (α) obtains, so does the equivalence whose two sides are equiform, which is obviously (according to (a) an equivalence of degree n whose left side is equiform with A and whose right side forms an equivalence whose right side is equiform with V. The said equivalence of degree n is demonstrable in SS on the basis of theorem T19, and is thus a required equivalence of clause 3), of whose correctness we wish to convince ourselves.

(g) If case (β) obtains, the equivalence (call it B), whose left side is equiform with A, but whose right side on the contrary is an equivalence whose left side is equiform with the right side of the equivalence A, and whose right side is equiform with the left side of the equivalence A, forms a required equivalence: the equivalence B is demonstrable in SS on the basis of T7.

(h) If case (γ) obtains, according to the fact that by (d) the right side of the equivalence A contains less than n variables we can form such an equivalence (call it C) of one degree lower than n whose left side is equiform with the right side of the equivalence A, but whose right side to the contrary is an equivalence whose right side is equiform with V. In accordance with (c) the equivalence C is demonstrable in SS. By substituting into T69 propositions equiform with the left side of the right side of the equivalence C for 'p', propositions equiform with the left side of the equivalence A for 'q', propositions equiform with V for 'r', and propositions equiform with the right side of the equivalence A for 's', we obtain in this way a certain equivalence (call it D) which is demonstrable in SS on the basis of T69. On the basis of D and C we can, with the help of the 'detachment' directive, prove in SS the required equivalence of degree n whose left side is equiform with A and whose right side forms an equivalence whose right side is equiform with V.

(i) If case (δ) obtains, according to the fact that by (d) the left side of the equivalence A contains less than n variables, we can form such an equivalence (call it E) of one degree lower than n whose left side is equiform with the left side of the equivalence A, but whose right side to the contrary is an equivalence whose right side is equiform with V. In accordance with (c) the equivalence E is demonstrable in SS. By substituting into T70 propositions equiform with the left side of the right side of the equivalence E for 'p', propositions equiform with the right side of the equivalence A for 'q', propositions equiform with V for 'r', and propositions equiform with the left side of the equivalence A for 's', we obtain in this way a certain equivalence (call it F) which is demonstrable in SS on the basis of T70. On the basis of the equivalences F and E we can, with the help of the 'detachment' directive, prove in SS the required equivalence of degree n whose left side is equiform with A and whose right side forms an equivalence whose right side is equiform with V.

It follows from (e), (f), (g), (h), and (i) that an equivalence is demonstrable in SS whose left side is equiform with A and whose right side forms an equivalence whose right side is equiform with V.

4) If

(a) A is an equivalence of degree n $(n \geq 3)$

(b) every equivalence of one degree lower than n is demonstrable in SS,

then A is demonstrable in SS.

We can convince ourselves of this with the help of an inference represented roughly as follows:

Assume that for a certain A and n the conditions (a) and (b) are fulfilled. From (a) it follows that there is a V such that

(c) V is a variable belonging to the left side of the equivalence A.

As attested by the considerations in 3), a B can be found, in accordance with (a), (c), and (b), such that

(d) B is an equivalence of degree n,

(e) the left side of the equivalence B is equiform with the left side of the equivalence A,

(f) the right side of the right side of the equivalence B is equiform with V,

(g) B is demonstrable in SS.

According to (a) and (c), a V' can be found such that

(h) V is a variable belonging to the right side of the equivalence A,

(i) V' is equiform with V.

As attested by the considerations in 3), a C can be found, in accordance with (a), (h), and (b), such that

(k) C is an equivalence of degree n,

(l) the left side of the equivalence C is equiform with the right side of the equivalence A,

(m) the right side of the right side of the equivalence C is equiform with V',

(n) C is demonstrable in SS.

According to (a), (d), (e), (k), and (l), a D can be found such that

(o) D is an equivalence of degree n,

(p) the left side of the equivalence D is equiform with the right side of the equivalence B,

(q) the right side of the equivalence D is equiform with the right side of the equivalence C.

From (p), (f), (i), (m), and (q) it follows that

(r) the right side of the left side of the equivalence D is equiform with the right side of the right side of the equivalence D.

According to (o) and (r), an E can be found such that

(s) E is an equivalence of one degree lower than n,

(t) the left side of the equivalence E is equiform with the left side of the left side of the equivalence D,

(u) the right side of the equivalence E is equiform with the left side of the right side of the equivalence D.

From (b) and (s) it follows that

(v) E is demonstrable in SS.

By substituting into T79 propositions equiform with V for 'p', propositions equiform with the left side of the equivalence E for 'q', propositions equiform with the right side of the equivalence E for 'r', propositions equiform with the left side of the equivalence A for 's', and propositions equiform with the right side of the equivalence A for 't', we obtain in this way a new equivalence F which fulfills the conditions:

(w) the left side of the equivalence F is equiform with B (according to (e), (t), (p), and (f)),

(x) the left side of the right side of the equivalence F is equiform with C (according to (l), (u), (q), (i), and (m)),

(y) the left side of the right side of the right side of the equivalence F is equiform with E,

(z) the right side of the right side of the right side of the equivalence F is equiform with A,

(aa) the equivalence F is demonstrable in SS on the basis of T79.

From (aa), (w), and (g) it follows that

(ab) a proposition equiform with the right side of the equivalence F is demonstrable in SS with the help of 'detachment'.

From (ab), (x), and (n) it follows that

(ac) by the same way, a proposition equiform with the right side of the right side of the equivalence F is demonstrable in SS.

From (ac), (y), (v), and (z) it follows that A (again with the help of 'detachment') is demonstrable in SS.

5) From 2) and 4) it follows that every equivalence of any natural degree is demonstrable in SS.

6) If

(a) A is an equivalence proposition,

(b) if some V is a variable belonging to A, and the number of variables equiform with V (V included) which belong to A is even

then A is demonstrable in SS.

A rough outline of the proof:

Assume that for a certain A the conditions (a) and (b) are fulfilled. From (a) it follows that an n has been found such that

(c) A contains exactly n (not necessarily different) variables.

According to (a), (b), and (c), a B can be found such that

(d) B is an equivalence of degree n,

(e) the left side of the equivalence B is an equivalence of degree $n/2$,

(f) the right side of the equivalence B is equiform with A.

From (d) and (e) it follows in accordance with 5) that

(g) B and the left side of the equivalence B are demonstrable in SS.

From (g) and (f) it follows that A is also demonstrable in SS (with the help of 'detachment').

7) If any proposition is a thesis belonging to SS, then it is an equivalence proposition demonstrable in the ordinary theory of deduction, because:

(a) both axioms of SS are equivalence propositions demonstrable in the ordinary theory of deduction;

(b) applying the 'substitution' and 'detachment' directives to the equivalence propositions demonstrable in the ordinary theory of deduction, we always obtain only equivalence propositions demonstrable in the ordinary theory of deduction.

8) No absurdity of type 1) is demonstrable in the ordinary theory of deduction, because of the consistency of this theory. (I say of an expression X that it is an absurdity of type n if the following conditions are fulfilled:

(a) X is an equivalence proposition containing exactly n variables,

(b) no variable contained in X is equiform with another variable in X.)

9) If

(a) A is an absurdity of type n ($n \geq 2$),

(b) no absurdity of type one less than n is demonstrable in the ordinary theory of deduction,

then A is not demonstrable in the ordinary theory of deduction.

A rough sketch of the proof:

Assume that for certain A and n, conditions (a) and (b) are fulfilled, and suppose at the same time that

(c) A is demonstrable in the ordinary theory of deduction.

According to (a), a B can be found such that

(d) B is an equivalence of degree n,

(e) the left side of the equivalence B is equiform with A,

(f) the right side of the equivalence B is an absurdity of type n,

(g) the left side of the right side of the equivalence B is a propositional variable,

(h) the right side of the right side of the equivalence B is an absurdity of type $n - 1$.

According to (d) it follows from 5) and 7) that

(i) B is demonstrable in the ordinary theory of deduction.

According to (i), (e), and (c), a C can be found such that

(k) C is equiform with the right side of the equivalence B,

(l) C is demonstrable in the ordinary theory of deduction.

From (g) and (k) it follows that

(m) the left side of the equivalence C is a propositional variable.

Substituting theorem T19 into C for the variable which, according to (m), forms the left side of the equivalence C and which does not correspond to any variable equiform with it in the right side of this equivalence according to (k) and (f), we obtain a new equivalence D which fulfills the following conditions:

(n) D is demonstrable in the ordinary theory of deduction on the basis of C (in accordance with (l)),

(o) the left side of the equivalence D is equiform with T19,

(p) the right side of the equivalence D is an absurdity of type $n - 1$ (in accordance with (k) and (h)).

From (n) and (o) it follows that

(r) the right side of the equivalence D is demonstrable in the ordinary theory of deduction,

but from (b) and (p) it follows that

(s) the right side of the equivalence D is not demonstrable in the ordinary theory of deduction.

(s) contradicts (r). So our assumption that conditions (a), (b), and (c) are fulfilled together must be false.

10) From 8) and 9) it follows that no absurdity of a natural type is demonstrable in the ordinary theory of deduction.

11) If

(a) A is an equivalence proposition,

(b) V is a variable belonging to A,

(c) the number of variables equiform with V (including V) belonging to A is odd

then A is not demonstrable in the ordinary theory of deduction.

A rough sketch of the proof

Assume that for certain A and V, conditions (a)–(c) are fulfilled, and suppose at the same time that

(d) A is demonstrable in the ordinary theory of deduction.

In accordance with 10) it follows from (d) that

(e) A is not an absurdity of a natural type.

According to (a), (b), (c), and (e), a B can be found such that

(f) B is an equivalence of a natural degree,

(g) the left side of the equivalence B is equiform with A,

(h) the left side of the right side of the equivalence B is an equivalence of a natural degree,

(i) the right side of the right side of the equivalence B is an absurdity of a natural type.

In accordance with (f) and (h) it follows from 5) and 7) that

(k) B is demonstrable in the ordinary theory of deduction,

(l) the left side of the right side of the equivalence B is demonstrable in the ordinary theory of deduction.

From (k), (g), and (d) we conclude that

(m) the right side of the equivalence B is demonstrable in the ordinary theory of deduction.

From (m) and (l), that

(n) the right side of the right side of the equivalence B is demonstrable in the ordinary theory of deduction.

In accordance with (i) it follows from 10) that

(o) the right side of the right side of the equivalence B is not demonstrable in the ordinary theory of deduction.

(o) contradicts n. So our assumption that conditions (a)–(c) and (d) are fulfilled together must be false.

12) If

(a) A is an equivalence proposition,

(b) A is demonstrable in the ordinary theory of deduction,

then A is demonstrable in SS.

Commentary: Suppose that for a certain A, conditions (a) and (b) are fulfilled. According to (a) and (b) it follows from 11) that

(c) if any V is a variable belonging to A, the number of variables equiform with V (including V) belonging to A is even.

We conclude from (a) and (c), in accordance with the considerations in 6), that A is demonstrable in SS.

13) It follows from 12) and 7) that SS contains all equivalence propositions demonstrable in the ordinary theory of deduction, and no other propositions.

§ 4. I arrived at a further stage in the development of Protothetic by considering this question: through which axioms and directives should one strengthen the system SS[26] discussed in the preceding section, in order to obtain from it a system of the ordinary propositional calculus to which has been added the theses

$$[p, q, f] \therefore p \equiv q \,.\, \supset : f(p) \,.\, \equiv \,.\, f(q),$$

together with all its consequences? I needed to construct a system in which, among other things, just such a thesis would be demonstrable, because ever since 1922 this thesis has had for me as much validity as any thesis of propositional calculus generally. (In a subsequent section I shall, primarily on technical

[26] In one of the subsequent sections of this article I shall be engaged with certain theses, any one of which could be taken as the single axiom of the system SS instead of the axiom system consisting of the axioms A1 and A2, as Wajsberg proved in 1926.

and editorial grounds, be occupied somewhat more directly with various theoretical doubts that might be raised about this thesis.) For the sake of brevity I shall here name the system of a strengthened propositional calculus I constructed in this manner SS1; this system is characterized by, among other things, the following properties:

(a) Variables appear in SS1 of all such 'semantical categories' that are represented in it by any 'constants'; in particular, in SS1 we have variable function signs with arguments which are propositions, in contrast to the system of Whitehead and Russell.[27]

(b) Of the two kinds of variables, the so-called apparent and real variables, which were still accepted by Whitehead and Russell in the first edition of *Principia Mathematica*,[28] only the 'apparent' variables have been retained in SS1. (As early as 1920 I had given an account of the redundancy of introducing 'real' variables into mathematics. In his 'Two-Valued Logic', published in 1921, and constructed only with the help of 'apparent' variables, Łukasiewicz wrote, "In recognizing only apparent variables I have followed the opinion of Prof. Leśniewski."[29] Whitehead and Russell recognize the redundancy of distinguishing 'real' from 'apparent' variables in the second edition of Vol. I of *Principia Mathematica*, 1925.[30]

Following the notation of Whitehead and Russell, which represents, as is well known, a development of Peano's notation,[31] the axioms of SS1 could be written down in the following way:

Ax. I. $[p, q, r] \therefore p \equiv r \,.\equiv .\, q \equiv p \,:\equiv .\, r \equiv q$[32]

[27] Cf. Whitehead–Russell [1], p. 7.

[28] Cf. Whitehead [1], pp. 16–17.

[29] Łukasiewicz [1], p. 3

[30] Cf. Whitehead–Russell [1], Secs. XIII, XVIII.

[31] Whitehead–Russell [1], p. 4

[32] Cf. A1 in §3.

Ax. II. $[p, q, r] \therefore p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r$[33]

Ax. III. $[q, p] :: [f] :: g(p, p) . \equiv \therefore [r] : f(r, r) . \equiv . g(p, p) : \equiv : [r] : f(r, r) . \equiv . g(p \equiv . [q] . q, p) :: \equiv . [q] . g(q, p).$

Relying upon Tarski's Th 10, which was mentioned above,and upon the results of his investigations dating from 1922 concerning the equivalences which exist between the thesis $[p, q, f] : p \equiv q . f(p) . \supset . f(q)$ and various other theses from the propositional calculus,[34] I convinced myself that Ax. III is demonstrable in ordinary propositional calculus strengthened by adding the thesis

(a) $[p, q, f] \therefore p \equiv q . \supset : f(p) . \equiv . f(q).$

Using these inference methods of Tarski's in a perfectly straightforward way, I established in the propositional calculus strengthened by the addition of thesis (a), that

(b) $[g, p] : g(p, p) \cdot g(\sim (p), p) . \equiv . [q] \cdot g(q, p)$[35]

and

(c) $[g, p] :: \sim (p) . \equiv : p \equiv . [q] . q \therefore \supset : g(\sim (p), p) . \equiv . g(p \equiv . [q] . q, p).$ [36]

Considering that

(d) $[p] \therefore \sim (p) . \equiv : p \equiv . [q] . q,$ [37]

I inferred from (c) and (d) that

(e) $[g, p] : g(\sim (p), p) . \equiv g(p \equiv . [q] . q, p),$

and further, from (b) and (e) that

(f) $[g, p] : g(p, p) . g(p \equiv . [q] . q, p) . \equiv . [q] . g(q, p).$

In a way fully analogous to the inference method used by Tarski in the proof of his Th. 10, I convinced myself that

(g) $[p, q] :: p . q . \equiv :: [f] :: p \equiv \therefore [r] : f(r, r) . \equiv p : \equiv : [r] : f(r, r) . \equiv q.$

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[33] Cf. A2 in §3.

[34] Cf. Tarski [2], pp. 22, 24. Tarski [3], pp. 64, 65, 67, 68, 71–73.

[35] Cf. Tarski [2], pp. 18, 19, 24. Tarski [3], pp. 67, 68, 73.

[36] Cf. Tarski [2], p. 24. Tarski [3], p. 73.

[37] Cf. Schröder [1], p. 77, Tarski [1], p. 197.

From (f) and (g) I obtained Ax. III. Since it will follow from considerations included in further sections of this article, I shall mention here that the expressions which appear in Ax. I–III belong to only two semantical categories: the semantical category represented, e.g., by the variables 'p' and 'r' in the expression '$p \equiv r$'; and the semantical category represented, e.g., by the sign '$\equiv$', or also by the variable 'f' in the expression 'f(r,r)'. Had I decided in formulating the axioms of SS1 to operate with expressions belonging to yet a third semantical category, namely, one with variable function-signs of only one propositional argument, I could have given to Ax. III a somewhat shorter form, as I already well knew when I was formulating the axiom system Ax. I–III. I wanted, however, to avoid bringing a third semantical category into the axioms of SS1. My motives were more or less the same as those which guide the efforts of numerous investigators in their attempts to reduce, e.g., the number of axioms, or the number of primitive terms of various theories.

In the symbolism I devised the axioms of SS1 have the following form, which is easily deciphered by comparing both versions of the corresponding axioms. (Expressions of the form '$\phi(p, q)$' here represent the corresponding expressions of the form '$p \equiv q$' in Axioms I–III):

A1. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big)\urcorner$,

A2. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big)\urcorner$,

$$\text{A3.}\ \llcorner gp \lrcorner \ulcorner \phi\Bigg(\llcorner f \lrcorner \ulcorner \phi\bigg(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g\big(\phi(p\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner\Big)\bigg)\urcorner \llcorner q \lrcorner \ulcorner g(qp)\urcorner\Bigg)\urcorner.$$

Using the following six directives, I obtained new propositions in SS1 on the basis of propositions already belonging to the system (for the present I limit myself in this section to a quite general characterization of these directives):

(α) the 'detachment' directive — as in SS,

(β) the 'substitution' directive,[38]

(γ) the 'distribution of quantifiers' directive by which, in case some thesis T compounded of a universal quantifier Q and an equivalence A standing under the quantifier already belongs to the system, it is permissible to add to the system a new thesis which is formed from thesis T through 'transferring' — by an exactly determined (but in practice, no doubt, somewhat artificial) way — all or only some variables contained in the quantifier into the quantifiers in front of the left and in front of the right side of the equivalence A. (On the basis of this directive it is permissible, e.g., to add to the system the thesis

$$\phi\left(\llcorner pqr\lrcorner\ \ulcorner\phi\left(\phi(pr)\phi(qp)\right)\urcorner\ \llcorner qr\lrcorner\ \ulcorner\phi(rq)\urcorner\right)$$

or the thesis

$$\llcorner r\lrcorner\ \ulcorner\phi\left(\llcorner pq\lrcorner\ \ulcorner\phi\left(\phi(pr)\phi(qp)\right)\urcorner\ \llcorner q\lrcorner\ \ulcorner\phi(rq)\urcorner\right)\urcorner,$$

as soon as one already has A1 in the system. It is also permissible according to this directive to derive in the system the thesis

$$\phi\left(\llcorner fgp)\ \ulcorner\phi\left(g(pp)\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)g(pp)\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)g\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\right)\urcorner\ \llcorner gpq\lrcorner\ \ulcorner g(qp)\urcorner\right)$$

[38] Cf. below in §11 TE XLVII and TE XLVIII and further below in the same §11 point 4) of the direction concerning the method of construction of SS5.

or the thesis

$$\llcorner p \lrcorner \ulcorner \phi\left(\llcorner fg \lrcorner \ulcorner \phi\left(g(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f (rr)g\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner \right)\right)\urcorner \llcorner qg \lrcorner \ulcorner g(qp)\urcorner \right)\urcorner ,$$

as soon as A3 is already at one's disposal, etc.[39]),[40]

(δ) the directive for writing out definitions which by satisfying certain exactly stated conditions, possess the form of equivalences that contain the definiendum in their left side,

(ϵ) the directive for writing out definitions which by satisfying certain exactly stated conditions, are composed of a universal quantifier and an equivalence standing under this quantifier that contains the definiendum in its left side,

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[39] Cf. Whitehead–Russell [1], Theorems *10.271 and *11.33.

[40] Cf. below in §11 TE XLV and point 2) of the direction concerning the method of construction of SS5: To avoid any possible misunderstanding I might mention that, taken alone, none of the directives of SS1 in adding new theses to the system permit the use of the method of inferring according to the schema

1) $\llcorner ab \ldots kl \ldots xy \ldots \lrcorner \ulcorner \phi\big(f(ab \ldots kl \ldots)g(ab \ldots xy \ldots)\big)\urcorner$

and

2) $\llcorner ab \ldots kl \ldots \lrcorner \ulcorner f(ab \ldots kl \ldots)\urcorner$

therefore

3) $\llcorner ab \ldots xy \ldots \lrcorner \ulcorner g(ab \ldots xy \ldots)\urcorner$.

A derivation of the corresponding thesis of type 3) from any theses of types 1) and 2) can be carried out step by step in the system considered by first obtaining from thesis 1), in accordance with directive (γ), the appropriate thesis which implies that

1*) $\phi\big(\llcorner ab \ldots kl \ldots \lrcorner \ulcorner f(ab \ldots kl \ldots)\urcorner \llcorner ab \ldots xy \ldots \lrcorner \ulcorner g(ab \ldots xy \ldots)\urcorner \big)$

,

and afterwards applying the usual 'detachment' in accordance with directive (α) upon theses 1*) and 2).

(ζ) the directive concerning quantifiers which in conjunction with the remaining directives permits the carrying out in practice of all generally known operations with the universal quantifier (I shall go somewhat further into this directive in §6).

Later sections of this article will enable the reader to determine just how concrete the derivations of the various theorems of ordinary propositional calculus looked in SS1.

§ 5. The idea introduced into the axiom system of SS1 by Axiom A3 came to me under the direct influence of Łukasiewicz's *Two-Valued Logic* mentioned earlier. In it he continues the tradition of verifying particular theses of the propositional calculus by means of substituting 'zeroes' and 'ones' for their variables[41] It contains among its directives one that runs as follows:

"(d) I admit any expression which contains variables with universal quantifiers and from which there arises a clearly recognized expression through the substitution of the values 0 and 1 into the positions of the variables."[42] Axiom A3 came about as a result of my inclination to capture within the scope of some special axioms the theoretical possibilities that issue from this directive. In practice A3 enabled me (as will be shown in a later section) to apply in SS1, from a certain place in the system onwards, a method for proving or, respectively, refuting theses that begin with universal quantifiers containing propositional variables, by referring to those theses already proved or refuted in SS1 which could be constructed from the theses to be proved or refuted by appropriately substituting for their propositional variables the expressions

$$\llcorner q \lrcorner \ \ulcorner q \urcorner \text{ and } \phi(\llcorner q \lrcorner \ \ulcorner q \urcorner \ \llcorner q \lrcorner \ \ulcorner q \urcorner),$$[43]

which correspond in SS1 to the 'zeroes' and 'ones' of the traditional propositional calculus.

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[41] Cf. Łukasiewicz [1], p. 3.

[42] *Op. cit., p. 11.*

[43] Cf. Tarski [1], p. 197; Tarski [2], p. 6.

I did not know how to prove a number of theses in SS1 which are meaningful in this system and which for me are as valid as any known thesis of ordinary propositional calculus. In particular, I could not devise any means in this respect for dealing with the numerous theses containing variables which are not propositional variables, but variable functions-signs. It was completely unknown to me, e. g., whether the thesis which could be fixed in a symbolism given in the style of Whitehead and Russell's in the form of the proposition

$$[f,g] \therefore [p,q] : f(p,q) . \equiv . g(p,q) : \equiv : [\Phi] : \Phi\{f\} . \equiv . \Phi\{g\}$$

is derivable in SS1. Since I wanted to construct a system of propositional calculus which while containing SS1, at the same time should possess the property that I would not know how to construct in this system any meaningful thesis which I would not know how to prove or refute in it, I completed SS1 in 1922 by means of a new directive (η), which is formed according to the example of Łukasiewicz's directive (d) quoted just above, and which concerns all variables that appear in SS1 other than propositional variables.

Directive (η) allowed me to add to the system a new thesis T beginning with a universal quantifier containing variable function-signs of any semantical category if the system already included those theses which could be obtained from T if, for the variables mentioned, certain constant function-signs were substituted into it whose method of definition for all semantical categories is exactly determined completely in advance. For the sake of brevity I shall here name the system formed from SS1 through the completion of it by means of directive (η), the system SS2. Of the numerous possible systems of the theory mutually equivalent to one another, it is this one I call Protothetic. Speaking very loosely, I could say that SS2 is an absolutely 'finitistic' system, for it permits us to establish an exactly determined finite number of different possible values for the variables of any semantical category appearing in the system (two values for propositional variables ('zero' and 'one' of the traditional propositional calculus), four values for variable function-signs within propositional functions of one propositional argument,[44] sixteen values for variable function-signs within propositional functions of one argument that belongs to the same semantical category to which the function-signs within the propositional functions of one propositional argument belong, etc.). Just these possible values of the variables of any given semantical category correspond to the constant function-signs mentioned above which directive (η) is about. The reader will be able to give himself a more exact account of the character of the directive mentioned, which I have represented here only quite generally, in further sections of this article.

To formulate precisely the directive (η) discussed here I needed a complicated apparatus of numerous supplementary terminological explanations which, because of the nature of things, would have inflicted a significant hinderance upon the eventual reader in understanding the construction of my system. This fact led me to look for some other directive which would yield the same theoretical effect as directive (η), yet which could be precisely formulated in some easier fashion. In particular, I reflected upon the question of whether one could not construct a system of Protothetic equivalent to SS2 if, instead of directive (η), a directive were accepted which allowed a direct statement in some form or other of the 'extensionality' of every kind of function appearing in Protothetic without consideration of the semantical category of the function-signs concerned. I have not achieved any concrete results on this question, but I shall return to it in §7 below.

§ 6. My aim in formulating the directive (ζ) in §4 was to make legitimate my practice in SS1 of guaranteeing certain propositions contained within theses already belonging to the system: those

[44] Cf. Tarski [2], p. 2; Tarski [3], p. 61.

propositions which in the style of Whitehead and Russell's symbolism could be written as formulae of the type

$p \supset . [q] . f(q),$
$p \supset . [q, r] . f(q, r),$
$p \supset . [q, r, s] . f(q, r, s),$ etc.

(The number and semantical categories of the terms appearing in the concluding proposition of this series are meant to be quite arbitrary.) I also wanted to be able to replace these traditional propositions with correspondingly equivalent propositions which can be written out in the same symbolism by means of corresponding formulae of the type

$[q] : p \supset . f(q),$
$[q, r] : p \supset . f(q, r),$
$[q, r, s] : p \supset . f(q, r, s),$ etc.,

and, *vice versa*, to replace the latter propositions by the former. I realized that I could legitimately do all of this by using the thesis which says

$$[f, p, q] \therefore p \equiv q . \supset : f(p) . \equiv . f(q).$$

I would not need to make use of directive (ζ) in the derivation of this thesis from the axioms of SS1 if I were able to obtain in this system propositions corresponding to the following:

(A) $[f, p] \therefore p \supset . [q] . f(q) : \equiv : [q] : p \supset . f(q),$
$[f, p] \therefore p \supset . [q, r] . f(q, r) : \equiv : [q, r] : p \supset . f(q, r),$
$[f, p] \therefore p \supset . [q, r, s] . f(q, r, s) : \equiv : [q, r, s] : p \supset . f(q, r, s),$
etc.[45]

I therefore concentrated my efforts on the problem of which would be the simplest possible directive to adopt sufficient to guarantee the provability of these propositions in the system under consideration. Since I did not have the conditional sign among the primitive terms of my system, and since at the same time I

[45] Cf. Whitehead–Russell [1], Theorems *10.21 and *11.3.

did not want to mention specifically in the directives of the system particular signs that would first be defined in the system, I formulated directive (ζ) in such a way that this directive, while it concerned various relations of both equivalences and quantifiers, did not allude to the conditional sign at all. Constructing the directive in question in this way, I made use of Tarski's result already discussed in §1, which concerns the definability of all known functions of the theory of deduction, and in particular, of the conditional function, by means of only the equivalence function taken as primitive.

Tarski had somewhat simplified the directive (ζ) I had adopted in SS1, and the even simpler directive that arose in the way just described I simplified somewhat further still. These results concerning various stages in the formulation of directive (ζ) were deprived of any topical interested by the fact, proved by Tarski in 1922, that any special directive is actually quite superfluous, because the propositions (A) discussed above are already demonstrable in SS1 with the help of the remaining directives of the system. Tarski based his view concerning this upon, among other things, the possibility I had already established earlier of obtaining in SS1 the following results without the help of directive (ζ):

(a) theses corresponding to all the theses of SS;

(b) theses corresponding to theorems well known in the tradition as: $[p] . 0 \supset p$ and $[p] : 1 \supset p . \equiv p$;

(c) the applicability in practice in SS1 of the method of proving theses beginning with quantifiers containing propositional variables by reference to corresponding theorems already proved in the system which can be formed from the theses to be proved by substituting 'zeroes' and 'ones' for the propositional variables belonging to these theses.

In presenting a rough sketch of the method Tarski applied for proving propositions (A) in SS1 without the help of directive (ζ), I shall for the time being use a symbolism in the style of Whitehead and Russell, in view of the fact that the reader has not yet been sufficiently prepared for understanding the proper symbolism of SS1. The proof of the thesis which says that

$$[f,p] \therefore p \supset . [q] . f(q) : \equiv : [q] : p \supset . f(q)$$

could be outlined in agreement with Tarski as follows: it is permissible to assert (cf. points (a) and (b) above)

(1) $[p,q] \therefore p \equiv : q \equiv . p \equiv q,$[46]

(2) $[p,q,r] \therefore p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r,$

(3) $[p] . 0 \supset p,$

(4) $[p] : 1 \supset p . \equiv p.$

From thesis (1) we conclude on the basis of directive (β) that

(5) $[f] :: 0 \supset . [q] . f(q) : \equiv :: [q] : 0 \supset . f(q) : \equiv \therefore 0 \supset . [q] . f(q) : \equiv : [q] : 0 \supset . f(q).$

From (5) on the basis of dir. (γ) that

(6) $[f] : 0 \supset . [q] . f(q) : \equiv :: [f] :: [q] : 0 \supset . f(q) : \equiv \therefore 0 \supset . [q] . f(q) : \equiv : [q] : 0 \supset . f(q).$

From (3) and dir. (β) that

(7) $[f] : 0 \supset . [q] . f(q),$

(8) $[f,q] : 0 \supset . f(q).$

From (6), (7), and dir. (α) that

(9) $[f] :: [q] : 0 \supset . f(q) : \equiv \therefore 0 \supset . [q] . f(q) : \equiv : [q] : 0 \supset . f(q).$

From (9) and dir. (γ) that

(10) $[f,q] : 0 \supset . f(q) : \equiv \therefore [f] \therefore 0 \supset . [q] . f(q) : \equiv : [q] : 0 \supset . f(q).$

From (10), (8), and dir. (α) that

(11) $[f] \therefore 0 \supset . [q] . f(q) : \equiv : [q] : 0 \supset . f(q).$

From (2) and dir. (β) that

(12) $[f] :: 1 \supset . [q] . f(q) : \equiv . [q] . f(q) \therefore \equiv :: [q] : 1 \supset . f(q) : \equiv . [q] . f(q) \therefore \equiv \therefore 1 \supset . [q] . f(q) : \equiv : [q] : 1 \supset . f(q).$

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[46] I might point out that the thesis according to which $p \equiv : q \equiv . p \equiv q$ was proved by Tarski in the ordinary theory of deduction before 1922.

From (12) and dir. (γ) that

(13) $[f] \therefore 1 \supset . [q] . f(q) : \equiv . [q] . f(q) \therefore \equiv :: [f] :: [q] : 1 \supset . f(q) : \equiv . [q] . f(q) \therefore \equiv \therefore 1 \supset .. [q] . f(q) : \equiv : [q] : 1 \supset . f(q).$

From (4) and dir. (β) that

(14) $[f] \therefore 1 \supset . [q] . f(q) : \equiv . [q] . f(q),$

(15) $[f, q] \therefore 1 \supset . f(q) : \equiv . f(q).$

From (13), (14), and dir. (α) that

(16) $[f] :: [q] : 1 \supset . f(q) : \equiv . [q] . f(q) \therefore \equiv \therefore 1 \supset . [q] . f(q) : \equiv : [q] : 1 \supset . f(q).$

From (16) and dir. (γ) that

(17) $[f] \therefore [q] : 1 \supset . f(q) : \equiv . [q] . f(q) \therefore \equiv \therefore 1 \supset . [q] . f(q) : \equiv : [q] : 1 \supset . f(q).$

From (15) and dir. (γ) that

(18) $[f] \therefore [q] : 1 \supset . f(q) : \equiv . [q] . f(q).$

From (17), (18), and dir. (α) that

(19) $[f] \therefore 1 \supset . [q] . f(q) : \equiv : [q] : 1 \supset . f(q).$

From (11) and (19) that (cf. point (c) above)

$[f, p] \therefore p \supset . [q] . f(q) : \equiv : [q] : p \supset . f(q).$

The proof of theses saying that

$[f, p] \therefore p \supset . [q, r] . f(q, r) : \equiv : [q, r] : p \supset . f(q, r),$

$[f, p] \therefore p \supset . [q, r, s] . f(q, r, s) : \equiv : [q, r, s] : p \supset . f(q, r, s),$

etc.,

can be represented in a completely analogous way.

By proving in the way characterized here that accepting directive (ζ) in SS1 is entirely unnecessary, Tarski established at the same time that for inferences analogous to the one above, any directive analogous to (ζ) which is also in the area of theories based on SS1 — in particular, in the area of my Ontology, which will be discussed further below — is rendered completely and entirely superfluous. Moreover, Tarski noticed that all analogous meaningful theses can be proved in a fashion fully similar to the one used above in which, instead of expressions of the type '$f(q)$', '$f(q, r)$', '$f(q, r, s)$', etc. considered above, expressions of any structure appear.

§ 7. I was already convinced while drawing up the first outline of SS2 discussed quite generally in §5 that in addition to the whole of SS1, including all of the ordinary theory of deduction, one can obtain, among other things, the following results:

a) theses which establish in general way the 'extensionality' of all functions appearing in the system independently of the semantical category of particular expressions appearing in these functions;

b) theses which establish in a general way, with respect to each propositional function appearing in the system of the type '$\Phi\{f\}$', '$\Phi\{f, g\}$', '$\Phi\{f, g, h\}$', etc., in which at least one argument is not a proposition, that if the appropriate logical product of those propositions which are values of such a function is satisfied by *certain* values for its arguments — values I knew in advance to be finite in number for each semantical category — then the function itself is satisfied by all possible values of its variables. (Even those constants of different semantical categories spoken of in §5 appear as values of these arguments.)

The following thesis can serve as an example of a thesis falling under heading (b):

$$[\Phi] : [f] . \Phi\{f\} . \equiv . \Phi\{vr\} . \Phi\{as\} . \Phi\{\sim\} . \Phi\{fl\},$$

in which the term '$\sim$' is the ordinary sign for propositional negation, and the terms 'vr', 'as', and 'fl' are three other constant function signs for propositional functions of one propositional argument.[47] This thesis, like theses in category (b) generally, is formed with the help of constant function signs in a manner quite similar to the way in which the following theorems considered by Tarski are formed with the help of the terms 'Fl' and 'Vr', which correspond to the 'zeroes' and 'ones' of the traditional propositional calculus:

$$[f] : [p] . f(p) . \equiv . f(Vr) . f(Fl),^{48}$$

[47] Cf. Tarski [2], p. 7; Tarski [3], p. 61.

[48] Cf. Tarski [2], p. 17; Tarski [3], p. 66.

$$[f] \mathbin{:} [p,q] \,.\, f(p,q) \,.\, \equiv \,.\, f(Vr,Vr) \,.\, f(Vr,Fl) \,.\, f(Fl,Vr) \,.\, f(Fl,Fl),$$[49] etc.

These theorems concern propositional functions all of whose arguments are propositions, and they have to do with the 'lower limits' of such functions.

In formulating directive (η) of SS2, I thought I would be able to replace it, without altering the theoretical effect, by some other directive which would insure the possibility of obtaining in SS2 all the propositions of group (b). For me this matter was closely tied to the fact that the problem mentioned in §5 — namely whether a system of Protothetic equivalent to SS2 could not be constructed if instead of directive (η) some other directive were adopted having the character of an 'extensionality directive' — amounted in practice to the problem of whether adding to SS1 all the theses under heading (a) would also enable one to obtain in SS1 all those theses under heading (b) without using directive (η). I was skeptical about the possibility of a positive solution to this problem. I did not suppose that it would be possible in SS1 to replace A3, which (using the equivalence sign instead of the logical product sign) asserts that

$$[g,p] \mathbin{:} g(p,p) \,.\, g(p \equiv .\, [q] \,.\, q, p) \,.\, \equiv \,.\, [q] \,.\, g(q,p),$$[50]

by the axiom which says

(A3*) $[p,q] \mathbin{\therefore} p \equiv q \,.\, \equiv \mathbin{:} [f,r] \mathbin{:} f(p,r) \,.\, \equiv \,.\, f(q,r),$

or for that matter by any other axioms of a similar kind. And I thought that the thesis which says

$$[f] \mathbin{:} [p] \,.\, f(p) \,.\, \equiv \,.\, f(Vr) \,.\, f(Fl)$$

could not be proven at all on the basis of axioms A1, A2, and A3* (or, respectively, with any other axiom of a kind similar to A3*), in accordance with the directives of SS1. Again, I did not expect it to be possible to obtain in SS1 any theses of category (b) even with the help of theses from category (a), without using some new

[49] Cf. Tarski [2], pp. 23–24; Tarski [3], pp. 72–73.

[50] Cf. above in §4, the theses (f) and (g).

directive. In particular, e. g., I did not expect that it would be possible to derive the thesis according to which

$$[\Phi]:[f].\Phi\{f\}.\equiv.\Phi\{vr\}.\Phi\{as\}.\Phi\{\sim\}.\Phi\{fl\}$$

from the thesis which says

$$[f,g]\therefore[p]:f(p).\equiv.g(p):\equiv:[\Phi]:\Phi\{f\}.\equiv.\Phi\{g\}.$$

In spite of my skepticism, however, in 1922 Tarski presented a certain general method which could be used for proving particular theorems from category (b) provided that appropriate theses from category (a) had already been added to SS1. He indicated how to use this method in his sketch of a proof for the thesis mentioned above which says that

$$[\Phi]:[f].\Phi\{f\}.\equiv.\Phi\{vr\}.\Phi\{as\}.\Phi\{\sim\}.\Phi\{fl\}.$$ [51]

Incidentally, I might mention that concerning the formulation of this thesis, Tarski said: "As I have written, my reasoning is irreproachable from the point of view of Leśniewski's theory of types, which has exercised on the exterior form of the present work an influence that manifests itself, e. g., in the use of special parentheses after function-signs that do not have propositions for arguments. Cf. Def. 6 and Def. 7 in §1."[52] In constructing the proper symbolism of SS1 and SS2 I adopted from my 'theory of types', mentioned here by Tarski and spoken of above in §2, a variability in parentheses that depends upon the semantical categories of the particular expressions involved. Although, historically considered, the appropriate sections of these works by Tarski had as their source the problem I had been dealing with in SS2, they were the outcome of observations he made several months after he had already formulated the rest of his results — from which I have been able to derive considerable profit in constructing SS2.

Taking into consideration the possibility Tarski showed of proving theses belonging to category (b) on the basis of theses belonging to category (a) without using directive (η), I decided, on

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[51] Cf. Tarski [2], pp. 24 and 25; Tarski [3], pp. 73 and 74.

[52] Cf. Tarski [3], p. 60; cf. Tarski [2], p. 4.

grounds already represented in §5, to replace directive (η) in SS2 by a directive bearing the character of an 'extensionality' directive. I shall here name this directive (η^*); further below I shall take care to formulate it with as high a degree of precision as I can possibly attain.[53]

The system of Protothetic constructed from Axioms A1–A3 with the help of directives (α), (β), (γ), (δ), (ϵ), and (η^*) I shall name, for the sake of brevity, SS3.

§ 8. I have already mentioned in §2 that I took care to formulate my directives in such a way that they can easily be adapted to different systems of Protothetic, depending upon the primitive terms from which they are to be constructed. I was also inclined to assume that anyone who could properly feel his way into the axioms and directives of my system of Protothetic based on the equivalence sign as the sole primitive term, would almost automatically understand the axioms and directives mentioned when transformed in this way, and that SS3 could be turned into any other equivalent system, so long as it is based upon some single primitive term on which a system of Protothetic could be built. Already while constructing SS2 I had realized that if one would like to formulate an equivalent system of Protothetic patterned after SS2 and based upon the conditional sign, which together with appropriate operations for universal quantifiers can be used to define all known functions of the ordinary theory of deduction, then those axioms and directives would be sufficient which can be roughly characterized as follows:

A) Directives

(α_1) a 'detachment' directive, which permits the addition of a proposition S to the system if a conditional proposition [*Konditionalsatz — tr.*] K whose consequent is equiform to S, and a

[53] Cf. TE49 in §11 and point 5 of the instruction concerning the method of constructing SS5.

proposition equiform to the antecedent of K both already belong to the system

(β_1) a 'substitution' directive

(γ_1) a 'distribution of quantifiers' directive,

analogous to directive (γ) of SS2, but used upon conditionals instead of equivalences[54]

(δ_1), (ϵ_1) two directives which warrant writing out definitions analogous to those authorized by directives (δ) and (ϵ), but not conceived in the form of an equivalence or, respectively, an equivalence preceded by the universal quantifier as in directives (δ) and (ϵ), but in the form of some other function established in advance for all definitions, expressed by means of the conditional sign, and equivalent to the equivalence concerned or, respectively, to the equivalence preceded by the universal quantifier.[55] As Tarski observed, such a simple function can easily be established in advance in connection with the well known theorem saying

$$[p,q] :: p . q . \equiv .\therefore [r] .\therefore p \supset . q \supset r : \supset r,$$

and from which, according to the fact that

$$[p,q] : p \equiv q . \equiv . p \supset q . q \supset p,$$

it follows that

$$[p,q] \therefore\!: p \equiv q . \equiv :: [r] :: p \supset q . \supset : q \supset p . \supset r .\therefore \supset r.$$

Earlier I myself had used a somewhat more complicated function here.

(ζ_1) a directive analogous to directive (ζ) of SS2, which aims at the same goals as (ζ) does, but which is stated in terms of conditional propositions instead of equivalences as in directive (ζ).

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54 Cf. Whitehead and Russell [10], Theorems *9.21, *10.27, and *11.32. The comments I added to directive (γ) in §4 above concerning 'detachment under the quantifier' relate *mutatis mutandis* to the system of Protothetic I am reporting here, constructed on the conditional sign.

55 Cf. the section in §1 above concerning the writing out of definitions by means of the only primitive term of a given system.

(η_1) a directive completely analogous to directive (η) of SS2.

B) Axioms

I) Any combination of axioms consisting of theses holding in ordinary theory of deduction and containing no constants besides the conditional sign, and on the basis of which all of the ordinary theory of deduction could be derived using the directives of the system.

At that time the simplest combination of such axioms I knew of was one due to Tarski, consisting of three axioms. In 1921 he had already proved that if, besides being able to use detachment and substitution, one could also introduce definitions according to my theory of types, and could as well make use of traditional methods for dealing with quantifiers within conditional propositions, then his axioms would suffice for the construction of all of the ordinary theory of deduction. Using symbolism in the style of Whitehead and Russell, Tarski's three axioms can be expressed in formulae which say

$$[p, q] : p \supset . q \supset p,$$
$$[p, q, r] \therefore p \supset q . \supset : q \supset r . \supset . p \supset r,$$
$$[p, q, r] \therefore p \supset q . \supset r : \supset : p \supset r . \supset r.$$ [56]

In the proper symbolism of SS3 these axioms have the following form, which can easily be deciphered by comparing both versions of the corresponding axioms (expressions of the type '$\phi\!\!-\!(pq)$' correspond to expressions of the type '$p \supset q$'):

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[56] Incidentally, I might point out that in 1926 Tarski proved 'completeness' for a system of theses containing no quantifiers which is based on the axioms

$$p \supset . q \supset p$$
$$p \supset q . \supset : q \supset r . \supset . p \supset r$$
$$p \supset q . \supset r : \supset : p \supset r . \supset r,$$

and which is constructed using a detachment directive and an appropriately formulated insertion directive.

$$(1)\quad \llcorner pq \lrcorner \ulcorner \phi\!\text{-}\big(p\phi\!\text{-}(qp)\big),$$

$$(2)\quad \llcorner pqr \lrcorner \ulcorner \phi\!\text{-}\Big(\phi\!\text{-}(pq)\phi\!\text{-}\big(\phi\!\text{-}(qr)\phi\!\text{-}(pr)\big)\Big)\urcorner,$$

$$(3)\quad \llcorner pqr \lrcorner \ulcorner \phi\!\text{-}\Big(\phi\!\text{-}\big(\phi\!\text{-}(pq)r\big)\phi\!\text{-}\big(\phi\!\text{-}(pr)r\big)\Big)\urcorner.$$

II) Some axiom analogous to A3 of SS2, which makes it possible to use in practice — from a certain place in the system onwards — a method of proving (or, respectively, refuting) theses beginning with universal quantifiers containing propositional variables by reference to those appropriate theses already proved (or refuted) in SS3 which can be formed from the thesis to be proved (or refuted) by appropriately substituting for its propositional variables the expressions

$$\llcorner q \lrcorner \ulcorner q \urcorner \quad \text{and} \quad \phi\!\text{-}(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner q \lrcorner \ulcorner q \urcorner),$$

which correspond in SS3 to the 'zeroes' and 'ones' of the traditional propositional calculus. An example of such an axiom, written in the style of Whitehead and Russell's symbolism, would be the formula

$$[g,p,q] \therefore g(p,p) \,.\supset\!:\, g(p \supset .\, [q] \,.\, q,p) \,.\supset .\, g(q,p),$$

which in the proper symbolism of SS3 has the form

$$(4)\quad \llcorner gpq \lrcorner \ulcorner \phi\!\text{-}\Big(g(pp)\phi\!\text{-}\Big(g\big(\phi\!\text{-}(p\llcorner q \lrcorner \ulcorner q \urcorner)p\big)g(qp)\Big)\Big)\urcorner.$$

With the formation of SS3, which is equivalent to, but at the same time simpler than SS2, it had become clear that the system of Protothetic considered in this section, which is based on the conditional sign, could be considerably simplified if it were formed on the model of SS3 instead of SS2; i.e., if directive (ζ_1) were completely rejected, and directive (η_1) replaced by (η_1*), which has the character of an 'extensionality' directive and which would permit the 'extensionality' of functions of every kind to be directly established by means of theses containing no constant terms except the conditional sign. For the sake of brevity I shall here give the name SS4 to the system constructed from Axioms 1–4 with the help of directives (α_1), (β_1), (γ_1), (δ_1), (ϵ_1) (the latter two using Tarski's function mentioned above), and (η_1*). In the interest of a complete historical picture I shall mention a few facts concerning the subsequent fate of this system.

In 1922 Tarski realized that from two theses — one equiform to Axiom 1 of SS4, and the other some thesis of the type

$$\llcorner r \lrcorner \ulcorner \phi\Big(\phi\big(P\phi(Qr)\big)r\Big) \urcorner$$

(one of the ways to express the logical product of P and Q using the conditional sign[57] in which P and Q are any propositions having meaning in Protothetic and which contain no variables dependent upon the initial quantifier $\llcorner r \lrcorner$ — the propositions P and Q themselves can be derived using the directives of SS4. Considering this fact Tarski established that:

1) Without having to alter the directives of SS4, the axioms, however many, of any system of Protothetic constructed using these directives could be replaced by an appropriate combination of only two axioms. One is equiform to Axiom 1 of SS4; the other is a logical product, expressed using the function $\phi(pq)$, of all the axioms distinct from axiom 1.

2) The axioms of SS4 could themselves be replaced in this way by combining axiom 1 with the logical product of its remaining three axioms.

In one of my conversations with Tarski in 1922 I expressed the conviction that it would still be possible to simplify considerably the two-axioms system he had devised. Emphasizing certain constructive similarities and distinctions existing between SS3 and SS4, I even put forth the concrete hypothesis that for constructing a system of Protothetic with the directives of SS4, two axioms would suffice, one having a form more or less resembling Axiom 4 of SS4, the other being an uncomplicated thesis valid in the ordinary theory of deduction (it seemed to me then,

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[57] Cf. the commentary on directives (δ_1) and (ϵ_1).

completely without justification as it later turned out, that for constructing a system of Protothetic with the directives of SS4 or SS3, at least two or three axioms, respectively, would be necessary). During this conversation I suggested to Tarski that as an expert on relations existing among theses containing no constants other than conditional signs, he might reflect upon the question of the possibility of simplifying SS4 in the sense of my hypothesis outlined above. That same year Tarski acquainted me with the following two results he had obtained in this area:

A) Two axioms suffice for constructing a system of Protothetic with the help of the directives of SS4, which run as follows

$$(1)\quad \llcorner pq \lrcorner \ulcorner \phi\!\!-\!\Big(p\phi\!\!-\!(qp)\Big)\urcorner,$$

$$(2)\quad \llcorner pqrf \lrcorner \ulcorner \phi\!\!-\!\left(f(rp)\phi\!\!-\!\Big(f\big(r\phi\!\!-\!(p\llcorner s\lrcorner\ulcorner s\urcorner)\big)f(rq)\Big)\right)\urcorner.$$

(This result represented a complete confirmation of my hypothesis.)

B) A system of Protothetic can be constructed on the basis of a single axiom if, besides directives (α_1), (β_1), (γ_1), and (η_1*) of SS4, two definition directives (δ_1*) and (ϵ_1*) are adopted which introduce definitions not constructed as would be expected from directives (δ_1) and (ϵ_1) of SS4, but instead two mutually reciprocal conditional propositions representing, therefore, the *one* corresponding equivalence (directive (δ_1*)), and two such conditional propositions with their preceding universal quantifiers (directive (ϵ_1*)) — Axiom 2 of the system just given in A) can serve as an example of an axiom of this kind.

§ 9. When Tarski analyzed SS3 in 1923 he observed that from two theses — one saying

$$(1)\quad \llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pq)\phi(qp)\Big)\urcorner$$

(the law of commutativity for equivalences), and the other being some thesis of the type

(2) $\llcorner fp \lrcorner \ulcorner \phi\Big(f(Pp)\phi\big(f(Qq)P\big)\Big)\urcorner$

(one of the forms of the logical product of P and Q in SS1–SS3 in which P and Q are any propositions having meaning in Protothetic and which contain no variables dependent upon the initial quantifier $\llcorner fp \lrcorner$ — the propositions P and Q themselves can be derived using the directives of SS3, according to the following schema:

(3) $\llcorner pq \lrcorner \ulcorner \phi\Big(\Phi(pq)\big(\phi(pq)\phi(qp)\big)\Big)\urcorner$

(a definition according to directive (ϵ)). From (2) we conclude on the basis of directive (β) that

(4) $\phi\Big(\Phi(PP)\phi\big(\Phi(QP)P\big)\Big)$.

From (1) on the basis of dir. (β) that

(5) $\phi\Bigg(\phi\Big(\llcorner pq \lrcorner \ulcorner \Phi(pq)\ \llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big)\urcorner \urcorner\Big)\phi\Big(\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)$
$\phi(qp)\big)\urcorner\ \llcorner pq \lrcorner \ulcorner \Phi(pq)\urcorner\Big)\Bigg)$.

From (3) and dir. (γ) that

(6) $\phi\Big(\llcorner pq \lrcorner \ulcorner \Phi(pq)\urcorner\ \llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big)\urcorner\Big)$.

From (5), (6), and dir. (α) that

(7) $\phi\Big(\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big)\urcorner\ \llcorner pq \lrcorner \ulcorner \Phi(pq)\urcorner\Big)$.

From (7), (1), and dir. (α) that

(8) $\llcorner pq \lrcorner \ulcorner \Phi(pq)\urcorner$.

From (8) and dir. (β) that

(9) $\Phi(PP)$,

(10) $\Phi(QP)$,

(11) $\Phi(QQ)$.

From (4), (9), and dir. (α) that

(12) $\phi\Big(\Phi(QP)P\Big)$.

From (12), (10), and dir. (α) that

(13) P,

(14) $\llcorner pq \lrcorner \ulcorner \phi\Big(X(pq)\big(\Phi(qp)p\big)\Big)\urcorner$

(a definition according to directive (ϵ)). From (14) and dir. (β) that

(15) $\phi\Big(X(QQ)\phi\big(\Phi(QQ)Q\big)\Big)$,

(16) $\phi\Big(X(PQ)\phi\big(\Phi(QP)P\big)\Big)$.

From (1) and dir. (β) that

(17) $\phi\Big(\phi\big(X(QQ)P\big)\phi\big(PX(QQ)\big)\Big)$,

(18) $\phi\Big(\phi\big(X(PQ)\phi\big(\Phi(QP)P\big)\big)\phi\big(\phi\big(\Phi(QP)P\big)X(PQ)\big)\Big)$

.

From (2) and dir. (α) that

(19) $\phi\Big(X(PQ)\phi\big(X(QQ)P\big)\Big)$.

From (18), (16), and dir. (α) that

(20) $\phi\Big(\phi\big(\Phi(QP)P\big)X(PQ)\Big)$.

From (20), (12), and dir. (α) that

(21) $X(PQ)$.

From (19), (21), and dir. (α) that

(22) $\phi\Big(X(QQ)P\Big)$.

From (17), (22), and dir. (α) that

(23) $\phi\Big(PX(QQ)\Big)$.

From (23), (13), and dir. (α) that

(24) $X(QQ)$.

From (15), (24), and dir. (α) that

(25) $\phi\Big(\Phi(QQ)Q\Big)$.

From (25), (11), and dir. (α) that

(26) Q.

From the derivability of P and Q from theses (1) and (2) using the directives of SS3, Tarski concluded that however many axioms there are in any given set A of axioms for a particular system which is equivalent to SS3 and which is constructed with its directives, A could be derived using the directives of SS3 from an appropriate combination K of two new axioms. One is the law of commutativity for equivalences, and the other is a logical product, expressed by means of the function $\phi(pq)$, of all the members of A distinct from the law of commutativity for equivalences. He established that the axiom combination K is derivable in SS3, and therefore also in the system equivalent to it constructed on the basis of the set of axioms A together with the directives of SS3, by taking into consideration the fact that in 1922 I proved that if any propositions P and Q hold in Protothetic, their conjunction formed according to the example of thesis (2) holds in it as well (this will become clear to the reader in connection with further sections of this article). On the basis of this fact Tarski observed that this logical product of all the propositions, except for the law of commutativity for equivalences, which belong to A (and which are therefore in accordance with the assumptions of SS3 is demonstrable in SS3, as is also the law of commutativity for equivalences itself, which I had already obtained in SS1 by 1922. In connection with the fact proved with the help of the considerations represented above, that any set of axioms for a system of Protothetic, where that system is constructed according to the directives of SS3, can without changing the directives be replaced by an appropriate combination of only two axioms, Tarski realized that if these considerations were applied to the axioms of SS3 itself, A1–A3 could be replaced by a combination of two axioms, one being the law of commutativity for equivalences, and the other being obtainable from the expression

$$\lfloor hs \rfloor \ulcorner \phi\left(h(Ps)\phi\left(h\left(\lfloor kt \rfloor \ulcorner \phi\left(k(Qt)\phi\left(k(Rt)Q\right)\right)\urcorner s\right)P\right)\right)\urcorner$$

by inserting into it Axioms A1, A2, and A3 for the terms P, Q, and R, respectively, in its full reading.

§ 10. Using this result of Tarski's I observed that if, instead of the definition directives (δ) and (ϵ) of SS3, I were to introduce definition directives (δ^*) and (ϵ^*) in which the definiendum would be on the right side of an equivalence instead of on the left as in (δ) and (ϵ),[58] and if I had at my disposal some thesis of the type

(a) $\lfloor fp \rfloor \ulcorner \phi\left(f(Pp)\phi\left(f(Qp)P\right)\right)\urcorner$,

in which P and Q are any meaningful propositions in Protothetic containing no variables dependent upon the quantifier $\lfloor fp \rfloor$ that begins the given thesis, then P could be derived from this thesis, with the directives modified as above in accordance with the following schema:

(b) $\lfloor pq \rfloor \ulcorner \phi\left(\phi(pq)\Psi(qp)\right)\urcorner$,

[58] In connection with directives (δ^*) and (ϵ^*), cf. TE XLIV in §11 and point 1 of the instruction concerning the method of constructing SS5.

(a definition according to directive (ϵ^*)).

(c) $\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pq)\Psi(qp)\big)\Phi(pq)\Big)\urcorner$,

(a definition according to directive (ϵ^*)). From (c) we conclude by dir. (γ) that

(d) $\phi\Big(\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\Psi(qp)\big)\urcorner \llcorner pq \lrcorner \ulcorner \Phi(pq)\urcorner\Big)$.

From (d), (b), and dir. (α) that

(e) $\llcorner pq \lrcorner \ulcorner \Phi(pq)\urcorner$.

Already having premises (a) and (e), which correspond to theses (2) and (8) of §9, at our disposal, we derive theses corresponding to (4), (9), (10), (12), and (13) of §9 from these premises using the corresponding directives specified in §9, and in this way obtain *P*.

The fact that, while I knew how to derive *P* from thesis (a) alone with the directives of SS3 modified as above, I did not know how to get this result using the proper directives of SS3, led me to examine more precisely the system based on the axioms A1–A3 of SS3, but constructed using the directives (α), (β), (γ), (δ^*), (ϵ^*), and (η^*), instead of the directives (α), (β), (γ), (δ), (ϵ), and (η^*) used in SS3. For the sake of brevity I shall here name the system thus constructed SS5.

Considering that the law of commutativity for equivalences is already derivable from axioms A1 and A2 using (α), (β), and (γ), and hence without using any definition directives (as the reader will be convinced further below), I established that by using this law I could obtain in SS5, from definitions written out according to the directives (δ^*) and (ϵ^*), all propositions corresponding to the definitions allowed by (δ) and (ϵ). Likewise, I established that I could use this law to obtain in SS5, from definitions written out according to directives (δ) and (ϵ) all propositions corresponding to the definitions allowed by (δ^*) and (ϵ^*). I convinced myself in this way that SS5 is a system of Protothetic equivalent to SS3.

Realizing that I could derive P from thesis (a) using the directives of SS5 according to the schema given above, I noticed that if, besides thesis (a), I had at my disposal the law of commutativity for equivalences which says that

$$(a_1) \quad \lfloor pq \rfloor \ulcorner \phi\Big(\phi(pq)\phi(qp)\Big) \urcorner,$$

it would also be possible to obtain Q by developing SS5 somewhat further. I could conclude

$$(f) \quad \lfloor pq \rfloor \ulcorner \phi\Big(\phi\big(\Phi(qp)p\big)X(pq)\Big) \urcorner$$

(a definition according to directive (ϵ^*)). From (a_1) and dir. (β) I could conclude that

$$(g) \quad \phi\Bigg(\phi\bigg(\phi\Big(\Phi(QQ)Q\Big)X(QQ)\bigg)\phi\bigg(X(QQ)\phi\Big(\Phi(QQ)Q\Big)\bigg)\Bigg).$$

From (f), and dir. (β) that

$$(h) \quad \phi\bigg(\phi\Big(\Phi(QQ)Q\Big)X(QQ)\bigg),$$

$$(i) \quad \phi\bigg(\phi\Big(\Phi(QP)P\Big)X(PQ)\bigg).$$

From (g), (h), and dir. (α) that

$$(k) \quad \phi\bigg(X(QQ)\phi\Big(\Phi(QQ)Q\Big)\bigg).$$

Having now at our disposal, besides the theses already mentioned corresponding to theses (2), (8), (12), and (13) of §9, the premises (a_1), (k), and (i) corresponding to theses (1), (15), and (20) of §9, we conclude in turn, with the help of the appropriate directives specified in §9, theses corresponding to (11), (17), (19), (21), (22), (23), (24), (25), (26), and Q.

From the possibility of deriving P from (a) and both P and Q from (a) and (a_1) using only the directives of SS5, I concluded that however many axioms there are in any given set A of axioms for any system which is equivalent to SS5 and which is constructed using its directives, A could be derived using the same directives from an appropriate single Axiom B. This axiom can be obtained by forming the logical product of two factors in accordance with the example of thesis (a); one such factor, corresponding to P in (a), is equiform to thesis (a_1); the other, corresponding to Q in (a), is a logical product C expressed by theses formed according to the example of theses (a), of all the propositions in A distinct from (a_1). (In order to derive all the axioms of A from axiom B using the directives of SS5, we first deduce (a_1) from B, which corresponds in B to P in (a); we then derive the logical product C of all the propositions belonging to A which are distinct from (a_1), from axiom B and (a_1) already attained; and finally we obtain from C and (a_1), all propositions in turn belonging to A which are distinct from (a_1). Taking into consideration the fact that[59] (a_1) and C, all of whose members are in A and therefore in accordance with the assumptions of SS5, are both demonstrable in SS5, which is a system of Protothetic, I established that Axiom B, which is formed on the model of (a) as a logical product of (a_1) and C, is derivable in SS5, and therefore also in the system equivalent to it constructed by using the directives of SS5 on the axioms of A. Taking all this into consideration I established that any set of axioms for a system of Protothetic constructed using the directives of SS5, and therefore even the axioms A1–A3 of SS5 itself, can be replaced by an appropriate single axiom without any change in the directives. (Perhaps I should add that in connection with the analysis I carried out on SS5 I became convinced that an axiom can easily be constructed which would by itself suffice to form a system of Protothetic, if besides directives (β), (γ), (δ), (ϵ), and (η^*), a directive (α^*) were accepted instead of directive (α). This directive (α^*) is a unnatural variety of the detachment directive: it would permit adding a proposition S to the system

[59] Cf. above in §9 the paragraph concerning the provability of the axiom combination K in SS3.

if an equivalence A whose left side is equiform to S, and a proposition equivalent to the right side of A already belonged to the system. I know of no axiom which, taken alone, would suffice to form a system of Protothetic using directives (α), (β), (δ), (ϵ^*), and (η^*) of SS3, or also using directives (α^*), (β), (γ), (δ^*), (ϵ^*), and (η^*).)

Despite the fact that I shall be departing somewhat from the mainly chronological way of presenting my own and others' results in the area of Protothetic, I would like to mention here a certain result Tarski gave in 1925, because of its relevance to sections of this article which are concerned with the problem of finding a single axiom for Protothetic constructed with equivalence as the only primitive function, and because I wish to prevent the possibility that anyone should want to solve, in light of the facts given in §8 about systems of Protothetic constructed with the directives of SS4, certain problems which have already been solved. Tarski specified a method for permitting the axioms of any system of Protothetic constructed with the directives of SS4, and which has the conditional sign as its only primitive term, to be replaced by an appropriate single axiom. By applying this method, *mutatis mutandis*, to systems of the ordinary theory of deduction containing the conditional sign among their primitive terms, he showed how one could also construct these systems on the basis of a single axiom. I shall not discuss these results in more detail, since they had no influence on the outcome of my own investigations.

§ 11. I shall postpone until further sections of this article questions connected with a number of successive and far-reaching simplifications of the single axiom of Protothetic constructed with the help of the directives of SS5. They were carried out by Wajsberg and me from 1923 to 1926 and finally led me to the simplest axiom so far of such a Protothetic; it has the form of the proposition

$$\llcorner fpqrst \lrcorner \ \phi \ulcorner\left(\phi(pq) \llcorner g \lrcorner \ \phi \ulcorner\left(f\left(pf(p \llcorner u \lrcorner \ulcorner u \urcorner)\right) \phi \left(\llcorner u \lrcorner \ulcorner f(qu) \right.\right.\right.$$

$$\left.\left.\left.\urcorner \phi \left(g\left(\phi\left(\phi(rs)t\right)q\right) g\left(\phi\left(\phi(st)r\right)p\right)\right)\right)\right)\urcorner\right)\urcorner .$$

From now on I shall be concerned with the precise formulation of the directives of Protothetic announced above. I shall do this explicitly for the directives of SS5. I might mention that I have twice presented these directives in detail in lectures at the University of Warsaw, once in my lectures on *Logistic* during 1924/25 and 1925/26, and again, in a considerably simplified and hence more perfect form, in my lectures on the *Foundations of Ontology* in 1926/27.

Since directives do not themselves belong to the system of Protothetic which they affect, I usually formulated them in ordinary colloquial language. I comment on particular terms of ordinary language appearing in the directives in a series of terminological explanations, which are also formulated in ordinary language. In this article, however, I shall, in order to save space, formulate the directives and terminological explanation in a symbolic fashion which will be easily understood by anyone who knows the symbolism that Whitehead and Russell adopted for the theory of deduction. The symbolic formulation of the terminological explanations and directives should be regarded as typographical abbreviations which would be replaced by expressions of ordinary speech had I more space at my disposal.

In the following table, expressions in the first column are supposed to be abbreviations of corresponding expressions in the second column:

$A \colon \in b$	A is a b[60]

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[60] Cf. Peano [1], p. 20.

Id(A)	same object as A
$\sim(a)$	object which is not a
$a \cap b \cap \ldots \cap k$	object which is a and b and ... and k
$a \cup b \cup \ldots \cup k$	object which is a or b or ... or k
$a \infty b$	there are just as many objects a as there are objects b
$a{\scriptstyle\infty}b$	there are less objects a than there are objects b
vrb	word
expr	expression
prnt	parenthesis
prntl	left-sided parenthesis
prntsym(A)	parenthesis symmetrical to A
cnf(A)	expression equiform to A
A1	axiom A1
thp	thesis in this system of Protothetic
ingr(A)	belonging to A
prcd(A)	preceding A
scd(A)	following A
Uprcd(A)	final word preceding A
Uingr(A)	final word belonging to A
lingr(A)	first word belonging to A
2ingr(A)	second word belonging to A
etc.	

I shall only add a few comments on the expressions cited in the table just given, in order to reduce for both the reader and author the likelihood of any misunderstanding:

'$a \infty b$'. Expressions of this type also cover situations where there is neither an object a nor an object b.

'$a{\scriptstyle\infty}b$'. Expressions of this type also cover situations where there is neither an object a nor an object b.

'vrb'. The expressions, 'man', 'word', 'p', 'ϕ', ' ⌞', '⌝ ', '(', '}' are examples of words. The expressions 'the man', '(p)', 'f ⌞) word' are examples of objects which are collections of words, but not themselves words. The expression ‘the man’ consists of two words, the expression ‘(p)’ of three words, the expression ‘f ⌞) word’ of four words. Axiom A3 consists of 80 words. Individual letters of words consisting of at least two letters are not words. Expressions consisting of at least two words are not words.

‘expr’. Every word is an expression. The collection of any number of successive words of any expression is an expression. The collection of words consisting of the first, third, and fourth words of any expression is not an expression. Every expression consists of words. I would not call a collection consisting of infinitely many words an expression.

‘prnt’ and ‘prntl’. The words ‘)’, ‘(’, ‘(’, ‘]’, ‘{’ are examples of parentheses; the second, third, and fifth are examples of left-sided parentheses. The words ‘⌞’ and ‘⌝’ are examples of words which are not parentheses.

‘prntsym (A)’. Each of the parentheses ‘(’, ‘(’, ‘(’ is symmetrical to each of the parentheses ‘)’, ‘)’, ‘)’, and reversed; likewise with ‘}’, ‘}’ and ‘{’, ‘{. None of the parentheses ‘(’, ‘]’, ‘}’ is symmetrical to the parenthesis ‘(’.

‘cnf (A)’. Every expression is equiform to itself. The seventh word of Axiom A1 is equiform to the ninth word of A1. The collection of the first five words of Axiom A1 is equiform to the collection of the first five words of Axiom A2. Parenthesis ‘(’ is equiform to the parenthesis ‘(’. (I give different lengths to parentheses equiform to one another in order to make the formulae I write perspicuous; all these lengths can be varied in any way without changing the meaning of these formulae.)[60a] The parenthesis '(' is not equiform to the parenthesis '['. The word '⌜' is equiform to the word '⌜'. (The different heights at which I attach the particular words '⌜' and '⌝' in accordance with the demands of perspicuity have no bearing upon the meaning of the formulae I write.) Two expressions equiform to each other written in two different places are never the *same* expression.[61] (Not taking these facts into consideration could lead the reader to a completely wrong interpretation of my terminological explanations given further below.)

'ingr (A)'. I use expressions of this type in a way that permits me to assert of any expression A that it is an ingr (A).

I usually give to the terminological explanations, which I am just approaching, the form of propositions of the type 'I say of an object A that it is a b if and only if p'. Corresponding expressions of the type '$A \in b . = p$' preceded by the universal quantifier should be regarded as abbreviations of such expressions.[62]

[60a] [All parentheses in the terminological explanations below are given in the same size — *tr.*]

[61] Cf. Frege [2], p. 107.

[62] [For ease of reference, the axioms of SS5 (from p. 37) are:

A1. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big) \urcorner$

A2. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big) \urcorner$

A3. $\llcorner gp \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g\Big(\phi$

$(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big) \urcorner \Big)\Big) \urcorner \llcorner q \lrcorner \ulcorner g(gp) \urcorner \Big) \urcorner.$

— tr.]

Terminological Explanation I. $[A] \colon A \in \mathrm{vrb1} \,.\, = \,.$
$A \in \mathrm{cnf}(\mathrm{1ingr}(\mathrm{A1}))$

T.E.II. $[A] \colon A \in \mathrm{vrb2} \,.\, = \,.\, A \in \mathrm{cnf}(\mathrm{5ingr}(\mathrm{A1}))$

T.E.III. $[A] \colon A \in \mathrm{vrb3} \,.\, = \,.\, A \in \mathrm{cnf}\ \mathrm{6ingr}(\mathrm{A1})$

T.E.IV. $[A] \colon A \in \mathrm{vrb4} \,.\, = \,.\, A \in \mathrm{cnf}(\mathrm{Uingr}(\mathrm{A1}))$

T.E.V. $[A] \colon A \in \mathrm{trm} \,.\, = \,.$
$A \in\sim (\mathrm{prnt}) \,.$
$A \in\sim (\mathrm{vrb1}) \,.$
$A \in\sim (\mathrm{vrb2}) \,.$
$A \in\sim (\mathrm{vrb3}) \,.$
$A \in\sim (\mathrm{vrb4}) \,.$[63]

T.E.VI. $[A, B] \colon A \in \mathrm{int}(B) \,.\, = \,.$
$B \in \mathrm{expr} \,.$
$A \in \mathrm{vrb} \,.$
$A \in \mathrm{ingr}(B) \,.$
$A \in\sim (\mathrm{1ingr}(B)) \,.$
$A \in (\mathrm{Uingr}(B))$

T.E.VII. $[A, a] \colon\colon A \in \mathrm{Cmpl}(a) \,.\, = \,.\!\cdot\,.$
$A \in \mathrm{expr} \,.\!\cdot\,.$
$[B] \colon B \in \mathrm{vrb} \,.\, B \in \mathrm{ingr}(A) \,.\, \supset [\exists C] \,.\, C \in a \,.\, B \in \mathrm{ingr}(C)$
$.\!\cdot\,.$
$[B, C, D,] \colon B \in a \,.\, C \in a \,.\, D \in \mathrm{vrb} \,.\, D \in \mathrm{ingr}(B) \,.$
$D \in \mathrm{ingr}(C) \,.\, \supset \,.\, B \in \mathrm{Id}(C) \,.\!\cdot\,.$
$[B] \colon B \in a \,.\, \supset \,.\, B \in \mathrm{expr} \cap \mathrm{ingr}(A)$

T.E.VIII. $[A] \colon\colon A \in \mathrm{qntf} \,.\, = \,.\!\cdot\,.$
$\mathrm{1ingr}(A) \in \mathrm{vrb1} \,.$
$\mathrm{Uingr}(A) \in \mathrm{vrb2} \colon$

63 For the sake of clarity I write in separate lines the individual 'logical factors' appearing after the sign '=' in my terminological explanations. I possess suitable examples for all the terminological explanations, which prove that all of the logical factors are independent of the logical product of the remaining factors.

$[\exists B] . B \in \mathrm{int}(A) .\because$
$[B] : B \in \mathrm{int}(A) . \supset . B \in \mathrm{trm} .\because$
$[B, C] : B \in \mathrm{int}(A) . C \in \mathrm{int}(A) . B \in \mathrm{cnf}(C) . \supset .$
$B \in \mathrm{Id}(C)$

T.E.IX. $[A] \because\!: A \in \mathrm{sbqntf} . = ::$
$[\exists B] . B \in \mathrm{int}(A) ::$
$[B] .\because B \in 1\mathrm{ingr}(A) . \vee . B \in \mathrm{int}(A) : \supset . (\mathrm{vrb3} \cap \mathrm{ingr}(A) \cap \mathrm{scd}(B)) \infty (\mathrm{vrb4} \cap \mathrm{ingr}(A) \cap \mathrm{scd}(B)) ::$
$[B] .\because B \in \mathrm{int}(A) . \vee . B \in \mathrm{Uingr}(A) : \supset . (\mathrm{vrb4} \cap \mathrm{ingr}(A) \cap \mathrm{prcd}(B)) \infty (\mathrm{vrb3} \cap \mathrm{ingr}(A) \cap \mathrm{prcd}(B))$[64]

T.E.X. $[A] :: A \in \mathrm{qnrl} . = .\because$
$[\exists B] . B \in \mathrm{qntf} . B \in \mathrm{ingr}(A) . 1\mathrm{ingr}(A) \in \mathrm{ingr}(B) :$
$[\exists B] . B \in \mathrm{sbqntf} . B \in \mathrm{ingr}(A) . \mathrm{Uingr}(A) \in \mathrm{ingr}(B) .\because$
$[B, C] : B \in \mathrm{qntf} . B \in \mathrm{ingr}(A) . C \in \mathrm{sbntf} . C \in \mathrm{ingr}(A) . 1\mathrm{ingr}(A) \in \mathrm{ingr}(B) . \mathrm{Uingr}(A) \in \mathrm{ingr}(C) . \supset .$
$A \in \mathrm{Cmpl}(B \cup C)$

T.E.XI. $[A, B] : A \in \mathrm{Qntf}(B) . = .$
$B \in \mathrm{gnrl} .$
$A \in \mathrm{qntf} \cap \mathrm{ingr}(B) .$
$\mathrm{Iingr}(B) \in \mathrm{ingr}(A)$[65]

T.E.XII. $[A, B] : A \in \mathrm{Sbqntf}(B) . = .$

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[64] Terminological Explanation IX is the result of a certain simplification I executed on the wording of the explanation proposed by Dr. Adolf Lindenbaum (then still a student at Warsaw University) which fixes a union of a certain four conditions necessary and sufficient for a given object to be a sbqntf, and which in turn was the result of a certain penetrating simplification Lindenbaum carried out upon my original wording, reported in 1924/25 in the lectures mentioned above on *Logistic*, which fixed a union of a certain five conditions necessary and sufficient for a given object to be a sbqntf.

[65] Relying on certain assumptions I consider absolutely binding concerning expressions, I have been able to prove with the help of terminological explanations already given that $[A, B] : B \in \mathrm{gnrl} . A \in \mathrm{qntf} . 1\mathrm{ingr}(B) \in \mathrm{ingr}(A) . \supset . A \in \mathrm{ingr}(B)$. I have not removed the superfluous (from this point of view) factor '$A \in \mathrm{ingr}(B)$', which is implicitly involved in two of the three independent factors appearing in T.E.XI after the '=' sign, since I also want

$B \in \text{gnrl}\,.$
$A \in \text{sbqntf}\,.$
$A \in \text{ingr}(B)\,.$
$\text{Uingr}(B) \in \text{ingr}(A)$

T.E.XIII. $[A, B] \therefore A \in \text{Essnt}(B)\,.=.$
$A \in \text{Cmpl}(\text{int}(\text{Sbqntf}(B)))\,.\vee.\,A \in \text{expr}\,.\,A \in \text{Id}(B)\,.$
$A \in \sim(\text{gnrl})$

T.E.XIV. $[A, B, C] :: A \in \text{var}(B, C)\,.=\therefore$
$B \in \text{int}(\text{Qntf}(C))\,.$
$A \in \text{cnf}(B)\,.$
$A \in \text{ingr}(\text{Essnt}(C) \therefore$
$[D, E] : D \in \text{ingr}(C)\,.\,E \in \text{int}(\text{Qntf}(D))\,.\,A \in \text{cnf}(E)\,.$
$A \in \text{ingr}(\text{D})\,.\supset.\in \text{Id}(C)$[66]

T.E.XV. $[A, B, C] \therefore A \in \text{cnvar}(B, C)\,.=:$
$[\exists D]\,.\,A \in \text{var}(D, C):$
$[\exists D]\,.\,B \in \text{var}(D, C):$
$A \in \text{cnf}(B)$[67]

T.E.XVI. $[A] \because A \in \text{prntm}\,.=::$
$[\exists B]\,.\,B \in \text{int}(A)::$
$[B] \therefore B \in \text{1ingr}(A)\,.\vee.\,B \in \text{int}(A) : \supset.\,(\text{ingr}(A) \cap \text{scd}(B) \cap \text{cnf}(\text{1ingr}(A))) \circ (\text{ingr}(A) \cap \text{scd}(B) \cap \text{prntsym}(\text{1ingr}(A)))$
$::$
$[B] \therefore B \in \text{int}(A)\,.\vee.\,B \in \text{Uingr}(A) : \supset.\,(\text{ingr}(A) \cap \text{prcd}(B) \cap \text{prntsym}(\text{1ingr}(A))) \circ (\text{ingr}(A) \cap \text{prcd}(B) \cap \text{prnt1} \cap \text{cnf}(\text{1ingr}(A)))$[68]

T.E.XVII. $[A, a, B] :: A \in \text{prntm}(B, a)\,.=\therefore$
$[C] : C \in a\,.\supset.\,C \in \text{prntm} \therefore$

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to give an exact account to the reader without the help of the assumption mentioned above that A can be a Qntf(B)only in case it is an ingr(B).

[66] Cf. Frege [1], p. 13.

[67] Cf. *loc. cit.*

[68] The comment added above to T.E.IX also refers in its entire extent to T.E.XVI.

$B \in \mathrm{Cmpl}(1\mathrm{ingr}(B) \cup a)$.
$1\mathrm{ingr}(B) \in \mathrm{trm}$.
$A \in a$

T.E.XVIII. $[A, B] : A \in \mathrm{prntm}(B) . = .$
$[\exists a] . A \in \mathrm{prntm}(B, a)$[69]

T.E.XIX. $[A] : A \in \mathrm{fnct} . = .$
$[\exists B] . B \in \mathrm{prntm}(A)$

T.E.XX. $[A, a, B] \because A \in \mathrm{arg}(B, a) . = ::$
$B \in \mathrm{prntm} ::$
$[C] \therefore C \in a . \supset : C \in \mathrm{trm} . \vee . C \in \mathrm{gnrl} . \vee . \in \mathrm{fnct} ::$
$\mathrm{Cmpl}(\mathrm{int}(B)) \in \mathrm{Cmpl}(a)$.
$A \in a$

T.E.XXI. $[A, B] : A \in \mathrm{arg}(B) . = .$
$[\exists a] . A \in \mathrm{arg}(B, a)$

T.E.XXII. $[A, B] : A \in \mathrm{Sgnfnct}(B) . = .$
$A \in \mathrm{expr}$.
$A \in \mathrm{ingr}(B)$.
$\mathrm{Cmpl}(\mathrm{vrb} \cap \mathrm{ingr}(B) \cap \sim (\mathrm{ingr}(A))) \in \mathrm{prntm}(B)$[70]

T.E.XXIII. $[A, B] : A \in \mathrm{simprntm}(B) . = .$
$A \in \mathrm{prntm}$.
$B \in \mathrm{prntm}$.
$1\mathrm{ingr}(A) \in \mathrm{cnf}(1\mathrm{ingr}(B))$.

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[69] Besides expressions of the type '$f(ab\ldots)$' we encounter, in the system of Protothetic considered, expressions of the type '$f[kl\ldots](ab\ldots)$', '$f\{xy\ldots\}[kl\ldots](ab\ldots)$', etc. This is a result of a generalization of the tendency which has found expression in such a form as, e.g., '$x\{\mathrm{Cnv}`(P \cap Q)\}y$'. (Cf. Whitehead and Russell [1], p. 239.)

[70] In wording T.E.XXII I made use of a remark of Lindenbaum's on a certain other terminological explanation which I gave in my already mentioned lectures on 'Logistic' in 1924/25, and which I now completely omit in connection with the whole of the sweeping simplification I had introduced in the meantime to the directives of Protothetic in the system of terminological explanations.

$\text{arg}(A) \infty \text{arg}(B)$

T.E.XXIV. $[A, B] :: A \in \text{genfnct}(B) . = .\therefore$

$A \in \text{fnct} :$

$\text{prntm}(A) \infty \text{prntm}(B) . \vee . \text{prntm}(A) \circ\!\!\circ \text{prntm}(B) .\therefore$

$[C, D] : C \in \text{prntm}(A) . D \in \text{prntm}(B) . (\text{prntm}(A) \cap \text{scd}(C)) \infty (\text{prntm}(B) \cap \text{scd}(D)) . \supset . C \in \text{simprntm}(D)$

T.E.XXV. $[A, B, C, D] : A \in \text{Anarg}(B, C, D) . = .$

$C \in \text{simprntm}(D) .$

$A \in \text{arg}(C) .$

$B \in \text{arg}(D) .$

$(\text{arg}(C) \cap \text{prcd}(A)) \infty (\text{arg}(D) \cap \text{prcd}(B))$

T.E.XXVI. $[A, B, C, D] \therefore \in \text{Ansgnfnct}(B, C, D) . = :$

$A \in \text{Sgnfnct}(C) .$

$B \in \text{Sgnfnct}(D) :$

$[\exists E, F] . E \in \text{prntm}(C) . E \in \text{scd}(A) . F \in \text{prntm}(D) .$

$F \in \text{scd}(B) . E \in \text{simprntm}(F)$

T.E.XXVII. $[A, B, C, D] \therefore A \in \text{An}(B, C, D) . = :$

$A \in \text{Anarg}(B, C, D) . \vee . A \in \text{Ansgnfnct}(B, C, D)$

T.E.XXVIII. $[A, B] : A \in \text{Argl}(B) . = .$

$[\exists C] . C \in \text{ingr}(\text{A1}) . A \in \text{Anarg}(13\text{ingr}(\text{A1}, B, C)$

T.E.XXIX. $[A, B] : A \in \text{Arg2}(B) . = .$

$[\exists C] . C \in \text{ingr}(\text{A1}) . A \in \text{Anarg}(14\text{ingr}(\text{A1}), B, C)$[71]

T.E.XXX. $[A, B] \therefore A \in \text{Eqv11}(B) . = :$

$\text{Sgnfnct}(B) \in \text{cnf}(7\text{ingr}(\text{A1})) :$

$[\exists C] . C \in \text{prntm}(B) . A \in \text{Argl}(C)$

T.E.XXXI. $[A, B] \therefore A \in \text{Eqv12}(B) . = :$

$\text{Sgnfnct}(B) \in \text{cnf}(7\text{ingr}(\text{A1})) :$

$[\exists C] . C \in \text{prntm}(B) . A \in \text{Arg2}(C)$

T.E.XXXII. $[A, B] \therefore A \in \text{thp}(B) . = :$

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[71] The comment added above to T.E.XI concerning factor '$A \in \text{ingr}(B)$' applies *mutatis mutandis* to the factor '$C \in \text{ingr}(\text{A1})$' appearing in T.E.XXVIII and T.E.XXIX.

$A \in \mathrm{thp}\,.$
$B \in \mathrm{thp}\,:$
$A \in \mathrm{prcd}(B)\,.\vee.\,A \in \mathrm{Id}(B)$

T.E.XXXIII. $[A, B] \therefore A \in \mathrm{frp}(B)\,.=:$
$A \in \mathrm{thp}(B)\,.\vee.\,[\exists C, D]\,.\,C \in \mathrm{thp}(B)\,.\,D \in \mathrm{ingr}(C)\,.$
$A \in \mathrm{Arg1}(D)\,.\vee.\,[\exists C, D]\,.\,C \in \mathrm{thp}(B)\,.\,D \in \mathrm{ingr}(C)$
$.\,A \in \mathrm{Arg2}(D)\,.\vee.\,[\exists C, D]\,.\,C \in \mathrm{thp}(B)\,.\,D \in \mathrm{sbqntf}\,.$
$D \in \mathrm{ingr}(C)\,.\,A \in \mathrm{Cmpl}(\mathrm{int}(D))$

T.E.XXXIV. $[A, B, C] \therefore A \in \mathrm{1homosemp}(B, C)\,.=:$
$A \in \mathrm{fpr}(C)\,.\,B \in \mathrm{frp}(C)\,.\vee.\,[\exists D, E]\,.\,D \in \mathrm{thp}(C)$
$.\,E \in \mathrm{ingr}(D)\,.\,A \in \mathrm{cnvar}(B, E)\,.\vee.\,[\exists D, E, F, G]\,.$
$D \in \mathrm{thp}(C)\,.\,E \in \mathrm{ingr}(D)\,.\,F \in \mathrm{thp}(C)\,.\,G \in \mathrm{ingr}(F)\,.$
$A \in \mathrm{An}(B, E, G)$

T.E.XXXV. $[A, B, C] :: A \in \mathrm{homosemp}(B, C)\,.=\because$
$A \in \mathrm{1homosemp}(A, C)\,.\,B \in \mathrm{1homosemp}(B, C) \because$
$[a] :: [D] : D \in a\,.\supset.\,D \in \mathrm{1homosemp}(D, C) \therefore [D, E] :$
$D \in a\,.\,E \in \mathrm{1homosemp}(D, C)\,.\supset.\,E \in a \therefore B \in a \therefore \supset.$
$A \in a$[72]

Loosely speaking, expressions of the type '$A \in$ *homosemp*(B, C)' could be read out by means of corresponding phrases of the type 'With regard to thesis C, which already belongs to the system of Protothetic, A is an expression of the same semantic category as B'. The letter 'p' appearing in the last place in the word 'homosemp' serves to indicate that the membership of any expression A and B in the same semantic category is always, in complete accordance with my full conception of the semantic categories, relative to a given thesis C belonging to Protothetic. In the terminological explanations for the directives of the system of

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72 The comment added above to T.E.XI applies *mutatis mutandis* to the factor '$A \in \mathrm{1homosemp}(A, C)$'. The situation is similar for the expression '$[D] : D \in a\,.\supset.\,D \in \mathrm{1homosemp}(D, C)$', which appears in the antecedent of the conditional proposition under the quantifier '[a]' in T.E.XXXV. In connection with T.E.XXXV, cf. Frege [1], pp. 40 and 41.

Ontology, in which, besides the semantic categories already appearing in Protothetic, various new semantic categories are represented, we shall meet the word 'homosemo' whose last letter 'o' will serve to indicate in a similar way that the membership of any expressions in the same semantic category is there relative to theses belonging to Ontology. The last letter 'm' of the word 'homosemm', which we shall meet in the terminological explanations of the system Mereology, will refer to a corresponding relativization in it. The letters 'p', 'o', and 'm' indicate correspondingly an analogous relativization, in the systems of Protothetic, Ontology, and Mereology, for a number of my other abbreviations of speech: besides the word 'thp' we shall have the words 'tho' and 'thm'; besides 'frm', etc.

T.E.XXXVI. $[A, B, C, D, E] :: A \in \text{constp}(B, C, D, E) . = .\therefore$
$D \in \text{homosemp}(E, B) .\therefore$
$[F, G] : G \in \text{thp}(B) . F \in \text{ingr}(G) . \supset . D \in\sim \text{cnvar}(D, F))$
$\therefore$
$A \in \text{cnf}(D) :$
$[\exists F, G, H] . F \in \text{ingr}(C) . G \in \text{thp}(B) . H \in \text{ingr}(G) .$
$A \in An(E, F, H)$

T.E.XXXVII. $[A, B, C] : A \in \text{constp}(B, C) . = .$
$[\exists D, E] . A \in \text{constp}(B, C, D, E)$

T.E.XXXVIII. $[A, B, C, D, E, F] \therefore A \in$
$\text{quasihomosemp}(B, C, D, E, F) . = :$
$E \in \text{homosemp}(F, C) :$
$[\exists G, H, I] . G \in \text{ingr}(D) . H \in \text{thp}(C) . I \in \text{ingr}(H) .$
$A \in An(E, G, I) :$
$[\exists G, H, I] . G \in \text{ingr}(D) . H \in \text{thp}(C) . I \in \text{ingr}(H) .$
$B \in An(F, G, I) :$

T.E.XXXIX. $[A, B, C, D, E] \therefore A \in \text{fnctp}(B, C, D, E) . = :$
$D \in \text{homosemp}(E, B) .$
$A \in \text{genfnct}(D) :$

$[\exists F, G, H] \,.\, F \in \text{ingr}(C) \,.\, G \in \text{thp}(B) \,.\, H \in \text{ingr}(G) \,.$
$A \in An(E, F, H)$

T.E.XL. $[A, B, C, D, E, F] \therefore A \in \text{varp}(B, C, D, E, F) \,.=:$
$E \in \text{homosemp}(B, C):$
$[\exists G, H, I] \,.\, G \in \text{ingr}(D) \,.\, H \in \text{thp}(C) \,.\, I \in \text{ingr}(H) \,.$
$F \in An(E, G, I):$
$F \in \text{ingr}(\text{Eqv11}(\text{Essnt}(D))) \,.$
$A \in \text{cnvar}(F, D)$

T.E.XLI. $[A, B, C, D, E] :: A \in \text{prntmp}(B, C, D, E) \,. = \therefore$
$D \in \text{homosemp}(B, B) \,.$
$E \in \text{prntm}(D) \,.$
$A \in \text{prntm}(\text{Eqv12}(\text{Essnt}(C))) \,.$
$\text{arg}(A) \infty \text{arg}(E) \therefore$
$[F, G] : F \in \text{arg}(A) \,.\, G \in \text{arg}(E) \,.\, (\text{arg}(A) \cap \text{prcd}(F)) \infty (\text{arg}(E) \cap \text{prcd}(G)) \,.\supset .\, [\exists H, I] \,.\, F \in \text{varp}(G, B, C, H, I)$

T.E.XLII. $[A, B, C, D, E] : A \in \text{1prntmp}(B, C, D, E) \,. = .$
$A \in \text{prntmp}(B, C, D, E) \,.$
$\text{Uingr}(D) \in \text{ingr}(E)$

T.E.XLIII. $[A, B, C, D, E, F, G] : A \in \text{2prntmp}((B, C, D, E, F, G) \,. = .$
$A \in \text{prntmp}(B, C, D, E) \,.$
$F \in \text{prntm}(D) \,.$
$\text{Uprcd}(F) \in \text{ingr}(E) \,.$
$G \in \text{simprntm}((F)$

T.E.XLIV. $[A, B] \because A \in \text{defp}(B)$[73] $\,. = ::$
$\text{1ingr}(\text{Essnt}(A)) \in \sim (\text{cnvar}(\text{1ingr}(\text{Essnt}(A)), A)) \,.$

[73] Loosely speaking, expressions of the type '$A \in \text{defp}(B)$' could be read off by means of corresponding phrases of the type 'A is an expression which could hold true as a definition in the system of Protothetic directly after thesis B'.

$1\text{ingr}(\text{Eqv}12(\text{Essnt}(A))) \in\!\sim$
$(\text{cnvar}(1\text{ingrEqv}12\text{Essnt}(A))), A))\,.$
$1\text{ingr}(\text{Eqv}12(\text{Essnt}(A))) \in\!\sim (\text{constp}(B, A))$[74] $:\,:$
$[C]\therefore C \in \text{trm}\,.\,C \in \text{ingr}(\text{Eqv}11(\text{Essnt}(A)))\,.\supset\,:[\exists D]$
$.\,D \in \text{qntf}\,.\,D \in \text{ingr}(A)\,.\,C \in \text{int}(D)\,.\vee.\,[\exists D, E]\,.$
$D \in \text{ingr}(A)\,.\,C \in \text{var}(E, D)\,.\vee.\,C \in \text{constp}(B, A)$[75]
$:\,:$
$[C, D]: D \in \text{qntf}\,.\,D \in \text{ingr}(A)\,.\,C \in \text{int}(D)\,.\supset.\,[\exists E, F]\,.$
$E \in \text{ingr}(A)\,.\,F \in \text{var}(C, E)$[76]
$[C, D, E]: C \in \text{int}(\text{Qntf}(A))\,.\,E \in \text{prntm}(\text{Essnt}(A))\,.$
$D \in \text{arg}(E)\,.\supset.\,[\exists F]\,.\,F \in \text{ingr}(D)\,.\,F \in \text{var}(C, A)$[77] $:\,:$
$[C, D, E]\therefore C \in \text{ingr}(Eqv11(\text{Essnt}(A))\,.\,E \in \text{ingr}(A)$
$.\,D \in \text{cnvar}(C, E)\,.\,D \in \text{ingr}(Eqv11(\text{Essnt}(A)))\,.\supset:$
$D \in \text{Id}(C)\,.\vee.\,[\exists F, G]\,.\,D \in \text{quasihomosemp}(C, B, A, F, G)$[78]
$:\,:$
$[C]: C \in \text{gnrl}\,.\,C \in \text{ingr}(A)\,.\,C \in\!\sim (\text{Id}(A))\,.\supset.\,[\exists D, E, F, G]$
$.$
$D \in \text{homosemp}(B, B)\,.\,E \in \text{thp}(B)\,.\,F \in \text{ingr}(E)\,.$
$G \in \text{ingr}(A)\,.\,D \in \text{Anarg}(C, F, G)\,:\,:$
$[C, D]\therefore C \in \text{gnrl}\,.\,C \in \text{ingr}(A)\,.\,D \in \text{Essnt}(C)\,.\supset: D \in \text{vrb}$
$.\vee.\,[\exists E]\,.\,E \in \text{frp}(B)\,.\,D \in \text{genfnct}(E)\,:\,:$
$[C]\therefore C \in \text{fnct}\,.\,C \in \text{ingr}(\text{Eqv}11(\text{Essnt}(A)))\,.\supset:[\exists D]$
$.\,D \in \text{gnrl}\,.\,D \in \text{ingr}(A)\,.\,C \in \text{Essnt}(D)\,.\vee.\,[\exists D, E]\,.$
$C \in \text{fnctp}(B, A, D, E)\,:\,:$
$[C]: C \in \text{prntm}(\text{Eqv}12(\text{Essnt}(A)))\,.\supset.\,[\exists D]\,.\,D \in \text{arg}(C)$[79]
$\therefore$
$[C, D]: C \in \text{prntm}(\text{Eqv}12(\text{Essnt}(A)))\,.\,D \in \text{arg}(C)\,.\supset.$

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74 Cf. Frege [1], p. 51.
75 Cf. Frege [1], pp. 41, 45, and 51.
76 Cf. Frege [1], pp. 45 and 51.
77 Cf. Frege [1], p. 52.
78 Cf. Frege [1], pp. 43–45, and 51.
79 Cf. Frege [1], pp. 51 and 52.

$[\exists E] . D \in \text{var}(E, A)$[80] $.\therefore$
$[C, D] : C \in \text{trm} . C \in \text{ingr}(\text{Eqv12}(\text{Essnt}(A))) . D \in \text{trm} .$
$D \in \text{ingr}(\text{Eqv12}(\text{Essnt}(A))) . C \in \text{cnf}(D) . \supset . C \in (\text{Id}(D)$
$\therefore$
$[C, D] : C \in \text{prntm}(\text{Eqv12}(\text{Essnt}(A))) . D \in$
$\text{prntm}(\text{Eqv12}(\text{Essnt}(A))) . C \in \text{simprntm}(D) . \supset . C \in \text{Id}(D)$
$\therefore$
$[C, D, E] : C \in \text{1prntmp}(B, A, D, E) . \text{Uingr}(\text{Eqv12}(\text{Essnt}$
$(A))) \in \text{ingr}(C) . \supset . C \in \text{simprntm}(E) .\therefore$
$[C, D, E, F, G] : C \in \text{2prntmp}(B, A, D, E, F, G) . G \in \text{ingr}(A)$
$.$
$\text{Uprcd}(G) \in \text{ingr}(C) . \supset . C \in \text{simprntm}(E) .\therefore$
$[C, D, E] : C \in \text{prntm}(\text{Eqv12}(\text{Essnt}(A))) .$
$\text{Uingr}(\text{Eqv12}(\text{Essnt}(A))) \in \text{ingr}(C) . D \in \text{thp}(B)$
$. E \in \text{ingr}(D) . C \in \text{simprntm}(E) . \supset . [\exists F, G] .$
$C \in \text{1prntmp}(B, A, F, G) .\therefore$
$[C, D, E, F] : C \in \text{prntm}(\text{Eqv12}(\text{Essnt}(A))) . D \in \text{prntm}$
$. D \in \text{ingr}(A) . \text{Uprcd}(D) \in \text{ingr}(C) . E \in \text{thp}(B)$
$. F \in \text{ingr}(E) . C \in \text{simprntm}(F) . \supset . [\exists G, H, I] .$
$C \in \text{2prntmp}(B, A, G, H, I, D)$

T.E.XLV. $[A, B] \therefore A \in \text{cnsqrprtqntf}(B)$[81] $. = ::$
$\text{Essnt}(\text{Eqv11}(\text{Essnt}(A))) \in \text{cnf}(\text{Essnt}(\text{Eqv11}(\text{Essnt}(B))))$
$.$
$\text{Essnt}(\text{Eqv12}(\text{Essnt}(A))) \in \text{cnf}(\text{Essnt}(\text{Eqv12}(\text{Essnt}(B))))$
$\therefore$
$[C] : C \in \text{int}(\text{Qntf}(A)) . \supset . [\exists D] . D \in \text{cnf}(C) .$
$D \in \text{ingr}(\text{Qntf}(B)) ::$

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80 Cf. *loc. cit.*

81 Loosely speaking, expressions of the type '$A \in \text{cnsqrprtqntf}(B)$' could be read off by means of corresponding phrases of the type 'A is derivable from B by means of a corresponding distribution of the quantifiers' (cf. dir. (γ)).

$[C, D, E, F, G, H] \therefore F \in \text{prntm}(\text{Essnt}(A)) \,.\, G \in$
$\text{prntm}(\text{Essnt}(B)) \,.\, C \in \text{Anarg}(D, F, G) \,.\, E \in \text{var}(H, B) \,.$
$E \in \text{ingr}(D) \,.$
$\supset : [\exists I] : I \in \text{cnf}(E) : I \in \text{int}(\text{Qntf}(A)) \,.\, \vee \,.\, I \in \text{int}(\text{Qntf}(C))$
$::$
$[C, D, E, F, G] : F \in \text{prntm}(\text{Essnt}(A)) \,.\, G \in \text{prntm}(\text{Essnt}(B))$
$.\, C \in \text{Anarg}(D, F, G) \,.\, E \in \text{int}(\text{Qntf}(D)) \,.\, \supset \,.$
$[\exists H] \,.\, H \in \text{cnf}(E) \,.\, H \in \text{ingr}(\text{Qntf}(C)) ::$
$[C, D, E, F, G] F \in \text{prntm}(\text{Essnt}(A)) \,.\, G \in \text{prntm}(\text{Essnt}(B)) \,.$
$C \in \text{Anarg}(D, F, G) \,.\, E \in \text{int}(\text{Qntf}(C)) \,.\, \supset :$
$[\exists H] : H \in \text{cnf}(E) \,.\, H \in \text{ingr}(D) : [\exists I] \,.\, H \in \text{var}(I, B) \,.$
$\vee \,.\, H \in \text{int}(\text{Qntf}(D)) ::$
$[C, D, E, F, G, H] : F \in \text{prntm}(\text{Essnt}(A)) \,.\, G \in$
$\text{prntm}(\text{Essnt}(B)) \,.\, C \in \text{Anarg}(D, F, G) \,.\, H \in \text{int}(\text{Qntf}(A))$
$.\, E \in \text{cnf}(H) \,.\, E \in \text{ingr}(\text{Qntf}(C)) \,.\, \supset \,.\, [\exists I] \,.\, I \in \text{cnf}(E) \,.$
$I \in \text{ingr}(\text{Qntf}(D))$

T.E.XLVI. $[A, B, C] : A \in \text{cnsqeqvl}(B, C) \,.=\,.$
$C \in \text{cnf}(\text{Eqvl1}(B)) \,.$
$A \in \text{cnf}(\text{Eqvl2}(B))$

T.E.XLVII. $[A, a, B, C] \because A \in \text{cnsqsbstp}(B, C, a)$[82] $. = ::$
$\text{Essnt}(A) \in \text{Cmpl}(a) \,.$
$a \infty \text{int}(\text{Sbqntf}(C)) ::$
$[D, E] \therefore D \in \text{int}(\text{Sbqntf}(C)) \,.\, E \in a \,.\, (a \cap$
$\text{prcd}(E)) \infty (\text{int}(\text{Sbqntf}(C)) \cap \text{prcd}(D)) \,.\, \supset : [\exists F] \,.$
$D \in \text{var}(F, C) \,.\, \vee \,.\, D \in \text{cnf}(E) ::$
$[D, E] \therefore D \in \text{int}(\text{Sbqntf}(C)) \,.\, E \in a \,.\, (a \cap$
$\text{prcd}(E)) \infty \text{int}(\text{Sbqntf}(C)) \cap \text{prcd}(D)) \,.\, \supset : E \in \text{trm} \,.\, \vee \,.$
$E \in \text{gnrl} \,.\, \vee \,.\, E \in \text{fnct} \,.\, \vee \,.\, E \in \text{cnf}(D) ::$

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[82] Loosely speaking, expressions of the type '$A \in \text{cnsqsbstp}(B, C, a)$' could be read off by means of corresponding phrases of the type 'A is derivable from C with the help of the expression (a), by means of an insertion correct in Protothetic with regard to B' (cf. dir. (β)).

$[D, E, F, G] : D \in \text{cnvar}(E, C) . F \in a . G \in a$
$. (a \cap \text{prcd}(F)) \infty (\text{int}(\text{Sbqntf}(C)) \cap \text{prcd}(D)) .$
$(a \cap \text{prcd}(G)) \infty (\text{int}(\text{Sbqntf}(C)) \cap \text{prcd}(E)) . \supset . F \in \text{cnf}(G)$
$\therefore$
$[D, E, F, G, H, I, K, L] : D \in \text{ingr}(\text{Essnt}(C)) . E \in$
$\text{int}(Qntf(D)) . F \in \text{var}(K, C) . F \in \text{ingr}(D) . G \in a .$
$H \in a . (a \cap \text{prcd}(G)) \infty (\text{int}(\text{Sbqntf}(C)) \cap \text{prcd}(E))(a \cap$
$\text{prcd}(H)) \infty (\text{int}(\text{Sbqntf}(C)) \cap \text{prcd}(F)) . L \in \text{ingr}(A) .$
$I \in \text{var}(G, L) . \supset . I \in\sim (\text{ingr}(H))$[83] $::$
$[D, E] \therefore D \in \text{int}(\text{Qntf}(A)) . E \in \text{cnf} D) . E \in \text{ingr}(C) . \supset :$
$[\exists F] . F \in \text{qntf} . F \in \text{ingr}(C) . E \in \text{int}(F) . . [\exists F, G] .$
$F \in \text{ingr}(C) . E \in \text{var}(G, F) ::$
$B \in \text{expr} ::$
$[D] \therefore D \in \text{trm} . D \in \text{ingr}(A) . \supset : [\exists E] . E \in \text{qntf} .$
$E \in \text{ingr}(A) .$
$D \in \text{int}(E) . \vee . [\exists E, F] . E \in \text{ingr}(A) . D \in \text{var}(F, E) . \vee .$
$D \in \text{constp}(B, A) ::$
$[D, E] : E \in \text{qntf} . E \in \text{ingr}(A) . D \in \text{int}(E) . \supset . [\exists F, G] .$
$F \in \text{ingr}(A) . G \in \text{var}(D, F) ::$
$[D, E, F] \therefore E \in \text{ingr}(A) . F \in \text{cnvar}(D, E) . \supset : F \in \text{Id}(D) . \vee$
$.$
$[\exists G, H] . F \in \text{quasihomosemp}(D, B, A, G, H) ::$
$[D] : D \in \text{gnrl} . D \in \text{ingr}(A) . D \in\sim (\text{Id}(A)) . \supset .$
$[\exists E, F, G, H] .$
$E \in \text{homosemp}(B, B) . F \in \text{thp}(B) . G \in \text{ingr}(F) . H \in$
$\text{ingr}(A) . E \in \text{Anarg}(D, G, H) ::$
$[D, E] \therefore D \in \text{gnrl} . D \in \text{ingr}(A) . E \in \text{Essnt}(D) . \supset : E \in \text{vrb}$
$. \vee .$
$[\exists F] . F \in \text{frp}(B) . E \in \text{genfnct}(F) ::$

83 Cf. Frege [1], pp. 62 and 63.

$[D] \therefore D \in \text{fnct} . D \in \text{ingr}(A) . \supset : D \in \text{Id}(A) . \vee . [\exists E]$
$. E \in \text{gnrl} . E \in \text{ingr}(A) . D \in \text{Essnt}(E) . \vee . [\exists E, F] .$
$D \in \text{fnctp}(B, A, E, F)$

T.E.XLVIII. $[A, B, C] : A \in \text{cnsqsbstp}(B, C) . = .$
$[\exists a] . A \in \text{cnsqsbstp}(B, C, a)$

T.E.IL. $[A, B] :: A \in \text{extnsnlp}(B) . = ::$
$[\exists C, D] . C \in \text{int}(\text{Qntf}(A)) . D \in \text{int}(\text{Qntf}(A)) . C \in \text{prcd}(D)$
$\therefore$
$[C, D] : D \in \text{qntf} . D \in \text{ingr}(A) . C \in \text{int}(D) . \supset . [\exists E, F] .$
$E \in \text{ingr}(A) . F \in \text{var}(C, E) . F \in \sim (\text{cnf}(1\text{ingr}(\text{Essnt}(A)))$
$\therefore$
$[\exists C] . C \in \text{prntm}(\text{Eqvl1}(\text{Essnt}(\text{Eqvl2}(\text{Essnt}(A))))) .$
$1\text{ingr}(\text{Eqvl1}(\text{Cmpl}(\text{int}(\text{Sbqntf}(\text{Eqvl1}(\text{Essnt}(A))))))) \in$
$\text{cnvar}(\text{Cmpl}(\text{int}(C)), A) :$
$[\exists C] . C \in \text{prntm}(\text{Eqvl2}(\text{Essnt}(\text{Eqvl2}(\text{Essnt}(A))))) .$
$1\text{ingr}(\text{Eqvl2}(\text{Essnt}(\text{Eqvl1}(\text{Essnt}(A))))) \in$
$\text{cnvar}(\text{Cmpl}(\text{int}(C)), A) ::$
$[C] \therefore C \in \text{fnct} . C \in \text{ingr}(A) . \supset : [\exists D] . D \in \text{gnrl} . D \in$
$\text{ingr}(A) . C \in \text{Essnt}(D) . \vee . [\exists D, E] . C \in \text{fnctp}(B, A, D, E)$
$::$
$[C, D, E, F] : D \in \text{prntm}(\text{Eqvl1}(\text{Essnt}(\text{Eqvl1}(\text{Essnt}(A)))))$
$. E\ \text{prntm}(\text{Eqvl2}(\text{Essnt}(\text{Eqvl1}(\text{Essnt}(A))))) . F \in$
$\text{Anarg}(C, D, E) . \supset . F \in \text{cnvar}(C, \text{Eqvl1}(\text{Essnt}(A)))$
$\therefore$
$[C, D, E] : D \in \text{ingr}(A) . E \in \text{cnvar}(C, D) . \supset . [\exists F, G] .$
$E \in \text{quasihomosemp}(C, B, A, F, G) \therefore$
$[C, D] : D \in \text{cnvar}(C, \text{Eqvl1}(\text{Essnt}(A))) . \supset . [\exists E, F] .$
$E \in \text{ingr}(A) . F \in \text{ingr}(A) . D \in \text{Anarg}(C, E, F) \therefore$
$[C, D, E] : C \in \text{prntm}(\text{Essnt}(\text{Eqvl2}(\text{Essnt}(A))))$
$. D \in \text{arg}(C) . E \in \text{Sgnfnct}(D) . \supset . E \in$
$\text{var}(\text{Cmpl}(\text{int}(\text{Qntf}(\text{Eqvl2}(\text{Essnt}(A))))), \text{Eqvl2}(\text{Essnt}(A)))$

Using terms whose meaning I established in the terminological explanations given above, the formulation of the directives for SS5 can be reduced to stipulating the following instruction:

If a thesis A is the last thesis already belonging to the system, an expression B may be added to the system as a new thesis only if at least one of the five following conditions is fulfilled:

(1) $B \in \text{defp}(A)$ (in connection with dir. (δ^*) and (ε^*)

(2) $[\exists C] . C \in \text{thp}(A) . B \in \text{cnsqrprtqntf}(C)$ (dir. (γ))

(3) $[\exists C, D] . C \in \text{thp}(A) . D \in \text{thp}(A) . B \in \text{cnsqeqvl}(C, D)$ (dir. (α))

(4) $[\exists C] . C \in \text{thp}(A) . B \in \text{cnsqsbstp}(A, C)$ (dir. (β))

(5) $B \in \text{extnsnlp}(A)$ (dir. (η^*))

In §12 I derive, in the way just represented, a series of various theorems of SS5 from the axioms A1–A3. I would like here, however, in order to shed some light on the construction of SS5, to call the reader's attention to the following (loosely expressed) facts:

A) The directives of SS5 do not presuppose any special forms for its constant terms different from the forms of its variable terms: all possible words, with the exception of parentheses and words equiform with tie first, fifth, sixth, or last word of Axiom A1, can appear in SS5 at one time as a constant, at another as a variable. According to this, their character in any particular formula depends upon the kind and position of quantifiers in the given formula.

B) The directives of SS5 do not presuppose any special forms for the constant or variable words of any particular semantical category different from the forms of the words of any other semantical category. Words of any semantical category can be equiform with one another, even within the scope of any single thesis belonging to the system.

C) According to the directives of SS5, only those function signs can be, and they must be, placed in front of the parentheses which embrace the arguments to which these function signs apply.

D) According to the directives one does not need to separate individual arguments from one another by a comma. T.E.XXI allows one to ascertain, without the help of any commas, where one argument ends and another begins.

E) The definition directive (1) is formulated in such a way that it is impossible to introduce into SS5 any sort of quantifier other than the universal quantifiers (with any number of variables) already established in advance.

F) The directives do not enable one to obtain in the system any thesis which includes as a constituent part a universal proposition (gnrl) whose essnt is also a universal proposition. In the case in which one normally has to do with expressions of the type

$$\llcorner ab\ldots\lrcorner \ulcorner \llcorner kl\ldots\lrcorner \ulcorner f(ab\ldots kl\ldots)\urcorner\urcorner,$$

we encounter in my system only corresponding expressions of the type

$$\llcorner ab\ldots kl\ldots\lrcorner \ulcorner f(ab\ldots kl\ldots)\urcorner,$$

in which the variables appearing in the quantifier can be rearranged within it in any way.

G) The substitution directive (4) stated above, while authorizing various substitutions for variables, does not permit substituting something for a whole expression of the type '$f(ab\ldots)$' which, as is well known, was warranted by the substitution directive Frege formulated with far-reaching precision in his system of the foundations of arithmetic.[84] In this respect, therefore, my directive is weaker. From the beginning I have always thought of directive (β) — for SS1 as well as for all the other systems of

[84] Cf. *loc. cit.*

Protothetic which I have spoken of above — as just such a weaker substitution directive.[85]

Perhaps I should add that for many months I spent a great deal of time working systematically towards the formalization of these systems of Protothetic by means of a clear formulation of their directives using the various auxiliary terms whose meanings I have fixed in the terminological explanations given above. Having no predilection for various 'mathematical games' that consist in writing out according to one or another conventional rule various more or less picturesque formulae which need not be meaningful, or even — as some of the 'mathematical gamers' might prefer — which should necessarily be meaningless, I would not have taken the trouble to systematize and to often check quite scrupulously the directives of my system, had I not imputed to its theses a certain specific and completely determined sense, in virtue of which its axioms, definitions, and final directives (as encoded for SS5), have for me an irresistible intuitive validity. I see no contradiction, therefore, in saying that I advocate a rather radical 'formalism' in the construction of my system even though I am an obdurate 'intuitionist'. Having endeavored to express some of my thoughts on various particular topics by representing them as a series of propositions meaningful in various deductive theories, and to derive one proposition from others in a way that would harmonize with the way I finally considered intuitively binding, I know no method more effective for acquainting the reader with my logical intuitions than the method of formalizing any deductive theory to be set forth. By no means do theories under the influence of such a formalization cease to consist of genuinely meaningful propositions which for me are intuitively valid. But I always view

[85] Cf. Ajdukiewicz [1], pp. 209, 210, 213, and 214; Ajdukiewicz [2], p. 252. Cf. also v. Neumann [1], pp. 10, 16, and 42.

the method of carrying out mathematical deductions on an 'intuitionistic' basis of various logical secrets as a considerably less expedient method.[86]

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[86] The directives represented above of the system of Protothetic, together with the directives of Ontology and Mereology which I shall deal with in further sections of this article, i consider, in all due modesty, to be but one of the numerous attempts to formalize mathematics which have been undertaken on a larger scale by various investigators since the time of Frege, and which are represented today under the 'metamathematical' banner (as it is best known) of David Hilbert. But even among those works whose purpose is to construct a foundation for mathematics I do not know of a single one which actually stipulates, in a way that causes no doubts about its interpretation, a combination of directives sufficient for the derivation of all the theses effectively admitted into its system, and which at the same time would not lead to a contradiction in one way or another not foreseen by its author. In particular, I shall give here several examples of such inaccuracies unforeseen by the authors that seem to me to be contained in the systems which were published last year by Chwistek and v. Newmann (cf. Chwistek [1], Chwistek [2], and v. Neumann [1]), and which, as far as I know, have not yet undergone a sufficiently exhaustive critical analysis. (I intend, however, to omit critical remarks on works in the same area published by Hilbert and Ackermann until the recently announced work of Bernays and Hilbert appears, in which, one can expect, will appear a synthesis of previous investigations of the active 'metamathematical' school (cf. Hilbert – Ackerman [1], Section V and p. 115).)

I) In formulating the directives of his system, Chwistek has stipulated a certain combination of conditions which an expression is supposed to satisfy before it can be written out as a definition (cf. Chwistek [1], p. 28, point 0.3; Chwistek [2], p. 97, errata). Chwistek's conditions appear to admit, among others, the formula which stipulate that

(1) $. p \mho p . \underset{\mathrm{df}}{=} . p \supset q .$

which we write out as, e.g., point 2.002. (In connection with condition 2° of point 0.3 I should point out that the expression '$. p \mho p .$' cannot be considered by the author to be significant, for he makes use of the expression '.' in the 'defined symbols' of a number of his definitions supplied with higher numbers (cf. e.g., Chwistek [1], p. 34, point 3.01) which would be forbidden by condition 2° of point 0.3, if the expression '.' were significant. From Thesis 2.02, Def 2.001 (for both, cf. *loc. cit.*), and formula (1) above, it follows that

(2) $\vdash . q \supset . p \mho p . .$;

BIBLIOGRAPHY

The abbreviations following below will be used in bibliographical references.

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from formula (2), that

(3) $\vdash ..p \supset p . \supset p \mho p ..;$

from formula (3) and thesis 2.08 (cf. *loc. cit.*), that

(4) $\vdash . p \mho p .;$

from (4) and (1), that

(5) $\vdash . p \supset q .;$

from (5), that

(6) $\vdash .. p \mho p . \supset q .;$

from (6) and (4), that

(7) $\vdash q;$

from (7), that

(8) $\vdash \sim q.$

(8) contradicts (7).

II) The definition directive Chwistek stipulated appears to permit one to write the definition saying that

(a) $. \underset{\mathrm{df}}{=} . \sim$

as point 2.003. From Thesis 2.08 and Def. 1.01 (cf. Chwistek [1], p. 28) it follows that

(b) $\vdash . \sim p \vee p .;$

from (b) and (a), that

(c) $\vdash . p \vee p .;$

from Thesis 1.2 (cf. Chwistek [1], p. 33) and (3), that

(d) $\vdash p;$

and from (d), that

(e) $\vdash \sim p.$

(e) contradicts (d).

III) von Neumann intends to replace 'classical' mathematics and logic with a certain system of formulae which, he emphasizes, should be constructed with nothing but meaningless signs (cf. von Neumann [1], pp. 4 and 5). The construction of the formal system should proceed in the sense of certain rules he states which I have not, by the way, been able to understand in all particulars. As a starting point, in the construction of this formal system there are certain special inscriptions which he calls 'axiom-schemata', and which he distributes into groups I–VI in a certain way (cf. von Neumann [1], pp. 13–21). von Neumann conjectured in his work that the system

Ajdukiewicz [1]: Kazimierz Ajdukiewicz, 'Voraussetzungen der traditionellen Logik', *Przegląd Filozoficzny*, Yearbook 29 (for the year 1926), III–IV (1927). (Polish).

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in question is consistent in the sense that no two formulae are derivable, one having the form of any expression 'a' and the other on the other hand having the form of the corresponding expression '$\sim a$' (cf. von Neumann [1], pp. 12, 21, 36, and 37), and he sketched an argument which is supposed to prove that in every case a system would be consistent if it could be obtained from the preceding system by omitting the axiom-schemata he numbers 'group V' (cf. von Neumann [1], pp. 18, and 21–37). His line of reasoning on the consistency of even this weaker system does not seem to be sufficiently careful on all points, for as I am inclined to assume, even without the schemata of group V we can, with methods he allows, manage to obtain new formulae in his system in such a manner that we shall arrive at two mutually contradictory formulae. The following is perhaps one such way:

Considering the consequences of formulae $b_{v,v}^{(n)}$ which von Neumann discusses on p. 20 of his work, we can say (cf. *op. cit.*, p. 21) that a natural number k can be found such that the formula possessing the form of the expression 'x' is identical with formula $b_{k,1}^{(1)}$, which constitutes one of the elements of the consequences $b_{v,1}^{(1)}$. In accordance with the authorization he had obtained to add to the system the formulae which fall in a way he foresaw under schema 2 of group VI (*op. cit.*, p. 20), we are permitted to add to the system the corresponding formulae which say that

(α) $\Omega_{k,1}^{(1)}O = O$.

'Modifying' in a way the symbol '$\Omega_{k,1}^{(1)}$' (cf. *op. cit.*, p. 21) in complete agreement with von Neumann's instructions concerning 'modification', in which expressions of the type '$\Omega_{k,1}^{(1)}a_1$' change into corresponding expressions of the type '$1 + a_1$', we can obtain from formula (α) the formula stating that

(β) $1 + 0 = 0$.

Taking into consideration schema 3 of group III (*op. cit.*, p. 15) we can assert (cf. *op. cit*, the instruction on p. 9 concerning the use of parentheses, and the author's manner of designations on pp. 19 and 21) that

(γ) $\sim (0 + 1 = 0)$.

'Modifying' the symbol '0' — again in full accordance with the relevant instructions from von Neumann (cf. *op. cit.*, p. 8) — to the symbol '1', we obtain, corresponding to (β) and (γ), formulae which state that

$1 + 1 = 1$

Ajdukiewicz [2]: Kazimierz Ajdukiewicz, 'Berichtigung wichtigerer Irrtümer, die im Artikel u. d. T. 'Voraussetzungen der traditionellen Logik' enhalten sind' *Przegląd Filozoficzny*, Yearbook 29 III–IV pp. 200–229), *Przegląd Filozoficzny*, Yearbook 30 II–IV (1927). (Polish).

Chwistek [1]: Leon Chwistek, 'The Theory of Constructive Types (Principles of Logic and Mathematics), Part I, General Principals of Logic, Theory of Classes and Relations', extracted from the *Annales de la Société Polonaise de Mathématique*, Cracow 1923.

Chwistek [2]: Leon Chwistek, 'The Theory of Constructive Types (Principles of Logic and Mathematics), Part II, Cardinal Arithmetic', separately published in *Rocznik Polskiego Tow. Matematycznego, Annales de la Société Polonaise de Mathématique* III (1924–25).

Frege [1]: G. Frege, *Grundgesetze der Arithmetik, Begriffsschriftlich abgeleitet*, vol I, Jena 1893.

Frege [2]: G. Frege, *Grundgesetze der Arithmetik, Begriffsschriftlich abgeleitet*, vol II, Jena 1903.

Hilbert–Ackermann [1]: D. Hilbert and W. Ackermann, 'Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete' XXVII, *Grundzüge der theoretischen Logik*, Berlin 1928.

Husserl [1]: Edmund Husserl, *Logische Untersuchungen*, vol. 2, *Untersuchungen zur Phänomenologie und Theorie der Erkenntnis*, I, part 2, revised ed., Halle 1913.

Leśniewski [1]: Stanisław Leśniewski,'Über die Grundlagen der Mathematik', *Przegląd Filozoficzny*, Yearbook 30 II–III (1927). (Polish).

Łukasiewicz [1]: Jan Łukasiewicz, *Zweiwertige Logik*, separately published in commemorative book in honor of Prof. Twardowski, (*Przegląd Filozoficzny*, Yearbook 23), Lwów 1921. (Polish).

Nicod [1]: J. G. P. Nicod, 'A reduction in the number of the Primitive Propositions of Logic', *Proceedings of the Cambridge Philosophical Society* XIX, 30 Oct. 1916–24, Nov. 1919, 1920. (Communicated by G. H. Hardy).

Peano [1]: G. Peano, 'Logique mathematique', *Formulaire de Mathematiques*. II – §1 [$K, \in, \supset, \frown, =, -, \smile, \Lambda, \exists, \iota, \bar{\iota}, K', \smile ', \frown ', \text{Ɉ}$, Sim, rcp], II–VIII–1897.

and

$$\sim (1 + 1 = 1).$$

Post [1]: Emil L. Post, 'Introduction to a General Theory of Elementary Propositions', *American Journal of Mathematics.* XLIII, no. 3, July 1921.

Russell [1]: Bertrand Russell, *Introduction to Mathematical Philosophy*, second ed., London, New York, April 1920.

Schröder [1]: Ernst Schröder, *Vorlesungen über die Algebra der Logik (exacte Logik)*, vol. 2, first ed., Leipzig 1891.

Sheffer [1]: Henry Maurice Sheffer, 'A Set of Five Independent Postulates for Boolean Algebras, with Application to Logical Constants', *Transactions of the American Mathematical Society* 14, no. 4, October 1913.

Tarski [1]: Alfred Tajtelbaum, 'Sur le terme primitif de la logistique', *Fundamenta Mathematicae* IV (1923).

Tarski [2]: Alfred Tajtelbaum-Tarski, *Über das primitive Wort der Logistik* (doctoral thesis), separately published in *Przegląd Filozoficzny*, Yearbook 1923, Warsaw 1923. (Polish).

Tarski [3]: Alfred Tajtelbaum-Tarski, 'Sur les truth-functions au sens de MM. Russell et Whitehead', *Fundamenta Mathematicae* V (1924).

von Neumann [1]: J. von Neumann, 'Zur Hilbertschen Beweistheorie', separately published in *Mathematische Zeitschrift* 26, no. 1, Berlin 1927.

WPI [1]: 'Bericht des Warschaues Philosophischen Instituts für die Zeit vom 1 Juli 1920 bis 25 Juni 1921', *Przegląd Filozoficzny*, Yearbook 24 (1921), no. III and IV, 1921. (Polish).

Whitehead–Russell [1]: Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, vol. I, second ed., Cambridge 1925.

Żyliński [1]: Eustachy Żyliński, 'Über die Darstellbarkeit der einen 'truth-functions' durch andere', *Denkbuch des Ersten Polnischen Philosophischen Kongresses, Lemberg, 1923, Przegląd Filozoficzny*, Yearbook 30 (1927), no. IV. (Polish).

Żyliński [2]: E. Żyliński, 'Some remarks concerning the theory of deduction', *Fundamenta Mathematicae* VII (1925).

§ 12. The following theses can be obtained in the system SS5[1]:

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[1] I use T1, T2, T3, etc. to enumerate theses of Protothetic added to the system in accordance with conditions 2–5 of the instruction given in §11 concerning the method for constructing system SS5. Theses added to the system in accordance with condition 1 of that instruction are enumerated with D1, D2, D3, etc.

T1. $\lfloor qr \rfloor \ulcorner \phi\Big(\phi\big(\phi(qr)\phi(rq)\big)\phi(rr)\Big) \urcorner$[2] $[A1; q/p; r/q]$[3]

T2. $\phi\Big(\lfloor qr \rfloor \ulcorner \phi\big(\phi(qr)\phi(rq)\big) \urcorner \lfloor r \rfloor \ulcorner \phi(rr) \urcorner\Big)$ [T1][4]

T3. $\lfloor qr \rfloor \ulcorner \phi\Bigg(\phi\Big(\phi\big(r\phi(qr)\big)\phi\big(\phi(rq)r\big)\Big)\phi\Big(\phi(qr)\phi(rq)\Big)\Bigg) \urcorner$[5]
$[A1; r/p; \phi(rq)/q; \phi(qr)/r]$

T4. $\phi\Bigg(\lfloor qr \rfloor \ulcorner \phi\Big(\phi\big(r\phi(qr)\big)\phi\big(\phi(rq)r\big)\Big) \urcorner \lfloor qr \rfloor \ulcorner \phi\Big(\phi(qr)$
$\phi(rq)\Big) \urcorner\Bigg)$ [T3]

T5. $\lfloor qr \rfloor \ulcorner \phi\Big(\phi\big(r\phi(qr)\big)\phi\big(\phi(rq)r\big)\Big) \urcorner$[6] $[A2; r/p]$

T6. $\lfloor ps \rfloor \ulcorner \phi\Bigg(\phi\Big(\phi(ps)\phi(sp)\Big)\phi\Big(\phi\big(\phi(ps)s\big)p\Big)\Bigg) \urcorner$ $[A2; \phi(ps)$
$/p; s/q; p/r]$

---

[2] Cf. T1 in §3.

[3] In parentheses following each thesis added to the system in accordance with condition 4 of the instruction given in §11 concerning the method for constructing SS5 is the number of the thesis on which I have just made some 'substitution' or other, and an easily understood listing of these substitutions.

[4] In parentheses following each thesis added to the system in accordance with condition 2 of the instruction concerning the method for construction SS5, is the number of the thesis on which I have just carried out a 'distribution of the quantifiers', without any further addition.

[5] Cf. T2 in §3.

[6] Cf. T4 in §3.

T7. $\phi\Big(\llcorner ps \lrcorner\ \ulcorner \phi\big(\phi(ps)\phi(sp)\big)\urcorner\ \llcorner ps \lrcorner\ \ulcorner \phi\big(\phi(\phi(ps)s)p\big)\urcorner\Big)$ [T6]

T8. $\llcorner qr \lrcorner\ \ulcorner \phi\big(\phi(qr)\phi(rq)\big)\urcorner^{7}$ [T4, T5][8]

T9. $\llcorner ps \lrcorner\ \ulcorner \phi\big(\phi(ps)\phi(sp)\big)\urcorner$ $[T8; p/q; s/r]$

T10. $\llcorner qr \lrcorner\ \ulcorner \phi\Big(\phi\big(q\phi(r\phi(rq))\big)\phi\big(\phi(r\phi(rq))q\big)\Big)\urcorner$ $\Big[T8; \phi\big(r\phi(rq)\big)/r\Big]$

T11. $\phi\Big(\llcorner qr \lrcorner\ \ulcorner \phi\big(q\phi(r\phi(rq))\big)\urcorner\ \llcorner qr \lrcorner\ \ulcorner \phi\big(\phi(r\phi(rq))q\big)\urcorner\Big)$ [T10]

T12. $\llcorner pqr \lrcorner\ \ulcorner \phi\Big(\phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big)\phi\Big(\phi\big(\phi(pq)r\big)\phi\big(p\phi(qr)\big)\Big)\Big)\urcorner^{9}$ $\Big[T8; \phi\big(p\phi(qr)\big)/q; \phi\big(\phi(pq)r\big)/r\Big]$

T13. $\phi\Big(\llcorner pqr \lrcorner\ \ulcorner \phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big)\urcorner\ \llcorner pqr \lrcorner\ \ulcorner \phi\Big(\phi\big(\phi(pq)r\big)\phi\big(p\phi(qr)\big)\Big)\urcorner\Big)$ [T12]

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7 Cf. T7 in §3.

8 In parentheses following each thesis added to the system in accordance with condition 3 of the instruction concerning the method for constructing SS5, are the numbers of the two theses on the basis of which I have just made the 'detachment'.

9 Cf. T9 in §3.

T14. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\Big(\phi(pr)\phi\big(\phi(qp)\phi(qr)\big)\Big)\phi\Big(\phi\big(\phi(qp)\phi(qr)\phi(pr)\big)\Big)\urcorner$[10] $[\mathrm{T8}; \phi(pr)/q; \phi\big(\phi(qp)\phi(qr)\big)/r]$

T15. $\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\big(\phi(pr)\phi\big(\phi(qp)\phi(qr)\big)\big)\urcorner \llcorner pqr \lrcorner \ulcorner \phi\big(\phi\big(\phi(qp)\phi(qr)\big)\phi(pr)\big)\urcorner\Big)$ [T14]

T16. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big)\phi\Big(\phi(rq)\phi\big(\phi(pr)\phi(qp)\big)\Big)\Big)\urcorner$[11] $[\mathrm{T8}; \phi\big(\phi(pr)\phi(qp)\big)/q; \phi(rq)/r]$

T17. $\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\big)\urcorner \llcorner pqr \lrcorner \ulcorner \phi\big(\phi(rq)\phi\big(\phi(pr)\phi(qp)\big)\big)\urcorner\Big)$ [T16]

T18. $\llcorner r \lrcorner \ulcorner \phi(rr)\urcorner$[12] [T2, T8]

T19. $\llcorner s \lrcorner \ulcorner \phi(ss)\urcorner$ $[\mathrm{T18}; s/r]$

T20. $\llcorner ps \lrcorner \ulcorner \phi\big(\phi\big(\phi(ps)s\big)p\big)\urcorner$ [T7, T9]

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10 Cf. T11 in §3.
11 Cf. T12 in §3.
12 Cf. T19 in §3.

T21. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pq)r\big)\phi\big(p\phi(qr)\big)\Big)\urcorner$[13] [T13, A2]

T22. $\llcorner qr \lrcorner \ulcorner \phi\Bigg(\phi\bigg(\phi(qr)\phi(rq)\phi\Big(q\phi\big(r\phi(rq)\big)\Big)\bigg)\Bigg)\urcorner$
[T21; q/p; r/q; $\phi(rq)/r$]

T23. $\phi\Bigg(\llcorner qr \lrcorner \ulcorner \phi\big(\phi(qr)\phi(rq)\big)\urcorner \llcorner qr \lrcorner \ulcorner \phi\Big(q\phi\big(r\phi(rq)\big)\Big)\urcorner\Bigg)$
[T22]

T24. $\llcorner rs \lrcorner \ulcorner \phi\Bigg(\phi\bigg(\phi(rr)\phi(ss)\bigg)\phi\Big(r\phi\big(r\phi(ss)\big)\Big)\Bigg)\urcorner$
[T21; r/p; r/q; $\phi(ss)/r$]

T25. $\phi\Bigg(\llcorner rs \lrcorner \ulcorner \phi\big(\phi(rr)\phi(ss)\big)\urcorner \llcorner rs \lrcorner \ulcorner \phi\Big(r\phi\big(r\phi(ss)\big)\Big)\urcorner\Bigg)$
[T24]

T26. $\llcorner pqr \lrcorner \ulcorner \phi\Bigg(\phi\bigg(\phi(pr)\phi\Big(\phi\big(\phi(qp)\phi(qr)\big)\Big)\bigg)\phi\bigg(p\phi\Big(r\phi\big(\phi(qp)\phi$
$(qr)\big)\Big)\bigg)\Bigg)\urcorner$ [T21; r/q; $\phi\big(\phi(qp)\phi(qr)\big)/r$]

T27. $\phi\Bigg(\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pr)\phi\big(\phi(qp)\phi(qr)\big)\Big) \llcorner pqr \lrcorner \ulcorner \phi\bigg(p\phi\Big(r\phi\big(\phi$
$(qp)\phi(qr)\big)\Big)\bigg)\urcorner\Bigg)$ [T26]

[13] Cf. T13 in §3.

T28. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(qr)\Big)\phi\Big(\phi(pr)\phi\big(\phi(qp)\phi(qr)\big)\Big)\Big)\urcorner$[14] $[\text{T21}; \phi(pr)/p; \phi(qp)/q; \phi(qr)/r]$

T29. $\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(qr)\Big)\urcorner \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pr)\phi\big(\phi(qp)\phi(qr)\big)\Big)\urcorner\Big)$ [T28]

T30. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\big(\phi(qp)\phi(qr)\big)\phi(pr)\Big)\phi\Big(\phi(qp)\phi\big(\phi(qr)\phi(pr)\big)\Big)\Big)\urcorner$[15] $[\text{T21}; \phi(qp)/p; \phi(qr)/q; \phi(pr)/r]$

T31. $\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(qp)\phi(qr)\big)\phi(pr)\Big)\urcorner \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(qp)\phi\big(\phi(qr)\phi(pr)\big)\Big)\urcorner\Big)$ [T30]

T32. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(rq)\phi\big(\phi(pr)\phi(qp)\big)\Big)\urcorner$[16] [T17, A1]

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[14] Cf. T16 in §3.
[15] Cf. T17 in §3.
[16] Cf. T22 in §3.

T33. $\llcorner pqs \lrcorner \ulcorner \phi\Big(\phi\big(\phi(\phi(ps)s)p\big)\phi\Big(\phi\big(\phi(\phi(ps)q)\phi(\phi(ps)s)\big)\phi\big(p\phi(\phi(ps)q)\big)\Big)\Big)\urcorner$ $[T32; \phi(\phi(ps)q)/p; p/q; \phi(\phi(ps)s)/r]$

T34. $\phi\Big(\llcorner ps \lrcorner \ulcorner \phi\big(\phi(\phi(ps)s)p\big)\urcorner \llcorner pqs \lrcorner \ulcorner \phi\Big(\phi\big(\phi(\phi(ps)q)\phi(\phi(ps)s)\big)\phi\big(p\phi(\phi(ps)q)\big)\Big)\urcorner\Big)$ [T33]

T35. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(rq)\phi(\phi(pr)\phi(qp))\big)\phi\Big(\phi\big(\phi(qr)\phi(rq)\big)\phi\big(\phi(\phi(pr)\phi(qp))\phi(qr)\big)\Big)\Big)\urcorner$[17] $[T32; \phi(qr)/p; \phi(\phi(pr)\phi(qp))/q; \phi(rq)/r]$

T36. $\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\big(\phi(rq)\phi(\phi(pr)\phi(qp))\big)\urcorner \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(qr)\phi(rq)\big)\phi\big(\phi(\phi(pr)\phi(qp))\phi(qr)\big)\Big)\urcorner\Big)$ [T35]

17 Cf. T24 in §3.

$$\text{T37.}\quad \llcorner gpq \lrcorner \ulcorner \phi\Bigg(\phi\bigg(\llcorner f \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner$$

$$\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner \llcorner q \lrcorner \ulcorner g(qp) \urcorner\bigg)\phi$$

$$\bigg(\phi\Big(q \llcorner f \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f$$

$$(rr)g\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)\phi\big(\llcorner q \lrcorner \ulcorner g(qp) \urcorner q\big)\bigg)\Bigg)\urcorner$$

$$\Big[\text{T32}; q/p;\ \llcorner q \lrcorner \ulcorner g(qp) \urcorner /q;\ \llcorner f \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)$$

$$g(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner \Big/ r\Big]$$

$$\text{T38.}\quad \phi\bigg(\llcorner gp \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner \llcorner r \lrcorner$$

$$\ulcorner \phi\Big(f(rr)g\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner \llcorner q \lrcorner \ulcorner g(qp) \urcorner\bigg)\urcorner \llcorner gpq \lrcorner$$

$$\ulcorner\phi\Bigg(\phi\Bigg(q\llcorner f\lrcorner\ \ulcorner\phi\bigg(g(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f$$

$$(rr)g\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p)\big)\urcorner\Big)\bigg)\urcorner\Bigg)\phi\Big(\llcorner q\lrcorner\ \ulcorner g(qp)\urcorner q\Big)\Bigg)\Bigg)$$

[T37]

$$\text{T39.}\quad \llcorner pqs\lrcorner\ \ulcorner\phi\Bigg(\phi\Bigg(\llcorner f\lrcorner\ \ulcorner\phi\bigg(q\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)q\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f$$

$$(rr)\phi\big(\phi(ps)s\big)\big)\urcorner\Big)\bigg)\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\bigg(q\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)q\big)\urcorner\ \llcorner r\lrcorner$$

$$\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\bigg)\urcorner\Bigg)\phi\Bigg(\phi\Bigg(s\llcorner f\lrcorner\ \ulcorner\phi\bigg(q\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)$$

$$q\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\big(\phi(ps)s\big)\Big)\urcorner\Big)\bigg)\urcorner\Bigg)\phi\Bigg(\llcorner f\lrcorner\ \ulcorner\phi\bigg(q\phi\Big(\llcorner r\lrcorner$$

$$\ulcorner\phi\big(f(rr)q\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\bigg)\urcorner s\Bigg)\Bigg)\Bigg)\urcorner\quad\Bigg[\text{T32; } s/p;$$

$$\llcorner f\lrcorner\ \ulcorner\phi\bigg(q\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)q\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\bigg)\urcorner\Big/q;$$

$$\lfloor f \rfloor \ulcorner \phi(q\phi(\lfloor r \rfloor \ulcorner \phi(f(rr)q) \urcorner \lfloor r \rfloor \ulcorner \phi(f(rr)\phi(\phi(ps)s)) \urcorner$$

$$)) \urcorner / r]$$

T40. $$\phi(\lfloor pqs \rfloor \ulcorner \phi(\lfloor f \rfloor \ulcorner \phi(q\phi(\lfloor r \rfloor \ulcorner \phi(f(rr)q) \urcorner \lfloor r \rfloor \ulcorner \phi(f(rr)$$

$$\phi(\phi(ps)s)) \urcorner)) \urcorner \lfloor f \rfloor \ulcorner \phi(q\phi(\lfloor r \rfloor \ulcorner \phi(f(rr)q) \urcorner \lfloor r \rfloor \ulcorner \phi(f$$

$$(rr)p) \urcorner)) \urcorner) \urcorner \lfloor pqs \rfloor \ulcorner \phi(\phi(s \lfloor f \rfloor \ulcorner \phi(q\phi(\lfloor r \rfloor \ulcorner \phi(f(rr)$$

$$q) \urcorner \lfloor r \rfloor \ulcorner \phi(f(rr)\phi(\phi(ps)s)) \urcorner)) \urcorner)\phi(\lfloor f \rfloor \ulcorner \phi(q\phi(\lfloor r \rfloor$$

$$\ulcorner \phi(f(rr)q) \urcorner \lfloor r \rfloor \ulcorner \phi(f(rr)p) \urcorner)) \urcorner s)) \urcorner) \quad [T39]$$

T41. $\lfloor qr \rfloor \ulcorner \phi(q\phi(r\phi(rq))) \urcorner$ [T23, T8]

T42. $\lfloor rs \rfloor \ulcorner \phi(\phi(ss)\phi(\phi(rr)\phi(\phi(rr)\phi(ss)))) \urcorner$ [T41; $\phi(ss)$ $/q; \phi(rr)/r$]

$$\text{T43.}\quad \phi\Big(\llcorner s \lrcorner\ \ulcorner\phi(ss)\urcorner\ \llcorner rs \lrcorner\ \ulcorner\phi\Big(\phi(rr)\phi\big(\phi(rr)\phi(ss)\big)\Big)\urcorner\Big)$$

[T42]

$$\text{T44.}\quad \llcorner fgp \lrcorner\ \ulcorner\phi\Bigg(\phi\Big(\llcorner r \lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi\Big(f(rr)g\big(\phi$$

$$(p\,\llcorner q \lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\phi\Bigg(g(pp)\phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner\ \ulcorner\phi\big(f(rr)$$

$$g(pp)\big)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi\Big(f(rr)g\big(\phi(p\,\llcorner q \lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\Bigg)\Bigg)\urcorner$$

$$\Big[\text{T41};\ \phi\Big(\llcorner r \lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi\Big(f(rr)g\big(\phi(p\,\llcorner q \lrcorner$$

$$\ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big/ q;\ g(pp)/r\Big]$$

$$\text{T45.}\quad \llcorner g \lrcorner\ \ulcorner\phi\Bigg(\llcorner fp \lrcorner\ \ulcorner\phi\Big(\llcorner r \lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi\Big(f(rr)g\big(\phi$$

$$(p\,\llcorner q \lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\urcorner\ \llcorner fp \lrcorner\ \ulcorner\phi\Bigg(g(pp)\phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner\ \ulcorner\phi\big(f$$

$$(rr)g(pp)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)g\Big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\Big)\Big)\urcorner\Big)\Big)\Big)\Big)\urcorner\Big)\urcorner$$

[T44]

$$\text{T46.}\quad \llcorner p\lrcorner\ \ulcorner\phi\Bigg(\phi\Big(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi$$

$$\Big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\urcorner\Big)\Big)\urcorner\Big)\phi\Bigg(p\phi\Big(p\phi\Big(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi$$

$$\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\urcorner\Big)\Big)\urcorner$$

$$\Big)\Big)\Big)\Bigg)\urcorner\quad \Bigg[\text{T41};\phi\Big(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)$$

$$\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\urcorner\Big)\Big)\urcorner\Big)\Big/ q;p/r\Bigg]$$

$$\text{T47.}\quad \phi\Bigg(\llcorner p\lrcorner\ \ulcorner\phi\Big(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi$$

$$\Big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner\ \llcorner p\lrcorner\ \ulcorner\phi\Bigg(p\phi\Big(p\phi\Big(\llcorner q\lrcorner\ \ulcorner q\urcorner$$

$$\llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r\lrcorner \ulcorner\phi\big(f(rr)p\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)\big)\urcorner$$

$$\Big)\Big)\urcorner\Big)\Big)\Big)\Big)\urcorner\Big) \quad [T46]$$

T48. $$\llcorner pqs\lrcorner \ulcorner\phi\Big(\phi\Big(\phi\big(\phi(ps)q\big)\phi\big(\phi(ps)s\big)\Big)\phi\Big(p\phi\big(\phi(ps)q\big)\Big)\Big)\urcorner$$

[T34, T20]

T49. $$\llcorner p\lrcorner \ulcorner\phi\Big(\phi\Big(\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ulcorner\phi$$

$$\big(f(rr)\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\big)\urcorner \llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(\phi(p\llcorner q\lrcorner$$

$$\ulcorner q\urcorner)\llcorner q\lrcorner\ulcorner q\urcorner\Big)\Big)\phi\Big(p\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi$$

$$\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\Big)$$

$$\Big)\urcorner \quad \Big[T48;\ \llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner$$

$$\ulcorner q\urcorner)\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner \Big/ q;\ \llcorner q\lrcorner\ulcorner q\urcorner/s\Big]$$

T50. $$\phi\Big(\llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\Big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\llcorner f\lrcorner\ \ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi$$
$$\Big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)p\Big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(\phi(p\llcorner q\lrcorner$$
$$\ulcorner q\urcorner)\llcorner q\lrcorner\ \ulcorner q\urcorner\Big)\Big)\urcorner\ \llcorner p\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\llcorner f\lrcorner\ \ulcorner\phi\Big(\phi(p\llcorner q\lrcorner$$
$$\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)p\Big)\urcorner\Big)\Big)\urcorner$$
$$\Big)\Big)\Big)\urcorner\Big)$$ [T49]

T51. $$\llcorner pqr\lrcorner\ \ulcorner\phi\Big(\phi\Big(\phi(qr)\phi(rq)\Big)\phi\Big(\phi\Big(\phi(pr)\phi(qp)\Big)\phi(qr)\Big)\Big)\urcorner$$ [18]

[T36, T32]

T52. $$\phi\Big(\llcorner qr\lrcorner\ \ulcorner\phi\Big(\phi(qr)\phi(rq)\Big)\urcorner\ \llcorner pqr\lrcorner\ \ulcorner\phi\Big(\phi\Big(\phi(pr)\phi(qp)\Big)\phi(qr)$$
$$\Big)\urcorner\Big)$$ [T51]

T53. $$\llcorner qr\lrcorner\ \ulcorner\phi\Big(\phi\Big(r\phi(rq)\Big)q\Big)\urcorner$$ [T11, T41]

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[18] Cf. T26 in §3.

T54. $\llcorner st \lrcorner \ulcorner \phi\Big(\phi\big(t\phi(ts)\big)s\Big)\urcorner$ [T53; s/q; t/r]

T55. $\llcorner rs \lrcorner \ulcorner \phi\Big(\phi(rr)\phi\big(\phi(rr)\phi(ss)\big)\Big)\urcorner$ [T43, T19]

T56. $\phi\Big(\llcorner r \lrcorner \ulcorner\phi(rr)\urcorner \llcorner rs \lrcorner \ulcorner \phi\big(\phi(rr)\phi(ss)\big)\urcorner\Big)$ [T55]

T57. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(qr)\Big)\urcorner$[19] [T52, T8]

T58. $\llcorner rs \lrcorner \ulcorner \phi\big(\phi(rr)\phi(ss)\big)\urcorner$ [T56, T18]

T59. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pr)\phi\big(\phi(qp)\phi(qr)\big)\Big)\urcorner$[20] [T29, T57]

T60. $\llcorner fprs \lrcorner \ulcorner \phi\bigg(\phi\Big(\phi\big(\phi(ps)s\big)p\Big)\phi\Big(\phi\Big(f(rr)\phi\big(\phi(ps)s\big)\Big)\phi\big(f$
$(rr)p\big)\Big)\bigg)\urcorner$ [T59; $\phi\big(\phi(ps)s\big)/p$; $f(rr)/q$; p/r]

T61. $\phi\bigg(\llcorner ps \lrcorner \ulcorner \phi\Big(\phi\big(\phi(ps)s\big)p\Big)\urcorner \llcorner fprs \lrcorner \ulcorner \phi\Big(\phi\Big(f(rr)\phi\big(\phi(ps)$
$s\big)\Big)\phi\big(f(rr)p\big)\Big)\urcorner\bigg)$ [T60]

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[19] Cf. T27 in §3.

[20] Cf. T28 in §3.

T62. $$\llcorner pqrs \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(\phi(qp)\phi(qr))\big)\phi\big(\phi(s\phi(pr))\phi(s\phi(\phi(qp)\phi(qr)))\big)\Big)\urcorner$$ $[59; \phi(pr)/p; s/q; \phi(\phi(qp)\phi(qr))/r]$

T63. $$\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\big(\phi(pr)\phi(\phi(qp)\phi(qr))\big)\urcorner \llcorner pqrs \lrcorner \ulcorner \phi\big(\phi(s\phi(pr))\phi(s\phi(\phi(qp)\phi(qr)))\big)\urcorner\Big)$$ [T62]

T64. $$\llcorner gp \lrcorner \ulcorner \phi\Big(\phi\big(\llcorner f \lrcorner \ulcorner \phi(g(pp)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)g(pp))\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)g(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p))\urcorner))\urcorner \llcorner q \lrcorner \ulcorner g(qp)\urcorner\big)\phi\big(\phi(p \llcorner f \lrcorner \ulcorner \phi(g(pp)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)g(pp))\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)g(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner p))\urcorner))\urcorner)\phi(p \llcorner q \lrcorner \ulcorner g(qp)\urcorner)\big)\Big)\urcorner$$ $\Big[$T59; $\llcorner f \lrcorner$

$$\ulcorner\phi\Big(g(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)g\big(\phi(p\,\llcorner q\lrcorner$$

$$\ulcorner q\urcorner)p\big)\big)\urcorner\Big)\Big)\urcorner\Big/p;\,p/q;\ \llcorner q\lrcorner\ \ulcorner g(qp)\urcorner/r\Big]$$

$$\text{T65.}\quad \phi\Big(\llcorner gp\lrcorner\ \ulcorner\phi\Big(\llcorner f\lrcorner\ \ulcorner\phi\Big(g(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r\lrcorner$$

$$\ulcorner\phi\Big(f(rr)g\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\urcorner\Big)\Big)\urcorner\ \llcorner q\lrcorner\ \ulcorner g(qp)\urcorner\Big)\urcorner\ \llcorner gp\lrcorner\ \ulcorner\phi$$

$$\Big(\phi\Big(p\,\llcorner f\lrcorner\ \ulcorner\phi\Big(g(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f$$

$$(rr)g\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(p\,\llcorner q\lrcorner\ \ulcorner g(qp)\urcorner\Big)\urcorner\Big)\Big)$$

[T64]

$$\text{T66.}\quad \llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\Big(\llcorner f\lrcorner\ \ulcorner\phi\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi(p\,\llcorner q\lrcorner$$

$$\ulcorner q\urcorner)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\ \llcorner q\lrcorner\ \ulcorner q\urcorner\Big)\phi\Big(\phi\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)$$

$$\llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\big)\urcorner\llcorner r\lrcorner\ulcorner\phi$$

$$\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\phi\big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\llcorner q\lrcorner\ulcorner q\urcorner\big)\Big)\Big)\urcorner \quad \Big[\mathrm{T59};$$

$$\llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\big)\urcorner\llcorner r\lrcorner\ulcorner\phi$$

$$\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big/p;\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)/q;\ \llcorner q\lrcorner\ulcorner q\urcorner/r\Big]$$

T67. $$\phi\Big(\llcorner p\lrcorner\ulcorner\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner$$

$$\ulcorner q\urcorner)\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\llcorner q\lrcorner\ulcorner q\urcorner\Big)\urcorner\llcorner p\lrcorner\ulcorner\phi\Big(\phi\Big(\phi(p$$

$$\llcorner q\lrcorner\ulcorner q\urcorner)\llcorner f\lrcorner\ulcorner\phi\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)$$

$$\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\phi\big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\llcorner q\lrcorner\ulcorner q\urcorner\big)\Big)\Big)$$

[T66]

T68. $\llcorner rs\lrcorner\ulcorner\phi\big(r\phi\big(r\phi(ss)\big)\big)\urcorner$ [T25, T58]

T69. $\llcorner fp \lrcorner \ulcorner \phi\Big(p\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner\Big)\Big)\Big)\urcorner$ $[\mathrm{T68}; p/r; \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner/s]$

T70. $\llcorner p \lrcorner \ulcorner \phi\Big(p \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$ [T69]

T71. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(qp)\phi(qr)\big)\phi(pr)\Big)\urcorner$[21] [T15, T59]

T72. $\llcorner pqr \lrcorner \ulcorner \phi\Big(p\phi\Big(r\phi\big(\phi(qp)\phi(qr)\big)\Big)\Big)\urcorner$ [T27, T59]

T73. $\llcorner fgpr \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(g\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner p)\big)\phi\Big(\phi\big(f(rr)g(pp)\big)\phi\Big(f(rr)g\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\Big)\Big)\Big)\urcorner$ $[\mathrm{T72}; g(pp)/p; f(rr)/q; g\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)/r]$

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[21] Cf. T40 in §3.

T74. $\llcorner g \lrcorner \ulcorner \phi \left(\llcorner p \lrcorner \ulcorner g(pp) \urcorner \llcorner fpr \lrcorner \ulcorner \phi \left(g\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right) \phi \left(\phi \left(f(rr)g(pp)\right) \phi \left(f(rr) g\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\right)\urcorner \right)\urcorner$

[T73]

T75. $\llcorner fprs \lrcorner \ulcorner \phi \left(\phi \left(f(rr) \phi \left(\phi(ps)s \right) \right) \phi \left(f(rr)p \right) \right) \urcorner$

[T61, T20]

T76. $\llcorner fps \lrcorner \ulcorner \phi \left(\llcorner r \lrcorner \ulcorner \phi \left(f(rr) \phi \left(\phi(ps)s \right) \right) \urcorner \llcorner r \lrcorner \ulcorner \phi \left(f(rr)p \right) \urcorner \right) \urcorner$

[T75]

T77. $\llcorner pqrs \lrcorner \ulcorner \phi \left(\phi \left(s\phi(pr) \right) \phi \left(s\phi \left(\phi(qp)\phi(qr) \right) \right) \right) \urcorner$

[T63, T59]

T78. $\llcorner pqrs \lrcorner \ulcorner \phi \left(\phi \left(\phi(pr) \phi \left(\phi(qp)\phi(qr) \right) \right) \phi \left(\phi(pr) \phi \left(\phi \left(s\phi(qp) \right) \phi \left(s\phi(qr) \right) \right) \right) \right) \urcorner$ [T77; $\phi(qp)/p; s/q; \phi(qr)/r; \phi(pr)/s$]

T79. $\phi\left(\llcorner pqr \lrcorner \ulcorner \phi\left(\phi(pr)\phi\left(\phi(qp)\phi(qr)\right)\right)\urcorner \llcorner pqrs \lrcorner \ulcorner \phi\left(\phi(pr)\phi\left(\phi\left(s\phi(qp)\right)\phi\left(s\phi(qr)\right)\right)\right)\urcorner\right)$ [T78]

T80. $\llcorner pqr \lrcorner \ulcorner \phi\left(\phi(qp)\phi\left(\phi(qr)\phi(pr)\right)\right)\urcorner$ [22] [T31, T71]

T81. $\llcorner pqrs \lrcorner \ulcorner \phi\left(\phi(pr)\phi\left(\phi\left(s\phi(qp)\right)\phi\left(s\phi(qr)\right)\right)\right)\urcorner$ [T79, T59]

T82. $\llcorner fpqs \lrcorner \ulcorner \phi\left(\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\left(\phi(ps)s\right)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner\right)\phi\left(\phi\left(q\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)q\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\left(\phi(ps)s\right)\right)\urcorner\right)\right)\phi\left(q\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)q\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner\right)\right)\right)\right)\urcorner$ [T81; $\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\left(\phi(ps)s\right)\right)\urcorner / p$; $\llcorner r \lrcorner \ulcorner \phi\left(f(rr)q\right)\urcorner / q$; $\llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner / r$; q/s]

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22 Cf. T44 in §3.

T83. $$\phi\Big(\llcorner fps \lrcorner \ \ulcorner \phi\big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi(\phi(ps)s)\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)$$

$$\urcorner\big)\urcorner \ \llcorner fpqs \lrcorner \ \ulcorner \phi\Big(\phi\Big(q\phi\big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)q\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi$$

$$\big(\phi(ps)s\big)\big)\urcorner\big)\Big)\phi\Big(q\phi\big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)q\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner$$

$$\big)\Big)\Big)\urcorner\Big) \quad [\mathrm{T82}]$$

T84. $$\llcorner fpqs \lrcorner \ \ulcorner \phi\Big(\phi\Big(q\phi\big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)q\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi(\phi$$

$$(ps)s)\big)\urcorner\big)\Big)\phi\Big(q\phi\big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)q\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner\big)$$

$$\Big)\Big)\urcorner \quad [\mathrm{T83, T76}]$$

T85. $$\llcorner pqs \lrcorner \ulcorner \phi(\llcorner f \lrcorner \ulcorner \phi(q\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)q) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)\phi$$
$$(\phi(ps)s)) \urcorner)) \urcorner \llcorner f \lrcorner \ulcorner \phi(q\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)q) \urcorner \llcorner r \lrcorner \ulcorner \phi(f$$
$$(rr)p) \urcorner)) \urcorner) \urcorner \quad [T84]$$

T86. $$\llcorner pqs \lrcorner \ulcorner \phi(\phi(s \llcorner f \lrcorner \ulcorner \phi(q\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)q) \urcorner \llcorner r \lrcorner \ulcorner \phi(f$$
$$(rr)\phi(\phi(ps)s)) \urcorner)) \urcorner)\phi(\llcorner f \lrcorner \ulcorner \phi(q\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)$$
$$q) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)p) \urcorner)) \urcorner s)) \urcorner \quad [T40, T85]$$

T87. $$\llcorner p \lrcorner \ulcorner \phi(\phi(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)\phi$$
$$(p \llcorner q \lrcorner \ulcorner q \urcorner)) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)\phi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner q \lrcorner \ulcorner q \urcorner)) \urcorner))$$
$$\urcorner)\phi(\llcorner f \lrcorner \ulcorner \phi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)) \urcorner$$

$\llcorner r \lrcorner \ \ulcorner \phi(f(rr)p)^{\urcorner}))^{\urcorner} \ \llcorner q \lrcorner \ \ulcorner q^{\urcorner}))^{\urcorner}$ [T86; $\phi(p \llcorner q \lrcorner \ulcorner q^{\urcorner})/q;$
$\llcorner q \lrcorner \ulcorner q^{\urcorner}/s$]

T88. $\phi(\llcorner p \lrcorner \ \ulcorner \phi(\llcorner q \lrcorner \ \ulcorner q^{\urcorner} \ \llcorner f \lrcorner \ \ulcorner \phi(\phi(p \llcorner q \lrcorner \ \ulcorner q^{\urcorner} \phi(\llcorner r \lrcorner \ \ulcorner \phi(f(rr)\phi$
$(p \llcorner q \lrcorner \ \ulcorner q^{\urcorner}))^{\urcorner} \ \llcorner r \lrcorner \ \ulcorner \phi(f(rr)\phi(\phi(p \llcorner q \lrcorner \ \ulcorner q^{\urcorner}) \llcorner q \lrcorner \ \ulcorner q^{\urcorner}))^{\urcorner}))$
$^{\urcorner})^{\urcorner} \ \llcorner p \lrcorner \ \ulcorner \phi(\llcorner f \lrcorner \ \ulcorner \phi(\phi(p \llcorner q \lrcorner \ \ulcorner q^{\urcorner})\phi(\llcorner r \lrcorner \ \ulcorner \phi(f(rr)\phi(p \llcorner q \lrcorner$
$\ulcorner q^{\urcorner}))^{\urcorner} \ \llcorner r \lrcorner \ \ulcorner \phi(f(rr)p)^{\urcorner}))^{\urcorner} \ \llcorner q \lrcorner \ \ulcorner q^{\urcorner})^{\urcorner})$ [T87]

T89. $\llcorner g \lrcorner \ \ulcorner \phi(\llcorner fp \lrcorner \ \ulcorner \phi(g(pp)\phi(\llcorner r \lrcorner \ \ulcorner \phi(f(rr)g(pp))^{\urcorner} \ \llcorner r \lrcorner$
$\ulcorner \phi(f(rr)g(\phi(p \llcorner q \lrcorner \ \ulcorner q^{\urcorner})p))^{\urcorner}))^{\urcorner} \ \llcorner pq \lrcorner \ \ulcorner g(pp)^{\urcorner})$
[A3]

T90. $$\llcorner gpq \lrcorner\ \ulcorner \phi\Big(\phi\Big(q \llcorner f \lrcorner\ \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner\ \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)g\Big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(\llcorner q \lrcorner\ \ulcorner g(qp)\urcorner q\Big)\Big)\urcorner$$

[T38, A3]

T91. $$\llcorner gp \lrcorner\ \ulcorner \phi\Big(\phi\Big(p \llcorner f \lrcorner\ \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner\ \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)g\Big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(p \llcorner q \lrcorner\ \ulcorner g(qp)\urcorner\Big)\Big)\urcorner$$

[T65, A3]

D1. $$\llcorner p \lrcorner\ \ulcorner \phi\Big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\Big)\urcorner$$ [23]

T92. $$\llcorner pr \lrcorner\ \ulcorner \phi\Big(\phi\Big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\Big)\phi\Big(\phi(rr)\phi\Big(\phi(rr)\phi\Big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\Big)\Big)\Big)\Big)\urcorner$$ $\Big[$T41; $\phi\Big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\Big)\Big/q;\ \phi(rr)/r\Big]$

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[23] The sign '$\vdash$' acts here as the negation sign. Cf. thesis (d) in §4. [*tr*: thesis (d) says $[p] \therefore \sim (p) \,.\equiv\, :p \equiv .\, [q]q\, .$]

T93. $\phi\Big(\llcorner p \lrcorner\ \ulcorner \phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p)\big)\urcorner\ \llcorner pr \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\Big(\phi(rr)\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p)\big)\Big)\Big)\urcorner\Big)$ [T92]

T94. $\llcorner pr \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\Big(\phi(rr)\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p)\big)\Big)\Big)\urcorner$ [T93, D1]

T95. $\phi\Big(\llcorner r \lrcorner\ \ulcorner \phi(rr) \urcorner\ \llcorner pr \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p)\big)\Big)\urcorner\Big)$ [T94]

T96. $\llcorner pr \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p)\big)\Big)\urcorner$ [T95, T18]

D2. $\llcorner rs \lrcorner\ \ulcorner \phi\Big(\phi\big(r\phi(ss)\big)\text{-}\phi(rs)\Big)\urcorner$

T97. $\llcorner fpqr \lrcorner\ \ulcorner \phi\Big(\phi\big(q\text{-}\phi(qp)\big)\phi\Big(\phi\big(f(rr)q\big)\phi\big(f(rr)\text{-}\phi(qp)\big)\Big)\Big)\urcorner$ [T59; q/p; $f(rr)/q$; $\text{-}\phi(qp)/r$]

T98. $\phi\Big(\llcorner pq \lrcorner\ \ulcorner \phi\big(q\text{-}\phi(qp)\big)\urcorner\ \llcorner fpqr \lrcorner\ \ulcorner \phi\Big(\phi\big(f(rr)q\big)\phi\big(f(rr)\text{-}\phi(qp)\big)\Big)\urcorner\Big)$ [T97]

T99. $$\llcorner rs \lrcorner \ulcorner \phi\Big(\phi\big(\phi(r\phi(ss))\,\text{-}\varphi(rs)\big)\phi\Big(\phi\big(r\phi(r\phi(ss))\big)\phi\big(r\,\text{-}\varphi(rs)\big)\Big)\Big)\urcorner$$ [T59; $\phi(r\phi(ss))/p;\ r/q;\ \text{-}\varphi(rs)/r$]

T100. $$\phi\Big(\llcorner rs \lrcorner \ulcorner \phi\big(\phi(r\phi(ss))\,\text{-}\varphi(rs)\big)\urcorner\ \llcorner rs \lrcorner \ulcorner \phi\Big(\phi\big(r\phi(r\phi(ss))\big)\phi\big(r\,\text{-}\varphi(rs)\big)\Big)\urcorner\Big)$$ [T99]

T101. $$\llcorner pqrs \lrcorner \ulcorner \phi\Big(\phi\big(\phi(rq)\phi(\text{-}\varphi(rs)q)\big)\phi\Big(\phi\big(p\phi(rq)\big)\phi\big(p\phi(\text{-}\varphi(rs)q)\big)\Big)\Big)\urcorner$$ [T59; $\phi(rq)/p;\ p/q;\ \phi(\text{-}\varphi(rs)q)/r$]

T102. $$\phi\Big(\llcorner qrs \lrcorner \ulcorner \phi\big(\phi(rq)\phi(\text{-}\varphi(rs)q)\big)\urcorner\ \llcorner pqrs \lrcorner \ulcorner \phi\Big(\phi\big(p\phi(rq)\big)\phi\big(p\phi(\text{-}\varphi(rs)q)\big)\Big)\urcorner\Big)$$ [T101]

T103. $$\llcorner fpqr \lrcorner \ulcorner \phi\Big(\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner\ \llcorner r \lrcorner \ulcorner \phi(f(rr)\,\text{-}\varphi(qp))\urcorner\big)\phi\Big(\phi\big(r\,\llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner\big)\phi\big(r\,\llcorner r \lrcorner \ulcorner \phi(f(rr)\,\text{-}\varphi(qp))\urcorner\big)\Big)\Big)$$

$\urcorner$

$[\mathrm{T59};\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)q\Big)\urcorner\Big/p; r/q;\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\,{\text{-}}\varphi(qp)\Big)\urcorner$
$\Big/r]$

$$\mathrm{T104.}\ \phi\Bigg(\llcorner fpq \lrcorner\ \ulcorner \phi\Bigg(\llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)q\Big)\urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\,{\text{-}}\varphi(qp)\Big)\urcorner$$
$$\Bigg)\urcorner\ \llcorner fpqr \lrcorner\ \ulcorner \phi\Bigg(\phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)q\Big)\urcorner\Bigg)\phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\,{\text{-}}\varphi$$
$$(qp)\Big)\urcorner\Bigg)\Bigg)\urcorner\Bigg)\quad [\mathrm{T103}]$$

$$\mathrm{T105.}\ \llcorner fpqrs \lrcorner\ \ulcorner \phi\Bigg(\phi\Bigg(\phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)q\Big)\urcorner\Bigg)\phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)$$
$${\text{-}}\varphi(qp)\Big)\urcorner\Bigg)\Bigg)\phi\Bigg(\phi\Bigg(s\phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)q\Big)\urcorner\Bigg)\Bigg)\phi\Bigg(s\phi\Bigg(r\ \llcorner r \lrcorner$$
$$\ulcorner \phi\Big(f(rr)\,{\text{-}}\varphi(qp)\Big)\urcorner\Bigg)\Bigg)\Bigg)\Bigg)\urcorner\quad \Bigg[\mathrm{T59}; \phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)q\Big)\urcorner$$
$$\Bigg)\Big/p; s/q; \phi\Bigg(r\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\,{\text{-}}\varphi(qp)\Big)\urcorner\Bigg)\Big/r\Bigg]$$

T106. $$\phi\left(\llcorner fpqr\lrcorner\ \ulcorner\phi\left(\phi\left(r\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)q\right)\urcorner\right)\phi\left(r\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.{-}\!\varphi(qp)\right)\urcorner\right)\right)\urcorner\ \llcorner fpqrs\lrcorner\ \ulcorner\phi\left(\phi\left(s\phi\left(r\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)q\right)\urcorner\right)\right)\phi\right.$$

$$\left.\left(s\phi\left(r\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr){-}\!\varphi(qp)\right)\urcorner\right)\right)\right)\urcorner\right) \quad [\mathrm{T105}]$$

T107. $$\llcorner p\lrcorner\ \ulcorner\phi\left(\phi\left(\llcorner q\lrcorner\ \ulcorner{-}\!\varphi(qp)\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\left(p\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)p\right)\urcorner\ \llcorner r\lrcorner\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\ulcorner\phi\left(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\urcorner\right)\right)\urcorner\right)\phi\left(\phi\left(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner q\lrcorner\ \ulcorner{-}\!\varphi(qp)\urcorner\right)\right.\right.$$

$$\left.\left.\phi\left(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\left(p\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)p\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\phi(p\right.\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\urcorner\right)\right)\urcorner\right)\right)\right)\urcorner \quad \left[\mathrm{T59};\ \llcorner q\lrcorner\ \ulcorner{-}\!\varphi(qp)\urcorner/p;\ \llcorner q\lrcorner\ \ulcorner q\urcorner/\right.$$

$$q;\ \llcorner f\lrcorner\ \ulcorner\phi\left(p\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)p\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\right.\right.\right.$$

$$\left.\left.\left.\left.)\urcorner\right)\right)\urcorner\Big/ r\right]$$

T108.
$$\phi\Big(\llcorner p\lrcorner\ \ulcorner\phi\Big(\llcorner q\lrcorner\ \ulcorner\text{-}\varphi(qp)\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner$$
$$\ulcorner\phi\big(f(rr)\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner\ \llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\big(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner q\lrcorner\ \ulcorner\text{-}\varphi$$
$$(qp)\urcorner\big)\phi\Big(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\ \llcorner r\lrcorner\ \ulcorner\phi\big(f$$
$$(rr)\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\urcorner\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)$$
[T107]

T109.
$$\llcorner fp\lrcorner\ \ulcorner\phi\Big(\phi\Big(\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner$$
$$)\big)\urcorner\Big)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\text{-}\varphi(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\text{-}\varphi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)$$
$$p)\Big)\urcorner\Big)\Big)\phi\Big(\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi(p\,\llcorner q\lrcorner$$
$$\ulcorner q\urcorner)\big)\urcorner\Big)\Big)\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\text{-}\varphi(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\text{-}\varphi$$
$$\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\Big)\Big)\urcorner$$
$$\Big[\text{T59}; \phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner$$

$\llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \phi (p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)\urcorner \Big) \Big/ p; p/q; \phi \Big(\llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \text{-}\varphi$
$(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \text{-}\varphi \Big(\phi (p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner \Big) \Big/ r\Big]$

T110. $\phi \Bigg(\llcorner fp \lrcorner \ulcorner \phi \Bigg(\phi \Big(\llcorner r \lrcorner \ulcorner \phi \Big(f(rr)p\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \phi (p \llcorner q \lrcorner \ulcorner q \urcorner$
$)\Big)\urcorner \Big) \phi \Big(\llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \text{-}\varphi(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \text{-}\varphi \Big(\phi (p \llcorner q \lrcorner$
$\ulcorner q \urcorner)p\Big)\Big)\urcorner \Big)\Bigg)\urcorner \llcorner fp \lrcorner \ulcorner \phi \Bigg(\phi \Big(p\phi \Big(\llcorner r \lrcorner \ulcorner \phi \Big(f(rr)p\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi$
$\Big(f(rr) \phi (p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)\urcorner \Big)\Big) \phi \Big(p\phi \Big(\llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \text{-}\varphi(pp)\Big)\urcorner$
$\llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \text{-}\varphi \Big(\phi (p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner \Big)\Big)\Bigg)\urcorner \Bigg)$ [T109]

T111. $\llcorner qrs \lrcorner \ulcorner \phi \Big(\phi \Big(r \text{-}\varphi(rs)\Big) \phi \Big(\phi(rq) \phi \Big(\text{-}\varphi(rs)q\Big)\Big)\Big)\urcorner$ [T80; $\text{-}\varphi$ $(rs)/p; r/q; q/r$]

T112. $\phi \Big(\llcorner rs \lrcorner \ulcorner \phi \Big(r \text{-}\varphi(rs)\Big)\urcorner \llcorner qrs \lrcorner \ulcorner \phi \Big(\phi(rq) \phi \Big(\text{-}\varphi(rs)q\Big)\Big)\urcorner \Big)$
[T111]

T113. $\llcorner fpqr \lrcorner \ulcorner \phi\Big(\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\varphi(qp))\urcorner\big)\phi$

$\Big(\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner r\big)\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\varphi(qp))\urcorner r\big)\Big)\Big)\urcorner$

$[\mathrm{T}80;\ \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\varphi(qp))\urcorner / p;\ \llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner / q]$

T114. $\phi\Big(\llcorner fpq \lrcorner \ulcorner \phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\varphi(qp))\urcorner\big)\urcorner$

$\llcorner fpqr \lrcorner \ulcorner \phi\Big(\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)q)\urcorner r\big)\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\varphi(q\,.$

$p))\urcorner r\big)\Big)\urcorner\Big)$ [T113]

T115. $\llcorner p \lrcorner \ulcorner \phi\Big(\phi\Big(\llcorner f \lrcorner \ulcorner \phi\Big(p\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)p)\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)\phi$

$(p \llcorner q \lrcorner \ulcorner q \urcorner))\urcorner\big)\Big)\Big)\urcorner \llcorner f \lrcorner \ulcorner \phi\Big(-\!\varphi(pp)\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\varphi(pp))$

$\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr) -\!\varphi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p)\big)\urcorner\big)\Big)\urcorner\Big)\phi\Big(\llcorner q \lrcorner \ulcorner -\!\varphi(qp)$

$\urcorner \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr)p)\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)$

$$)^{\urcorner}))^{\urcorner}))^{\urcorner} \quad \Big[\mathrm{T90};\ -\phi/g;\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)^{\urcorner}$$

$$\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi(p\llcorner q\lrcorner\ \ulcorner q^{\urcorner})\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big/q\Big]$$

T116. $$\phi\Bigg(\llcorner p\lrcorner\ \ulcorner\phi\Bigg(\llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)^{\urcorner}\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi$$

$$(p\llcorner q\lrcorner\ \ulcorner q^{\urcorner})\big)^{\urcorner}\Big)\Big)^{\urcorner}\ \llcorner f\lrcorner\ \ulcorner\phi\Big(-\phi(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)-\phi(pp)\big)$$

$$^{\urcorner}\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)-\phi\big(\phi(p\llcorner q\lrcorner\ \ulcorner q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Bigg)^{\urcorner}\ \llcorner p\lrcorner\ \ulcorner\phi\Bigg(\llcorner q\lrcorner$$

$$\ulcorner -\phi(qp)^{\urcorner}\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)^{\urcorner}\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi(p$$

$$\llcorner q\lrcorner\ \ulcorner q^{\urcorner})\big)^{\urcorner}\Big)\Big)^{\urcorner}\Bigg)^{\urcorner}\Bigg)$$ [T115]

T117. $$\llcorner rs\lrcorner\ \ulcorner\phi\Big(\phi\Big(r\phi\big(r\phi(ss)\big)\Big)\phi\big(r-\phi(rs)\big)\Big)^{\urcorner}$$ [T100, D2]

T118. $$\phi\Big(\llcorner rs\lrcorner\ \ulcorner\phi\Big(r\phi\big(r\phi(ss)\big)\Big)^{\urcorner}\ \llcorner rs\lrcorner\ \ulcorner\phi\big(r-\phi(rs)\big)^{\urcorner}\Big)$$

[T117]

T119. $\llcorner rs \lrcorner \ulcorner \phi\left(r \neg\phi(rs)\right)\urcorner$ [T118, T68]

T120. $\llcorner pq \lrcorner \ulcorner \phi\left(q \neg\phi(qp)\right)\urcorner$ [T119; q/r; p/s]

T121. $\llcorner p \lrcorner \ulcorner \phi\left(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner q \lrcorner \ulcorner \neg\phi(qp)\urcorner\right)\urcorner$ [T120]

T122. $\llcorner qrs \lrcorner \ulcorner \phi\left(\phi(rq)\phi\left(\neg\phi(rs)q\right)\right)\urcorner$ [T112, T119]

T123. $\llcorner fpqr \lrcorner \ulcorner \phi\left(\phi\left(f(rr)q\right)\phi\left(f(rr)\neg\phi(qp)\right)\right)\urcorner$
[T98, T120]

T124. $\llcorner fpq \lrcorner \ulcorner \phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)q\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\neg\phi(qp)\right)\urcorner\right)\urcorner$
[T123]

T125. $\llcorner pqrs \lrcorner \ulcorner \phi\left(\phi\left(p\phi(rq)\right)\phi\left(p\phi\left(\neg\phi(rs)q\right)\right)\right)\urcorner$
[T102, T122]

T126. $\llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p$
$\llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)\Big)\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\neg\phi(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)$
$\neg\phi\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\Big)\urcorner\Big)\Big)\Big)\Big)\phi\Big(\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner$

$$\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner))\Big)\urcorner\Big)\Big)\phi\Big(\text{-}\varphi(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)$$

$$\text{-}\varphi(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\text{-}\varphi\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\Big)\Big)\urcorner$$

$$\Big[\mathrm{T125}; \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)p\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)$$

$$\Big)\urcorner\Big)\Big)\Big/ p; \phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\text{-}\varphi(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\text{-}\varphi\Big(\phi(p$$

$$\llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big/ q; p/r; p/s\Big]$$

T127. $$\phi\Big(\llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)p\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi(p \llcorner q \lrcorner$$

$$\ulcorner q \urcorner)\Big)\urcorner\Big)\Big)\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\text{-}\varphi(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\text{-}\varphi$$

$$\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Big)\Big)\Big)\Big)\urcorner \llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)$$

$$p\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner))\Big)\urcorner\Big)\Big)\phi\Big(\text{-}\varphi(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f$$

$$(rr)\,\text{-ϙ}(pp)\Big)^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\,\text{-ϙ}\Big(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\Big)\Big)^{\urcorner}\Big)\Big)\Big)\Big)^{\urcorner}\Big)$$

[T126]

T128. $$\llcorner fpqr\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\Big(r\,\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)q\Big)^{\urcorner}\Big)\phi\Big(r\,\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\,\text{-ϙ}\,(qp)\Big)^{\urcorner}\Big)\Bigg)^{\urcorner}$$ [T104, T124]

T129. $$\llcorner fpqr\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\Big(\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)q\Big)^{\urcorner}r\Big)\phi\Big(\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\,\text{-ϙ}(qp)\Big)^{\urcorner}r\Big)\Bigg)^{\urcorner}$$ [T114, T124]

T130. $$\llcorner fp\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\Big(\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)p\Big)^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})\Big)^{\urcorner}\Big)\phi\Big(\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\,\text{-ϙ}(pp)\Big)^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})\Big)^{\urcorner}\Big)\Bigg)^{\urcorner}$$ [T129; p/q; $\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})\Big)^{\urcorner}/r$]

T131. $$\llcorner fpqrs\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\Bigg(s\phi\Big(r\,\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)q\Big)^{\urcorner}\Big)\Bigg)\phi\Bigg(s\phi\Big(r\,\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\,\text{ϙ}(qp)\Big)^{\urcorner}\Big)\Bigg)\Bigg)^{\urcorner}$$ [T106, T128]

$$\text{T132.}\quad \llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner$$

$$))\urcorner\Big)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\neg(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)$$

$$\big)\urcorner\Big)\Big)\phi\Big(\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)$$

$$\urcorner\Big)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\neg(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\neg\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)$$

$$p\big)\Big)\urcorner\Big)\Big)\Big)\urcorner \quad \Big[\text{T131}; \phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/q; \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\neg(pp)$$

$$\big)\urcorner/r; \phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)$$

$$\Big/s\Big]$$

$$\text{T133.}\quad \phi\Big(\llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner$$

$$))\urcorner\Big)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\neg(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)$$

$$\urcorner\Big)\Big)\urcorner \llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner$$

$$\ulcorner q \urcorner))\urcorner)\phi(\llcorner r \lrcorner\ \ulcorner\phi(f(rr)-\phi(pp))\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi(f(rr)-\phi(\phi(p\,\llcorner q \lrcorner$$

$$\ulcorner q \urcorner)p))\urcorner)))\urcorner) \quad [T132]$$

T134. $$\llcorner fp \lrcorner\ \ulcorner\phi(\phi(\llcorner r \lrcorner\ \ulcorner\phi(f(rr)p)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi(f(rr)\phi(p\,\llcorner q \lrcorner\ \ulcorner q \urcorner))$$

$$\urcorner)\phi(\llcorner r \lrcorner\ \ulcorner\phi(f(rr)-\phi(pp))\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi(f(rr)-\phi(\phi(p\,\llcorner q \lrcorner\ \ulcorner q \urcorner)$$

$$p))\urcorner))\urcorner \quad [T133, T130]$$

T135. $$\llcorner fp \lrcorner\ \ulcorner\phi(\phi(p\phi(\llcorner r \lrcorner\ \ulcorner\phi(f(rr)p)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi(f(rr)\phi(p\,\llcorner q \lrcorner$$

$$\ulcorner q \urcorner))\urcorner)))\phi(p\phi(\llcorner r \lrcorner\ \ulcorner\phi(f(rr)-\phi(pp))\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi(f(rr)-\phi$$

$$(\phi(p\,\llcorner q \lrcorner\ \ulcorner q \urcorner)p))\urcorner)))\urcorner \quad [T110, T134]$$

T136. $$\llcorner fp \lrcorner \ulcorner \phi\Big(\phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)\Big)\phi\Big(-\phi(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)-\phi(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)-\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\big)\urcorner\Big)\Big)\Big)\urcorner$$ [T127, T135]

T137. $$\llcorner p \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)\Big)\urcorner \llcorner f \lrcorner \ulcorner \phi\Big(-\phi(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)-\phi(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)-\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$$ [T136]

T138. $$\llcorner p \lrcorner \ulcorner \phi\Big(\llcorner q \lrcorner \ulcorner -\phi(qp)\urcorner \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big) \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$$ [T116, T137]

T139. $$\llcorner p \lrcorner \ulcorner \phi \Bigg(\phi \Big(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner q \lrcorner \ulcorner -\phi(qp) \urcorner \phi \Bigg(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi \Big(p\phi \Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big) \urcorner \Big)\Big) \urcorner \Bigg)\Bigg) \urcorner$$

[T108, T138]

T140. $$\phi \Bigg(\llcorner p \lrcorner \ulcorner \phi \Big(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner q \lrcorner \ulcorner -\phi(qp) \urcorner \Big) \urcorner \llcorner p \lrcorner \ulcorner \phi \Bigg(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi \Bigg(p\phi \Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big) \urcorner \Big)\Bigg) \urcorner \Bigg)\Bigg)$$

[T139]

T141. $$\llcorner p \lrcorner \ulcorner \phi \Bigg(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi \Big(p\phi \Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big) \urcorner \Big)\Big) \urcorner \Bigg) \urcorner$$

[T140, T121]

T142. $$\llcorner p \lrcorner \ulcorner \phi\Big(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner q \lrcorner \ulcorner q \urcorner\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$$ [T141; $\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/p$]

T143. $$\llcorner p \lrcorner \ulcorner \phi\Big(p\phi\Big(p\phi\Big(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)\Big)\urcorner\Big)\Big)\Big)\urcorner$$ [T47, T141]

T144. $$\llcorner p \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \phi\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner \llcorner q \lrcorner \ulcorner q \urcorner\Big)\urcorner$$ [T88, T142]

T145. $$\ulcorner \llcorner p \lrcorner \; \phi\Bigg(\phi\Bigg(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)$$
$$\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)^{\urcorner} \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)^{\urcorner}\Big)\Bigg)^{\urcorner}$$
$$\Bigg)\phi\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner q \lrcorner \ulcorner q \urcorner\Big)\Bigg)^{\urcorner}$$ [T67, T144]

T146. $$\ulcorner \llcorner p \lrcorner \; \phi\Bigg(p\phi\Bigg(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi\Big(\llcorner r \lrcorner$$
$$\ulcorner \phi\big(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)^{\urcorner} \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)^{\urcorner}\Big)\Bigg)^{\urcorner}\Bigg)\Bigg)^{\urcorner}$$
[T50, T145]

D3. $$\ulcorner \llcorner pq \lrcorner \; \phi\Bigg(\phi\Bigg(p \llcorner f \lrcorner \ulcorner \phi\Bigg(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)^{\urcorner} \llcorner r \lrcorner \ulcorner \phi\big(f(rr)$$
$$q\big)^{\urcorner}\Big)\Bigg)^{\urcorner}\Bigg)\phi\!-(pq)\Bigg)^{\urcorner}$$[24]

24 In connection with the sign '$\phi\!-$' cf. §8. B. I. [*tr*: '$\phi\!-$' is '$\supset$'.]

$$\text{T147.}\quad \llcorner p \lrcorner\ulcorner\ \phi\Bigg(\phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ulcorner\ \phi\Bigg(\phi\text{-}(pp)\phi\Big(\llcorner r\lrcorner\ulcorner\ \phi\big(f$$

$$(rr)\phi\text{-}(pp)\big)\urcorner\ \llcorner r\lrcorner\ulcorner\ \phi\Big(f(rr)\phi\big(\phi\text{-}(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\Big)\urcorner\Big)\Bigg)\urcorner\Bigg)$$

$$\Bigg)\phi\Bigg(\phi\big(\phi\text{-}(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\ \llcorner f\lrcorner\ulcorner\ \phi\Bigg(\phi\text{-}(pp)\phi\Big(\llcorner r\lrcorner\ulcorner\ \phi\big(f(rr)\phi\text{-}$$

$$(pp)\big)\urcorner\ \llcorner r\lrcorner\ulcorner\ \phi\Big(f(rr)\phi\big(\phi\text{-}(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\Big)\urcorner\Big)\Bigg)\urcorner\Bigg)$$

$$\Bigg[\text{A2};\ \phi\text{-}(pp)/p;\ \llcorner q\lrcorner\ \ulcorner q\urcorner/q;\ \llcorner f\lrcorner\ulcorner\ \phi\Bigg(\phi\text{-}(pp)\phi\Big(\llcorner r\lrcorner\ulcorner\ \phi\big(f$$

$$(rr)\phi\text{-}(pp)\big)\urcorner\ \llcorner r\lrcorner\ulcorner\ \phi\Big(f(rr)\phi\big(\phi\text{-}(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\Big)\urcorner\Big)\Bigg)\urcorner\Big/$$

$$r\Bigg]$$

$$\text{T148.}\quad \phi\Bigg(\llcorner p\lrcorner\ulcorner\ \phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\llcorner q\lrcorner\ \ulcorner q\urcorner\ \llcorner f\lrcorner\ulcorner\ \phi\Bigg(\phi\text{-}(pp)\phi\Big(\llcorner r\lrcorner\ulcorner\ \phi$$

$$\big(f(rr)\phi\text{-}(pp)\big)\urcorner\ \llcorner r\lrcorner\ulcorner\ \phi\Big(f(rr)\phi\big(\phi\text{-}(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\Big)\urcorner\Big)\Bigg)\urcorner$$

$$\Big)\Big)^{\urcorner} \llcorner p \lrcorner \ \ulcorner \phi\Big(\phi\big(\phi\text{-}(pp) \ \llcorner q \lrcorner \ \ulcorner q \urcorner\big) \ \llcorner f \lrcorner \ \ulcorner \phi\Big(\phi\text{-}(pp)\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi$$

$$\big(f(rr)\phi\text{-}(pp)\big)^{\urcorner} \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi\big(\phi\text{-}(pp) \ \llcorner q \lrcorner \ \ulcorner q \urcorner\big)\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)$$

$$^{\urcorner}\Big) \quad [\mathrm{T147}]$$

$$\mathrm{T149.} \quad \llcorner pq \lrcorner \ \ulcorner \phi\Big(\phi\Big(\phi\Big(p \ \llcorner f \lrcorner \ \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)^{\urcorner} \ \llcorner r \lrcorner \ \ulcorner \phi\big(f$$

$$(rr)q\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\phi\text{-}(pq)\Big)\phi\Big(\phi\text{-}(pq)\phi\Big(p \ \llcorner f \lrcorner \ \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi$$

$$\big(f(rr)p\big)^{\urcorner} \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)q\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)\Big)\Big)^{\urcorner} \quad \Big[\mathrm{T8};\phi\Big(p \ \llcorner f \lrcorner$$

$$\ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)^{\urcorner} \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)q\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)q;\phi\text{-}(pq)$$

$$/r\Big]$$

T150. $\phi\Bigg(\llcorner pq\lrcorner\ulcorner\ \phi\Bigg(\phi\Bigg(p\llcorner f\lrcorner\ulcorner\ \phi\Big(p\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ulcorner\phi$

$\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Bigg)\phi\!-(pq)\Bigg)\urcorner\ \llcorner pq\lrcorner\ulcorner\ \phi\Bigg(\phi\!-(pq)\phi\Bigg(p\llcorner f\lrcorner\ulcorner\ \phi$

$\Big(p\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ulcorner\phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Bigg)\Bigg)\urcorner\Bigg)$

[T149]

T151. $\llcorner ps\lrcorner\ulcorner\ \phi\Bigg(\phi\Bigg(\phi\Bigg(s\llcorner f\lrcorner\ulcorner\ \phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\ \llcorner r\lrcorner\ulcorner\phi\big(f$

$(rr)p\big)\urcorner\Big)\Big)\urcorner\Bigg)\phi\Bigg(s\llcorner f\lrcorner\ulcorner\ \phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\ \llcorner r\lrcorner\ulcorner\phi$

$\Big(f(rr)\phi\!-\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner\Bigg)\Bigg)\phi\Bigg(\phi\Bigg(s\llcorner f\lrcorner\ulcorner\ \phi\Big(s\phi\Big($

$\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\ \llcorner r\lrcorner\ulcorner\phi\Big(f(rr)\phi\!-\Big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)p\Big)\Big)\urcorner\Big)\Big)\urcorner$

$\Bigg)\phi\Bigg(s\llcorner f\lrcorner\ulcorner\ \phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\ \llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)$

$$\urcorner\Big)\Big)\Big)^{\urcorner} \quad \Big[\mathrm{T8};\ \phi\Big(s\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(s\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\big(f(rr)s\big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi$$

$$\big(f(rr)p\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big/ q;\ \phi\Big(s\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(s\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\big(f(rr)s\big)^{\urcorner}$$

$$\llcorner r\lrcorner\ {}^{\ulcorner}\phi\Big(f(rr)\phi\!-\!\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big/ r\Big]$$

$$\text{T152.}\ \phi\Big(\llcorner ps\lrcorner\ {}^{\ulcorner}\phi\Big(\phi\Big(s\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(s\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\big(f(rr)s\big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\big(f$$

$$(rr)p\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\phi\Big(s\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(s\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\big(f(rr)s\big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi$$

$$\Big(f(rr)\phi\!-\!\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\ \llcorner ps\lrcorner\ {}^{\ulcorner}\phi\Big(\phi\Big(s\ \llcorner f\lrcorner$$

$$^{\ulcorner}\phi\Big(s\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\big(f(rr)s\big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\Big(f(rr)\phi\!-\!\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)$$

$$\left.\right)^{\urcorner}\left.\right)\left.\right)^{\urcorner}\left.\right)\phi\left(s\,\llcorner f\lrcorner\ ^{\ulcorner}\phi\left(s\phi\left(\llcorner r\lrcorner\ ^{\ulcorner}\phi\left(f(rr)s\right)^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi\left(f(rr)\right.\right.\right.\right.$$

$$\left.p\right)^{\urcorner}\left.\right)\left.\right)^{\urcorner}\left.\right)\left.\right)^{\urcorner}\left.\right) \qquad [\mathrm{T151}]$$

T153. $$\llcorner fpr\lrcorner\ ^{\ulcorner}\phi\left(\phi\left(p\phi\!-\!\left(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\right)\right)\phi\left(\phi\left(f(rr)p\right)\phi\left(f(rr)\right.\right.\right.$$

$$\left.\phi\!-\!\left(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\right)\right)\left.\right)\left.\right)^{\urcorner} \qquad \left[\mathrm{T59};\, f(rr)/q;\, \phi\!-\!\left(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})\right.\right.$$

$$\left.\left.p\right)/r\right]$$

T154. $$\phi\left(\llcorner p\lrcorner\ ^{\ulcorner}\phi\left(p\phi\!-\!\left(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\right)\right)^{\urcorner}\ \llcorner fpr\lrcorner\ ^{\ulcorner}\phi\left(\phi\left(f(rr)p\right)\phi\right.\right.$$

$$\left.\left(f(rr)\phi\!-\!\left(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\right)\right)\right)^{\urcorner}\left.\right) \qquad [\mathrm{T153}]$$

T155. $$\llcorner pq\lrcorner\ ^{\ulcorner}\phi\left(\phi\left(\phi\left(p\,\llcorner f\lrcorner\ ^{\ulcorner}\phi\left(p\phi\left(\llcorner r\lrcorner\ ^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi\left(f\right.\right.\right.\right.\right.\right.$$

$$\left.(rr)q\right)^{\urcorner}\left.\right)\left.\right)^{\urcorner}\left.\right)\phi\!-\!(pq)\left.\right)\phi\left(\phi\left(q\phi\left(p\,\llcorner f\lrcorner\ ^{\ulcorner}\phi\left(p\phi\left(\llcorner r\lrcorner\ ^{\ulcorner}\phi\right.\right.\right.\right.\right.$$

$$\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)q\right)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)\phi\left(q\phi\!-(pq)\right)\Big)\Big)^{\urcorner}$$

$$\left[\mathrm{T59};\phi\Big(p\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(p\phi\Big(\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f\ \llcorner rr\lrcorner\ q\right)\right.$$

$$\left.{}^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big/p;\phi\!-(pq)/r\right]$$

$$\mathrm{T156.}\ \phi\Big(\ \llcorner pq\lrcorner\ {}^{\ulcorner}\phi\Big(\phi\Big(p\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(p\phi\Big(\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\Big(f$$

$$(rr)q\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\phi\!-(pq)\Big)^{\urcorner}\ \llcorner pq\lrcorner\ {}^{\ulcorner}\phi\Big(\phi\Big(q\phi\Big(p\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(p\phi\Big($$

$$\llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)q\right)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)\phi\left(q\phi\!-(pq)\right)\Big)^{\urcorner}$$

$$\Big)\qquad [\mathrm{T155}]$$

T157. $\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\Big(q \llcorner f \lrcorner \ulcorner \phi\Big(\phi\!\!-(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi\!\!-(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(\phi(pq)\phi\Big(p \llcorner f \lrcorner \ulcorner \phi\Big(\phi\!\!-(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi\!\!-(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Big)\Big)\Big)\urcorner$ $\Big[$T59; $q/p; p/q; \llcorner f \lrcorner \ulcorner \phi\Big(\phi\!\!-(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi\!\!-(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner \Big/ r\Big]$

T158. $\phi\Big(\llcorner pq \lrcorner \ulcorner \phi\Big(q \llcorner f \lrcorner \ulcorner \phi\Big(\phi\!\!-(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi\!\!-(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner \llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\Big(p \llcorner f \lrcorner \ulcorner \phi\Big(\phi\!\!-(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi\!\!-(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)$

[T157]

$$\text{T159.}\quad \llcorner ps \lrcorner\ulcorner \phi\Bigg(\phi\Bigg(\llcorner f\lrcorner\ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)$$

$$p\big)\urcorner\Big)\Big)\urcorner\llcorner f\lrcorner\ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\Big(f(rr)\phi\!-\!\big(\phi$$

$$(p\llcorner q\lrcorner\ulcorner q\urcorner p)\big)\urcorner\Big)\Big)\urcorner\Bigg)\phi\Bigg(\phi\Big(s\llcorner f\lrcorner\ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)$$

$$s\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(s\llcorner f\lrcorner\ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f$$

$$(rr)s\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\Big(f(rr)\phi\!-\!\big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner p)\big)\urcorner\Big)\Big)\urcorner\Big)\Bigg)\Bigg)\urcorner$$

$$\Big[\text{T59};\ \llcorner f\lrcorner\ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)$$

$$\urcorner\Big/p;\ s/q;\ \llcorner f\lrcorner\ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ulcorner\phi\big(f(rr)s\big)\urcorner\llcorner r\lrcorner\ulcorner\phi\Big(f(rr)\phi\!-$$

$$\big(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner p)\big)\urcorner\Big)\Big)\urcorner\Big/r\Big]$$

T160. $\phi\Big(\llcorner ps\lrcorner \ulcorner\phi\Big(\llcorner f\lrcorner \ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner \ulcorner\phi\big(f(rr)s\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\big(f(rr)$

$p\big)\urcorner\Big)\Big)\urcorner \llcorner f\lrcorner \ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner \ulcorner\phi\big(f(rr)s\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\Big(f(rr)\phi\!-\!\big(\phi$

$(p\llcorner q\lrcorner \ulcorner q\urcorner p)\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner \llcorner ps\lrcorner \ulcorner\phi\Big(\phi\Big(s\llcorner f\lrcorner \ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner \ulcorner\phi$

$\big(f(rr)s\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(s\llcorner f\lrcorner \ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner$

$\ulcorner\phi\big(f(rr)s\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\Big(f(rr)\phi\!-\!\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\Big)\urcorner$

$\Big)$ [T159]

T161. $\llcorner p\lrcorner \ulcorner\phi\Big(\phi\!-\!\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big) \llcorner f\lrcorner \ulcorner\phi\Big(\phi\!-\!\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\phi\Big($

$\llcorner r\lrcorner \ulcorner\phi\Big(f(rr)\phi\!-\!\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner \llcorner r\lrcorner \ulcorner\phi\Big(f(rr)\phi\!-\!\big(\phi(p\llcorner q\lrcorner$

$\ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$ [T70; $\phi\!-\!\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)/p$]

T162. $$\llcorner fps \lrcorner \ \ulcorner \phi\Big(\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi\text{-}\big(\phi(p\llcorner q \lrcorner \ \ulcorner q \urcorner)p\big)\big)\urcorner\Big)\phi\Big(\phi\Big(s\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)s\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner\Big)\Big)\phi\Big(s\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)s\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi\text{-}\big(\phi(p\llcorner q \lrcorner \ \ulcorner q \urcorner)p\big)\big)\urcorner\Big)\Big)\Big)\Big)\urcorner$$

$$\Big[\text{T81; } \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner / p; \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)s\big)\urcorner / q; \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi\text{-}\big(\phi(p\llcorner q \lrcorner \ \ulcorner q \urcorner)p\big)\big)\urcorner / r\Big]$$

T163. $$\phi\Big(\llcorner fp \lrcorner \ \ulcorner \phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi\text{-}\big(\phi(p\llcorner q \lrcorner \ \ulcorner q \urcorner)p\big)\big)\urcorner\Big)\urcorner \ \llcorner fps \lrcorner \ \ulcorner \phi\Big(\phi\Big(s\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)s\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)p\big)\urcorner\Big)\Big)\phi\Big(s\phi\Big(\llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)s\big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\big(f(rr)\phi\text{-}\big(\phi(p\llcorner q \lrcorner \ \ulcorner q \urcorner)p\big)\big)\urcorner\Big)\Big)\Big)\urcorner\Big)$$ [T162]

T164. $$\llcorner p \lrcorner \ulcorner \phi\Bigg(\phi\Bigg(p \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}(pp)\Big)\urcorner \llcorner r \lrcorner$$
$$\ulcorner \phi\Big(f(rr)\phi\text{-}\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Bigg)\Bigg)\urcorner\Bigg)\phi\Big(p \llcorner q \lrcorner \ulcorner \phi\text{-}(qp)\urcorner\Big)\Bigg)\urcorner$$

[T91; $\phi\text{-}/g$]

T165. $$\phi\Bigg(\llcorner p \lrcorner \ulcorner \phi\Bigg(p \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}(pp)\Big)\urcorner \llcorner r \lrcorner$$
$$\ulcorner \phi\Big(f(rr)\phi\text{-}\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\urcorner\Bigg)\Bigg)\urcorner\Bigg)\urcorner \llcorner p \lrcorner \ulcorner \phi\Big(p \llcorner q \lrcorner \ulcorner \phi\text{-}$$
$$(qp)\urcorner\Big)\urcorner\Bigg)$$

[T164]

T166. $$\llcorner p \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Bigg(\llcorner r \lrcorner$$
$$\ulcorner \phi\Big(f(rr)\phi\text{-}(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\Big(\phi\text{-}(pp) \llcorner q \lrcorner \ulcorner q \urcorner\Big)\Big)\urcorner\Bigg)\Bigg)$$
$$\Bigg)\Bigg)\Bigg)\urcorner$$

[T143; $\phi\text{-}(pp)/p$]

T167. $\phi\left(\llcorner p \lrcorner\ \ulcorner \phi\!-(pp) \urcorner\ \llcorner p \lrcorner\ \ulcorner \phi\left(\phi\!-(pp)\phi\left(\llcorner q \lrcorner\ \ulcorner q \urcorner\ \llcorner f \lrcorner\ \ulcorner \phi\left(\phi\!-(pp)\phi\right.\right.\right.\right.$

$\left(\llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)\phi\!-(pp)\right) \urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)\phi\left(\phi\!-(pp)\ \llcorner q \lrcorner\ \ulcorner q \urcorner\right)\right)\right.$

$\left.\left.\left.\left.\urcorner\right)\right) \urcorner\right)\right) \urcorner\Big)$ [T166]

T168. $\llcorner p \lrcorner\ \ulcorner \phi\left(\phi\left(p\ \llcorner f \lrcorner\ \ulcorner \phi\left(p\phi\left(\llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)p\right) \urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)\right.\right.\right.\right.\right.$

$\left.\left.\left.\left.\left.p\right) \urcorner\right)\right) \urcorner\right)\phi\!-(pp)\right) \urcorner$ [D3; p/q]

T169. $\phi\left(\llcorner p \lrcorner\ \ulcorner \phi\left(p\ \llcorner f \lrcorner\ \ulcorner \phi\left(p\phi\left(\llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)p\right) \urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)\right.\right.\right.\right.\right.$

$\left.\left.\left.\left.p\right) \urcorner\right)\right) \urcorner\right) \urcorner\ \llcorner p \lrcorner\ \ulcorner \phi\!-(pp) \urcorner\Big)$ [T168]

T170. $$\llcorner p \lrcorner \ulcorner \phi\Big(\phi\Big(p \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)$$

$$\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\phi\!-\!\Big(p\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\Big)\urcorner$$

$$\Big[\mathrm{D3}; \phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)/q\Big]$$

T171. $$\phi\Big(\llcorner p \lrcorner \ulcorner \phi\Big(p \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)$$

$$\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner \llcorner p \lrcorner \ulcorner \phi\!-\!\Big(p\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)$$

$$\Big)\urcorner\Big)\ [\mathrm{T170}]$$

T172. $$\llcorner p \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big) \llcorner f \lrcorner \ulcorner \phi\Big(\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi$$

$$\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\!-\!\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner\Big)\Big)$$

$$^{\urcorner}\Bigg)\phi\!-\Big(\phi\!-\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\Big)p\Big)\Bigg)^{\urcorner}$$ [D3; $\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)/$

$p; p/q$]

T173. $$\phi\Bigg(\llcorner p\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\!-\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\Big)\,\llcorner f\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\!-\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\Big)\phi$$

$$\Bigg(\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\phi\!-\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\Big)\Big)^{\urcorner}\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)p\Big)^{\urcorner}\Bigg)\Bigg)$$

$$^{\urcorner}\Bigg)^{\urcorner}\llcorner p\lrcorner\ ^{\ulcorner}\phi\!-\Big(\phi\!-\Big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\Big)p\Big)^{\urcorner}\Bigg)$$ [T172]

T174. $$\llcorner pq\lrcorner\ ^{\ulcorner}\phi\Bigg(\phi\!-(pq)\phi\Bigg(p\,\llcorner f\lrcorner\ ^{\ulcorner}\phi\Bigg(p\phi\Big(\llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)p\Big)^{\urcorner}\llcorner r\lrcorner\ ^{\ulcorner}\phi$$

$$\Big(f(rr)q\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Bigg)\Bigg)^{\urcorner}$$ [T150, D3]

T175. $$\llcorner gpq \lrcorner \ulcorner \phi\Big(\phi\!-\!\big(\phi(pq)\phi\big(g(p)g(q)\big)\big)\phi\Big(\phi(pq) \llcorner f \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(pq)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\big(g(p)g(q)\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\Big)\urcorner$$ $\Big[$T174; $\phi(pq)/p$; $\phi\big(g(p)g(q)\big)/q\Big]$

T176. $$\phi\Big(\llcorner gpq \lrcorner \ulcorner \phi\!-\!\Big(\phi(pq)\phi\big(g(p)g(q)\big)\Big)\urcorner \llcorner gpq \lrcorner \ulcorner \phi\Big(\phi(pq) \llcorner f \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\phi(pq)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\big(g(p)g(q)\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner\Big)$$ [T175]

T177. $$\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\Big(q\phi\Big(p \llcorner f \lrcorner \ulcorner \phi\Big(p\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)q\big)\urcorner\Big)\Big)\urcorner\Big)\Big)\phi\big(q\phi\!-\!(pq)\big)\Big)\urcorner$$ [T156, D3]

T178. $$\llcorner p \lrcorner \ulcorner \phi\left(\phi\left(p\phi\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner f \lrcorner \ulcorner \phi\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi\left(\llcorner r \lrcorner\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\ulcorner \phi\left(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner\right)\right)\urcorner\right)\right)\phi\left(p\phi\!-\right.$$

$$\left.\left.\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\urcorner \quad [\text{T177}; \phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/p; p/q]$$

T179. $$\phi\left(\llcorner p \lrcorner \ulcorner \phi\left(p\phi\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \llcorner f \lrcorner \ulcorner \phi\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi\left(\llcorner r \lrcorner\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\ulcorner \phi\left(f(rr)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner\right)\right)\urcorner\right)\right)\urcorner \llcorner p \lrcorner \ulcorner \phi$$

$$\left.\left(p\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right) \quad [\text{T178}]$$

T180. $\llcorner p \lrcorner \ulcorner \phi\!-(pp)\urcorner$ [T169, T70]

T181. $\llcorner p \lrcorner \ulcorner \phi\!-\left(\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner$

$\left[\text{T180}; \phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)/p\right]$

T182. $\llcorner p \lrcorner \ulcorner \phi\left(p\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner$ [T179, T146]

T183. $\llcorner p \lrcorner \ulcorner \phi\left(\phi\!-(pp)\phi\left(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner f \lrcorner \ulcorner \phi\left(\phi\!-(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\right.\right.\right.\right.\right.$

$\left.\phi\!-(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\left(\phi\!-(pp) \llcorner q \lrcorner \ulcorner q \urcorner\right)\right)\urcorner\Big)\Big)\urcorner\Big)\Big)\urcorner$

[T167, T180]

T184. $\llcorner fpr \lrcorner \ulcorner \phi\left(\phi\left(f(rr)p\right)\phi\left(f(rr)\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\urcorner$

[T154, T182]

T185. $\llcorner fp \lrcorner \ulcorner \phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\right.\right.\right.$

$\left.\left.\left.p\right)\right)\urcorner\right)\urcorner$ [T184]

T186. $\llcorner p \lrcorner \ulcorner \phi\left(\phi\left(\phi\!-(pp) \llcorner q \lrcorner \ulcorner q \urcorner\right) \llcorner f \lrcorner \ulcorner \phi\left(\phi\!-(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\right.\right.\right.\right.$

$\left.\phi\!-(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\left(\phi\!-(pp) \llcorner q \lrcorner \ulcorner q \urcorner\right)\right)\urcorner\Big)\Big)\urcorner\Big)\urcorner$

[T148, T183]

T187. $$\llcorner fps \lrcorner \ulcorner \phi\Big(\phi\Big(s\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)s\big)\urcorner \llcorner r \lrcorner \ulcorner f(rr)p\big)\urcorner\Big)\Big)\Big)\phi$$

$$\Big(s\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)s\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner$$

$$\Big)\Big)\Big)\urcorner$$ [T163, T185]

T188. $$\llcorner ps \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \ulcorner \phi\Big(s\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)s\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\big(f(rr)p\big)$$

$$\urcorner\Big)\Big)\urcorner \llcorner f \lrcorner \ulcorner \phi\Big(s\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)s\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}\big(\phi(p$$

$$\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$$ [T187]

T189. $$\llcorner ps \lrcorner \ulcorner \phi\Big(\phi\Big(s \llcorner f \lrcorner \ulcorner \phi\Big(s\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)s\big)\urcorner \llcorner r \lrcorner \ulcorner \phi$$

$$\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\phi\Big(s \llcorner f \lrcorner \ulcorner \phi\Big(s\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)$$

$$s)^{\urcorner} \; \llcorner r \lrcorner \; ^{\ulcorner}\phi\Big(f(rr)\phi\!-\!\big(\phi(p \, \llcorner q \lrcorner \; ^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\bigg)\bigg)^{\urcorner}$$

[T160, T188]

T190. $$\llcorner p \lrcorner \; ^{\ulcorner}\phi\Bigg(\phi\bigg(p \, \llcorner f \lrcorner \; ^{\ulcorner}\phi\Big(p\phi\big(\llcorner r \lrcorner \; ^{\ulcorner}\phi(f(rr)p)^{\urcorner} \; \llcorner r \lrcorner \; ^{\ulcorner}\phi(f(rr)$$

$$p)^{\urcorner}\big)\Big)^{\urcorner}\bigg)\phi\bigg(p \, \llcorner f \lrcorner \; ^{\ulcorner}\phi\Big(p\phi\big(\llcorner r \lrcorner \; ^{\ulcorner}\phi(f(rr)p)^{\urcorner} \; \llcorner r \lrcorner \; ^{\ulcorner}\phi(f$$

$$(rr)\phi\!-\!(\phi(p \, \llcorner q \lrcorner \; ^{\ulcorner}q^{\urcorner})p))^{\urcorner}\big)\Big)^{\urcorner}\bigg)\Bigg)^{\urcorner}$$ [T189; p/s]

T191. $$\phi\Bigg(\llcorner p \lrcorner \; ^{\ulcorner}\phi\bigg(p \, \llcorner f \lrcorner \; ^{\ulcorner}\phi\Big(p\phi\big(\llcorner r \lrcorner \; ^{\ulcorner}\phi(f(rr)p)^{\urcorner} \; \llcorner r \lrcorner \; ^{\ulcorner}\phi(f(rr)$$

$$p)^{\urcorner}\big)\Big)^{\urcorner}\bigg)^{\urcorner} \; \llcorner p \lrcorner \; ^{\ulcorner}\phi\bigg(p \, \llcorner f \lrcorner \; ^{\ulcorner}\phi\Big(p\phi\big(\llcorner r \lrcorner \; ^{\ulcorner}\phi(f(rr)p)^{\urcorner} \; \llcorner r \lrcorner$$

$$^{\ulcorner}\phi\Big(f(rr)\phi\!-\!\big(\phi(p \, \llcorner q \lrcorner \; ^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\big)\Big)^{\urcorner}\bigg)^{\urcorner}\Bigg)$$ [T190]

T192. $\llcorner ps\lrcorner\ \ulcorner\phi\Bigg(\phi\bigg(s\ \llcorner f\lrcorner\ \ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)s\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)$

$\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\urcorner\Big)\Big)\urcorner\bigg)\phi\bigg(s\ \llcorner f\lrcorner\ \ulcorner\phi\Big(s\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f$

$(rr)s\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\bigg)\Bigg)\urcorner$

T193. $\llcorner p\lrcorner\ \ulcorner\phi\Bigg(\phi\bigg(\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\ \llcorner f\lrcorner\ \ulcorner\phi\Big(\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\phi$

$\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p$

$\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\bigg)\phi\bigg(\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\ \llcorner f\lrcorner\ \ulcorner\phi\Big(\phi\!-\big(\phi(p$

$\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f$

$(rr)p\big)\urcorner\Big)\Big)\urcorner\bigg)\Bigg)\urcorner$ $\quad$ [T192; $\phi\!-\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)/s$]

T194. $$\phi\Big(\llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\ \llcorner f\lrcorner\ \ulcorner\phi\Big(\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\,\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner\ \llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\ \llcorner f\lrcorner\ \ulcorner\phi\Big(\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner\Big)$$ [T193]

T195. $$\llcorner p\lrcorner\ \ulcorner\phi\Big(p\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$$ [T191, T70]

T196. $\llcorner p \lrcorner \ulcorner \phi\left(\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right) \llcorner f \lrcorner \ulcorner \phi\left(\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner\right)\right)\urcorner\right)\urcorner$ [T194, T161]

T197. $\llcorner p \lrcorner \ulcorner \phi\!-\!\left(p\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner$ [T171, T195]

T198. $\llcorner p \lrcorner \ulcorner \phi\!-\!\left(\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)p\right)\urcorner$ [T173, T196]

D4. $\llcorner pq \lrcorner \ulcorner \phi\left(\phi\left(q \llcorner f \lrcorner \ulcorner \phi\left(p\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)q\right)\urcorner\right)\right)\urcorner\right)\underset{1}{\phi\!-}(qp)\right)\urcorner$

T199. $\llcorner pq \lrcorner \ulcorner \phi\left(\phi\left(\phi\left(q \llcorner f \lrcorner \ulcorner \phi\left(p\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)p\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)q\right)\urcorner\right)\right)\urcorner\right)\underset{1}{\phi\!-}(qp)\right)\phi\left(\underset{1}{\phi\!-}(qp)\phi\left(q \llcorner f \lrcorner \ulcorner \phi\left(p\phi\left(\llcorner r \lrcorner \ulcorner \phi\right.\right.\right.\right.\right.$

$$\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)q\right)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)\Big)^{\urcorner}. \quad \Big[\text{T8}; \phi\Big(q\ \llcorner f\lrcorner$$

$${}^{\ulcorner}\phi\Big(p\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)q\right)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big/q; \underset{1}{\phi}$$

$$(qp)/r\Big]$$

$$\text{T200.}\ \phi\Big(\llcorner pq\lrcorner\ {}^{\ulcorner}\phi\Big(\phi\Big(q\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi\Big(p\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi$$

$$\left(f(rr)q\right)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\underset{1}{\phi}(qp)\Big)^{\urcorner}\ \llcorner pq\lrcorner\ {}^{\ulcorner}\phi\Big(\underset{1}{\phi}(qp)\phi\Big(q\ \llcorner f\lrcorner\ {}^{\ulcorner}\phi$$

$$\Big(p\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)p\right)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\left(f(rr)q\right)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)$$

[T199]

$$\text{T201.}\ \phi\Big(\llcorner p\lrcorner\ {}^{\ulcorner}\underset{1}{\phi}(pp)^{\urcorner}\ \llcorner fpr\lrcorner\ {}^{\ulcorner}\phi\Big(\underset{1}{\phi}\Big(\phi(p\ \llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\Big)\phi\Big(\phi\Big(f(rr)$$

$$\underset{1}{\phi}(pp)\Big)\phi\Big(f(rr)\underset{1}{\phi}\Big(\phi(p\ \llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\Big)\Big)\Big)\Big)^{\urcorner}\Big) \quad \Big[\text{T74}; \underset{1}{\phi}/$$

$$g\Big]$$

T202. $$\llcorner fp \lrcorner \ \ulcorner \phi \left(\phi \left(\llcorner r \lrcorner \ \ulcorner \phi \left(f(rr) \underset{1}{\phi\!-} \left(\phi\!-(pp) \phi\!-(pp) \right) \right) \urcorner \ \llcorner r \lrcorner \ \ulcorner \phi \left(f \right. \right. \right.$$

$$\left. \left. (rr) \underset{1}{\phi\!-} \left(\phi \left(\phi\!-(pp) \ \llcorner q \lrcorner \ \ulcorner q \urcorner \right) \phi\!-(pp) \right) \right) \urcorner \right) \phi \left(\underset{1}{\phi\!-} \left(\phi\!-(pp) \phi\!-(pp) \right. \right.$$

$$\left. \right) \phi \left(\underset{1}{\phi\!-} \left(\phi\!-(pp) \phi\!-(pp) \right) \phi \left(\llcorner r \lrcorner \ \ulcorner \phi \left(f(rr) \underset{1}{\phi\!-} \left(\phi\!-(pp) \phi\!-(pp) \right) \right) \right. \right.$$

$$\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi \left(f(rr) \underset{1}{\phi\!-} \left(\phi \left(\phi\!-(pp) \ \llcorner q \lrcorner \ \ulcorner q \urcorner \right) \phi\!-(pp) \right) \right) \urcorner \left. \left. \left. \left. \right) \right) \right) \right) \urcorner$$

$$\left[\text{T41}; \phi \left(\llcorner r \lrcorner \ \ulcorner \phi \left(f(rr) \underset{1}{\phi\!-} \left(\phi\!-(pp) \phi\!-(pp) \right) \right) \urcorner \ \llcorner r \lrcorner \ \ulcorner \phi \left(f(rr) \right. \right. \right.$$

$$\left. \left. \underset{1}{\phi\!-} \left(\phi \left(\phi\!-(pp) \ \llcorner q \lrcorner \ \ulcorner q \urcorner \right) \phi\!-(pp) \right) \right) \urcorner \right) \Big/ q; \underset{1}{\phi\!-} \left(\phi\!-(pp) \phi\!-(pp) \right)$$

$$\left. \Big/ r \right]$$

T203. $$\phi\Big(\llcorner fp\lrcorner \ulcorner\phi\Big(\llcorner r\lrcorner \ulcorner\phi\big(f(rr)\underset{1}{\phi-}\big(\phi-(pp)\phi-(pp)\big)\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\big(f$$

$$(rr)\underset{1}{\phi-}\big(\phi\big(\phi-(pp)\,\llcorner q\lrcorner \ulcorner q\urcorner\big)\phi-(pp)\big)\big)\urcorner\Big)\urcorner \llcorner fp\lrcorner \ulcorner\phi\Big(\underset{1}{\phi-}\big(\phi-$$

$$(pp)\phi-(pp)\big)\phi\Big(\underset{1}{\phi-}\big(\phi-(pp)\phi-(pp)\big)\phi\Big(\llcorner r\lrcorner \ulcorner\phi\big(f(rr)\underset{1}{\phi-}\big(\phi-$$

$$(pp)\phi-(pp)\big)\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\big(f(rr)\underset{1}{\phi-}\big(\phi\big(\phi-(pp)\,\llcorner q\lrcorner \ulcorner q\urcorner\big)\phi-(pp)$$

$$\big)\big)\urcorner\Big)\Big)\Big)\urcorner\Big)$$ [T202]

T204. $$\llcorner p\lrcorner \ulcorner\phi\Big(\llcorner f\lrcorner \ulcorner\phi\Big(\underset{1}{\phi-}\big(\phi-(pp)\phi-(pp)\big)\phi\Big(\llcorner r\lrcorner \ulcorner\phi\big(f(rr)\underset{1}{\phi-}\big(\phi-$$

$$(pp)\phi-(pp)\big)\big)\urcorner \llcorner r\lrcorner \ulcorner\phi\big(f(rr)\underset{1}{\phi-}\big(\phi\big(\phi-(pp)\,\llcorner q\lrcorner \ulcorner q\urcorner\big)\phi-(pp)$$

$$\big)\big)\urcorner\Big)\Big)\urcorner \llcorner q\lrcorner \ulcorner\underset{1}{\phi-}\big(q\phi-(pp)\big)\urcorner\Big)$$ [A3; $\underset{1}{\phi-}/g; \phi-(pp)/p$]

T205. $$\phi\Big(\llcorner fp\lrcorner\ \ulcorner\phi\Big(\underset{1}{\phi\!-}\big(\phi\!-(pp)\phi\!-(pp)\big)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\underset{1}{\phi\!-}\big(\phi\!-(pp)\ \phi\!-(pp)\big)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\underset{1}{\phi\!-}\Big(\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\phi\!-(pp)\Big)\Big)\urcorner\Big)\Big)\urcorner\ \llcorner pq\lrcorner\ \ulcorner\underset{1}{\phi\!-}\big(q\phi\!-(pp)\big)\urcorner\Big)$$ [T204]

T206. $$\llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\Big(p\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\ p\big)\urcorner\Big)\Big)\urcorner\Big)\underset{1}{\phi\!-}(pp)\Big)\urcorner$$ [D4; p/q]

T207. $$\phi\Big(\llcorner p\lrcorner\ \ulcorner\phi\Big(p\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\ p\big)\urcorner\Big)\Big)\urcorner\Big)\urcorner\ \llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!-}(pp)\urcorner\Big)$$ [T206]

T208. $$\llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\Big(\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\ \llcorner f\lrcorner\ \ulcorner\phi\Big(\phi\!-(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\phi\!-(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\Big)\urcorner$$

$$\Big)\Big)^{\urcorner}\Big)\underset{1}{\phi\!-}\Big(\Big(\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\phi\!-(pp)\Big)\Big)^{\urcorner}\qquad\Big[\mathrm{D4};\ \phi\!-(pp)/$$

$$p;\ \phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)/q\Big]$$

T209. $$\phi\Big(\ \llcorner p\lrcorner\ \ulcorner\phi\Big(\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\ \llcorner f\lrcorner\ \ulcorner\phi\Big(\phi\!-(pp)\phi\Big(\ \llcorner r\lrcorner\ \ulcorner\phi\big(f$$

$$(rr)\phi\!-(pp)\big)^{\urcorner}\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)^{\urcorner}$$

$$\llcorner p\lrcorner\ \underset{1}{\phi\!-}\Big(\phi\big(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\big)\phi\!-(pp)\Big)^{\urcorner}\Big)\qquad[\mathrm{T208}]$$

T210. $$\llcorner pq\lrcorner\ \ulcorner\phi\Big(\underset{1}{\phi\!-}(qp)\phi\Big(q\ \llcorner f\lrcorner\ \ulcorner\phi\Big(p\phi\Big(\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)p\big)^{\urcorner}\ \llcorner r\lrcorner\ \ulcorner\phi$$

$$\big(f(rr)q\big)^{\urcorner}\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\qquad[\mathrm{T200, D4}]$$

T211. $\llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!-}(pp)^{\urcorner}$ [T207, T70]

T212. $$\llcorner fpr\lrcorner\ \ulcorner\phi\Big(\underset{1}{\phi\!-}\big(\phi(p\ \llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\phi\Big(\phi\big(f(rr)\underset{1}{\phi\!-}(pp)\big)\phi\Big(f(rr)$$

$$\underset{1}{\phi\!-}\big(\phi(p\ \llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\Big)\Big)^{\urcorner}\qquad[\mathrm{T201, T211}]$$

T213. $$\llcorner pq\lrcorner \ulcorner\phi\left(\underset{1}{\phi\text{-}}\left(q\phi\text{-}(pp)\right)\phi\left(q\llcorner f\lrcorner \ulcorner\phi\left(\phi\text{-}(pp)\phi\left(\llcorner r\lrcorner \ulcorner\phi\left(f(rr)\phi\text{-}(pp)\right)\urcorner \llcorner r\lrcorner \ulcorner\phi\left(f(rr)q\right)\urcorner\right)\right)\urcorner\right)\right)\urcorner$$ [T210; $\phi\text{-}(pp)/p$]

T214. $$\phi\left(\llcorner pq\lrcorner \ulcorner\underset{1}{\phi\text{-}}\left(q\phi\text{-}(pp)\right)\urcorner \llcorner pq\lrcorner \ulcorner\phi\left(q\llcorner f\lrcorner \ulcorner\phi\left(\phi\text{-}(pp)\phi\left(\llcorner r\lrcorner \ulcorner\phi\left(f(rr)\phi\text{-}(pp)\right)\urcorner \llcorner r\lrcorner \ulcorner\phi\left(f(rr)q\right)\urcorner\right)\right)\urcorner\right)\urcorner\right)$$ [T213]

T215. $$\llcorner p\lrcorner \ulcorner\underset{1}{\phi\text{-}}\left(\phi\text{-}(pp)\phi\text{-}(pp)\right)\urcorner$$ [T211; $\phi\text{-}(pp)/p$]

T216. $$\llcorner fpr\lrcorner \ulcorner\phi\left(\underset{1}{\phi\text{-}}\left(\phi\left(\phi\text{-}(pp)\llcorner q\lrcorner \ulcorner q\urcorner\right)\phi\text{-}(pp)\right)\phi\left(\phi\left(f(rr)\underset{1}{\phi\text{-}}\left(\phi\text{-}(pp)\phi\text{-}(pp)\right)\right)\phi\left(f(rr)\underset{1}{\phi\text{-}}\left(\phi\left(\phi\text{-}(pp)\llcorner q\lrcorner \ulcorner q\urcorner\right)\phi\text{-}(pp)\right)\right)\right)\right)\urcorner$$ [T212; $\phi\text{-}(pp)/p$]

T217. $$\phi\left(\llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!-}\left(\phi\left(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\right)\phi\!-(pp)\right)\urcorner\ \llcorner fpr\lrcorner\ \ulcorner\phi\left(\phi\left(f(rr)\underset{1}{\phi\!-}\left(\phi\!-(pp)\phi\!-(pp)\right)\right)\phi\left(f(rr)\underset{1}{\phi\!-}\left(\phi\left(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\right)\phi\!-(pp)\right)\right)\right)\urcorner\right)$$ [T216]

T218. $$\llcorner p\lrcorner\underset{1}{\phi\!-}\left(\phi\left(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\right)\phi\!-(pp)\right)\urcorner$$ [T209, T186]

T219. $$\llcorner fpr\lrcorner\ \ulcorner\phi\left(\phi\left(f(rr)\underset{1}{\phi\!-}\left(\phi\!-(pp)\phi\!-(pp)\right)\right)\phi\left(f(rr)\underset{1}{\phi\!-}\left(\phi\left(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\right)\phi\!-(pp)\right)\right)\right)\urcorner$$ [T217, T218]

T220. $$\llcorner fp\lrcorner\ \ulcorner\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{1}{\phi\!-}\left(\phi\!-(pp)\phi\!-(pp)\right)\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{1}{\phi\!-}\left(\phi\left(\phi\!-(pp)\ \llcorner q\lrcorner\ \ulcorner q\urcorner\right)\phi\!-(pp)\right)\right)\urcorner\right)\urcorner$$ [T219]

T221. $\llcorner fp \lrcorner \ulcorner \phi \Big(\underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big) \phi \Big(\underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big) \phi \Big(\llcorner r \lrcorner \ulcorner \phi \big(f(rr) \underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \underset{1}{\phi\text{-}} \Big(\phi\big(\phi\text{-}(pp) \llcorner q \lrcorner \ulcorner q \urcorner\big)\phi\text{-}(pp)\Big)\Big)\urcorner\Big)\Big)\Big)\urcorner$ [T203, T220]

T222. $\phi\Big(\llcorner p \lrcorner \ulcorner \underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big)\urcorner \llcorner fp \lrcorner \ulcorner \phi \Big(\underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big) \phi \Big(\llcorner r \lrcorner \ulcorner \phi \big(f(rr) \underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \underset{1}{\phi\text{-}} \Big(\phi\big(\phi\text{-}(pp) \llcorner q \lrcorner \ulcorner q \urcorner\big)\phi\text{-}(pp)\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)$ [T221]

T223. $\llcorner fp \lrcorner \ulcorner \phi \Big(\underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big) \phi \Big(\llcorner r \lrcorner \ulcorner \phi \big(f(rr) \underset{1}{\phi\text{-}} \big(\phi\text{-}(pp)\phi\text{-}(pp)\big)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) \underset{1}{\phi\text{-}} \Big(\phi\big(\phi\text{-}(pp) \llcorner q \lrcorner \ulcorner q \urcorner\big)\phi\text{-}(pp)\Big)\Big)\urcorner\Big)\Big)\urcorner$ [T222, T215]

T224. $\llcorner pq \lrcorner \ulcorner \phi\text{-}\Big(q\phi\text{-}(pp)\Big)\urcorner$ [T205, T223]

T225. $\llcorner pq \lrcorner \ulcorner \phi\Bigg(q \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi$
$\Big(f(rr)q\Big)\urcorner\Big)\Bigg)\urcorner\Bigg)\urcorner$ [T214, T224]

T226. $\llcorner pq \lrcorner \ulcorner \phi\Bigg(\phi(pq)\phi\Bigg(p \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}(pp)$
$\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)q\Big)\urcorner\Big)\Bigg)\urcorner\Bigg)\Bigg)\urcorner$ [T158, T225]

T227. $\llcorner p \lrcorner \ulcorner \phi\Bigg(\phi\Big(p\phi\text{-}\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)\phi\Bigg(p \llcorner f \lrcorner \ulcorner \phi\Bigg(\phi\text{-}(pp)\phi\Big($
$\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\phi\text{-}\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)$
$\Big)\urcorner\Big)\Bigg)\urcorner\Bigg)\Bigg)\urcorner$ $\Big[$T226; $\phi\text{-}\Big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)/q\Big]$

T228. $$\phi\left(\llcorner p \lrcorner \ulcorner \phi\left(p\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner \llcorner p \lrcorner \ulcorner \phi\left(p \llcorner f \lrcorner \ulcorner \phi\left(\phi\!-\!(pp)\right.\right.\right.$$

$$\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\!-\!(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right.\right.$$

$$\left.\left.\left.\left.\left.\left.\right)\urcorner\right)\right)\right)\urcorner\right)\urcorner\right)$$ [T227]

T229. $$\llcorner p \lrcorner \ulcorner \phi\left(p \llcorner f \lrcorner \ulcorner \phi\left(\phi\!-\!(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\phi\!-\!(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\right.\right.\right.$$

$$\left.\left.\left.\left(f(rr)\phi\!-\!\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\right)\urcorner\right)\urcorner$$ [T228, T182]

T230. $\llcorner p \lrcorner \ulcorner \phi\left(p \llcorner q \lrcorner \ulcorner \phi\!-\!(qp)\urcorner\right)\urcorner$ [T165, T229]

T231. $\llcorner p \lrcorner \ulcorner \phi\left(\phi\!-\!(pp) \llcorner q \lrcorner \ulcorner \phi\!-\!\left(q\phi\!-\!(pp)\right)\urcorner\right)\urcorner$ [T230; $\phi\!-\!(pp)/p$]

T232. $\phi\left(\llcorner p \lrcorner \ulcorner \phi\!-\!(pp)\urcorner \llcorner pq \lrcorner \ulcorner \phi\!-\!\left(q\phi\!-\!(pp)\right)\urcorner\right)$ [T231]

T233. $\llcorner r \lrcorner \ulcorner \phi\left(\phi(rr) \llcorner q \lrcorner \ulcorner \phi\!-\!\left(q\phi(rr)\right)\urcorner\right)\urcorner$ [T230; $\phi(rr)/p$]

T234. $\phi\left(\llcorner r \lrcorner \ulcorner \phi(rr) \llcorner qr \lrcorner \ulcorner \phi\!-\!\left(q\phi(rr)\right)\urcorner\right)$ [T233]

T235. $\llcorner st \lrcorner \ulcorner \phi\left(\phi\left(\phi\left(t\phi(ts)\right)s\right) \llcorner q \lrcorner \ulcorner \phi\!\text{-}\left(q\phi\left(\phi\left(t\phi(ts)\right)s\right)\right)\urcorner\right)\urcorner$

$\left[\mathrm{T230}; \phi\left(\phi\left(t\phi(ts)\right)s\right)\big/p\right]$

T236. $\phi\left(\llcorner st \lrcorner \ulcorner \phi\left(\phi\left(t\phi(ts)\right)s\right)\urcorner \llcorner qst \lrcorner \ulcorner \phi\!\text{-}\left(q\phi\left(\phi\left(t\phi(ts)\right)s\right)\right)\urcorner\right)$ [T235]

T237. $\llcorner ps \lrcorner \ulcorner \phi\left(\phi\!\text{-}\left(\phi\!\text{-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!\text{-}(sp)\right) \llcorner q \lrcorner \ulcorner \phi\!\text{-}\left(q\phi\!\text{-}\left(\phi\!\text{-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!\text{-}(sp)\right)\right)\urcorner\right)\urcorner$

$\left[\mathrm{T230}; \phi\!\text{-}\left(\phi\!\text{-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!\text{-}(sp)\right)\big/p\right]$

T238. $\phi\left(\llcorner ps \lrcorner \ulcorner \phi\!\text{-}\left(\phi\!\text{-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!\text{-}(sp)\right)\urcorner \llcorner pqs \lrcorner \ulcorner \phi\!\text{-}\left(q\phi\!\text{-}\left(\phi\!\text{-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!\text{-}(sp)\right)\right)\urcorner\right)$ [T237]

T239. $\llcorner pq \lrcorner \ulcorner \phi\!\text{-}\left(q\phi\!\text{-}(pp)\right)\urcorner$ [T232, T180]

T240. $\llcorner p \lrcorner \ulcorner \phi\!\text{-}\left(p\phi\!\text{-}(pp)\right)\urcorner$ [T239; p/q]

T241. $\llcorner p \lrcorner \ulcorner \phi\!- \Big(\phi\!- \big(\phi (p \llcorner q \lrcorner \ulcorner q \urcorner) p \big) \phi\!- (pp) \Big) \urcorner$ $\Big[$T239; $\phi\!- \big(\phi (p \llcorner q \lrcorner \ulcorner q \urcorner) p \big) / q \Big]$

T242. $\llcorner p \lrcorner \ulcorner \phi\!- \Big(p \phi\!- \big(\phi (p \llcorner q \lrcorner \ulcorner q \urcorner) \phi (p \llcorner q \lrcorner \ulcorner q \urcorner) \big) \Big) \urcorner$ $\big[$T239; $\phi (p \llcorner q \lrcorner \ulcorner q \urcorner) / p ; p / q \big]$

T243. $\llcorner qr \lrcorner \ulcorner \phi\!- \big(q \phi (rr) \big) \urcorner$ [T234, T18]

T244. $\llcorner gp \lrcorner \ulcorner \phi\!- \Big(\phi (pp) \phi \big(g(p) g(p) \big) \Big) \urcorner$ $\big[$T243; $\phi (pp) / q ; g(p) / r \big]$

T245. $\llcorner qst \lrcorner \ulcorner \phi\!- \Big(q \phi \Big(\phi \big(t \phi (ts) \big) s \Big) \Big) \urcorner$ [T236, T54]

D5. $\llcorner pq \lrcorner \ulcorner \phi \big(\phi\!- (pq) -\!\phi (qp) \big) \urcorner$

T246. $\llcorner pq \lrcorner \ulcorner \phi \Big(\phi \big(\phi\!- (pq) -\!\phi (qp) \big) \phi \big(-\!\phi (qp) \phi\!- (pq) \big) \Big) \urcorner$ $\big[$T8; $\phi\!- (pq) / q ; -\!\phi (qp) / r \big]$

T247. $\phi \Big(\llcorner pq \lrcorner \ulcorner \phi \big(\phi\!- (pq) -\!\phi (qp) \big) \urcorner \llcorner pq \lrcorner \ulcorner \phi \big(-\!\phi (qp) \phi\!- (pq) \big) \urcorner \Big)$ [T246]

T248. $$\llcorner fpqst \lrcorner \ulcorner \phi\Big(\phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(pp)) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(\phi(\phi(t\phi(ts))s)q)) \urcorner\big)\phi\big(-\!\phi(pp)\phi(-\!\phi(pp)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(pp)) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(\phi(\phi(t\phi(ts))s)q)) \urcorner))\big)\Big) \urcorner$$

$$\left[\text{T41}; \phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(pp)) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(\phi(\phi(t\phi(ts))s)q)) \urcorner\big) \big/ q; -\!\phi(pp)/r\right]$$

T249. $$\phi\Big(\llcorner fpqst \lrcorner \ulcorner \phi\big(\llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(pp)) \urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr) -\!\phi(\phi(\phi(t\phi(ts))s)q)) \urcorner\big) \urcorner \llcorner fpqst \lrcorner \ulcorner \phi\big(-\!\phi(pp)\phi(-\!\phi(pp)\phi($$

$$\llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\, {\text{-}}\phi(pp)\Big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Bigg(f(rr)\, {\text{-}}\phi\Bigg(\phi\Bigg(\phi\Big(t\phi(ts)\Big)s\Bigg)$$

$$q\Bigg)\Bigg)^{\urcorner}\Bigg)\Bigg)\Bigg)\Bigg)^{\urcorner}\Bigg) \quad [\text{T248}]$$

T250. $\llcorner fpqrst \lrcorner \ulcorner \phi\Bigg({\text{-}}\phi(pp)\phi\Bigg({\text{-}}\phi\Bigg(\phi\Bigg(\phi\Big(t\phi(ts)\Big)s\Bigg)q\Bigg)\phi\Bigg(\phi$

$$\Big(f(rr)\, {\text{-}}\phi(pp)\Big)\phi\Bigg(f(rr)\, {\text{-}}\phi\Bigg(\phi\Bigg(\phi\Big(t\phi(ts)\Big)s\Bigg)q\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)$$

$$\Bigg)^{\urcorner} \quad \Bigg[\text{T72; } {\text{-}}\phi(pp)/p;\ f(rr)/q;\ {\text{-}}\phi\Bigg(\phi\Bigg(\phi\Big(t\phi(ts)\Big)s\Bigg)q\Bigg)\Big/$$

$$r\Bigg]$$

$$\text{T251.}\quad \phi\left(\llcorner p\lrcorner\ \ulcorner\text{-}\phi(pp)\urcorner\ \llcorner fpqrst\lrcorner\ \ulcorner\phi\left(\text{-}\phi\left(\phi\left(\phi\left(t\phi(ts)\right)s\right)q\right)\phi\left(\phi\right.\right.\right.$$

$$\left.\left.\left.\left(f(rr)\text{-}\phi(pp)\right)\phi\left(f(rr)\text{-}\phi\left(\phi\left(\phi\left(t\phi(ts)\right)s\right)q\right)\right)\right)\right)\urcorner\right)$$

[T250]

$$\text{T252.}\quad \llcorner p\lrcorner\ \ulcorner\phi\left(\llcorner f\lrcorner\ \ulcorner\phi\left(\text{-}\phi\left(\phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\right)\right.\right.$$

$$\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\text{-}\phi\left(\phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\right)\right)\right.$$

$$\left.\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\text{-}\phi\left(\phi\left(\phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\llcorner q\lrcorner\ \ulcorner q\urcorner\right)\phi\left(p\phi(p\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\right)\right)\urcorner\right)\right)\urcorner\ \llcorner q\lrcorner\ \ulcorner\text{-}\phi\left(q\phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\right)\urcorner\left.\right)\urcorner$$

$\left[\text{A3};\ \text{-}\phi/g;\ \phi\left(p\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)/p\right]$

$$\text{T253.}\ \phi\Bigg(\llcorner fp\lrcorner\ \ulcorner\phi\Bigg(\text{-}\phi\Big(\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)\phi\Bigg($$

$$\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\text{-}\phi\Big(\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)\Big)$$

$$\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Bigg(f(rr)\text{-}\phi\Bigg(\phi\Big(\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\llcorner q\lrcorner\ \ulcorner q\urcorner\Big)\phi\Big(p\phi(p$$

$$\llcorner q\lrcorner\ \ulcorner q\urcorner)\Big)\Bigg)\Bigg)\urcorner\Bigg)\Bigg)\urcorner\ \llcorner pq\lrcorner\ \ulcorner\text{-}\phi\Big(q\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)\urcorner\Bigg)$$

[T252]

$$\text{T254.}\ \llcorner p\lrcorner\ \ulcorner\phi\big(\phi\text{-}(pp)\text{-}\phi(pp)\big)\urcorner \quad [\text{D5}; p/q]$$

$$\text{T255.}\ \phi\big(\llcorner p\lrcorner\ \ulcorner\phi\text{-}(pp)\urcorner\ \llcorner p\lrcorner\ \ulcorner\text{-}\phi(pp)\urcorner\big) \quad [\text{T254}]$$

$$\text{T256.}\ \llcorner qst\lrcorner\ \ulcorner\phi\Bigg(\phi\text{-}\Big(q\phi\Big(\phi\big(t\phi(ts)\big)s\Big)\Big)\text{-}\phi\Big(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\Big)$$

$$\Bigg)\urcorner \quad \Big[\text{D5}; q/p; \phi\Big(\phi\big(t\phi(ts)\big)s\Big)\Big/q\Big]$$

$$\text{T257.}\ \phi\Bigg(\llcorner qst\lrcorner\ \ulcorner\phi\text{-}\Big(q\phi\Big(\phi\big(t\phi(ts)\big)s\Big)\Big)\urcorner\ \llcorner qst\lrcorner\ \ulcorner\text{-}\phi\Big(\phi\Big(\phi\big(t\phi(ts)\big)$$

$$s\Big)q\Big)\urcorner\Bigg) \quad [\text{T256}]$$

T258. $\llcorner pq \lrcorner \ulcorner \phi\Big({-}\!\phi(qp)\phi\!{-}(pq)\Big)\urcorner$ [T247, D5]

T259. $\llcorner pq \lrcorner \ulcorner \phi\Bigg({-}\!\phi\bigg(q\phi\Big(p\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)\bigg)\phi\!{-}\bigg(\phi\Big(p\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)q\bigg)\Bigg)\urcorner$ $\Big[$T258; $\phi\Big(p\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)/p\Big]$

T260. $\phi\Bigg(\llcorner pq \lrcorner \ulcorner {-}\!\phi\bigg(q\phi\Big(p\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)\bigg)\urcorner \llcorner pq \lrcorner \ulcorner \phi\!{-}\bigg(\phi\Big(p\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\Big)q\bigg)\urcorner\Bigg)$ [T259]

T261. $\llcorner p \lrcorner \ulcorner {-}\!\phi(pp)\urcorner$ [T255, T180]

T262. $\llcorner fpqrst \lrcorner \phi\Bigg({-}\!\phi\bigg(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\bigg)\phi\bigg(\phi\Big(f(rr){-}\!\phi(pp)\Big)\phi\Big(f(rr){-}\!\phi\Big(\phi\big(\phi(t\phi(ts))s\big)q\Big)\Big)\bigg)\Bigg)\urcorner$ [T251, T261]

T263. $\phi\Bigg(\llcorner qst \lrcorner \ulcorner {-}\!\phi\bigg(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\bigg)\urcorner \llcorner fpqrst \lrcorner \ulcorner \phi\bigg(\phi\Big(f(rr){-}\!\phi(pp)\Big)\phi\Big(f(rr){-}\!\phi\Big(\phi\big(\phi(t\phi(ts))s\big)q\Big)\Big)\bigg)\urcorner\Bigg)$

[T262]

T264. $\llcorner qst \lrcorner \ \ulcorner \text{-}\phi\Big(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\Big)\urcorner$ [T257, T245]

T265. $\llcorner fpqrst \lrcorner \ \ulcorner \phi\Bigg(\phi\Big(f(rr)\,\text{-}\phi(pp)\Big)\phi\Bigg(f(rr)\,\text{-}\phi\Bigg(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\Bigg)\Bigg)\Bigg)\urcorner$ [T263, T264]

T266. $\llcorner fpqst \lrcorner \ \ulcorner \phi\Bigg(\llcorner r \lrcorner \ \ulcorner \phi\Big(f(rr)\,\text{-}\phi(pp)\Big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\Bigg(f(rr)\,\text{-}\phi\Big(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\Big)\Bigg)\urcorner\Bigg)\urcorner$ [T265]

T267. $\llcorner fpqst \lrcorner \ \ulcorner \phi\Bigg(\text{-}\phi(pp)\phi\Bigg(\text{-}\phi(pp)\phi\Bigg(\llcorner r \lrcorner \ \ulcorner \phi\Big(f(rr)\,\text{-}\phi(pp)\Big)\urcorner \ \llcorner r \lrcorner \ \ulcorner \phi\Bigg(f(rr)\,\text{-}\phi\Big(\phi\Big(\phi\big(t\phi(ts)\big)s\Big)q\Big)\Bigg)\urcorner\Bigg)\Bigg)\Bigg)\urcorner$

[T249, T266]

T268. $$\phi\left(\llcorner p \lrcorner\ \ulcorner -\!\phi(pp)\urcorner\ \llcorner fpqst \lrcorner\ \ulcorner \phi\left(-\!\phi(pp)\phi\left(\llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)-\!\phi(pp)\right)\urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)-\!\phi\left(\phi\left(\phi\left(t\phi(ts)\right)s\right)q\right)\right)\urcorner\right)\right)\urcorner\right)$$

[T267]

T269. $$\llcorner fpqst \lrcorner\ \ulcorner \phi\left(-\!\phi(pp)\phi\left(\llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)-\!\phi(pp)\right)\urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\left(f(rr)-\!\phi\left(\phi\left(\phi\left(t\phi(ts)\right)s\right)q\right)\right)\urcorner\right)\right)\urcorner$$ [T268, T261]

T270. $$\llcorner fp \lrcorner\ \ulcorner \phi\left(-\!\phi\left(\phi\left(p\phi(p\llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\phi\left(p\phi(p\llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\right)\phi\left(\llcorner r \lrcorner\right.\right.$$
$$\ulcorner \phi\left(f(rr)-\!\phi\left(\phi\left(p\phi(p\llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\phi\left(p\phi(p\llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\right)\right)\urcorner\ \llcorner r \lrcorner$$
$$\ulcorner \phi\left(f(rr)-\!\phi\left(\phi\left(\phi\left(p\phi(p\llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\llcorner q \lrcorner\ \ulcorner q \urcorner\right)\phi\left(p\phi(p\llcorner q \lrcorner\ \ulcorner q \urcorner\right.\right.\right.$$

$$\Big)\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner} \quad \Big[\mathrm{T269};\ \phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)/p;\ \phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner$$

$$\big)\big)/q;\ \llcorner q\lrcorner\ \ulcorner q\urcorner/s;\ p/t\Big]$$

T271. $\llcorner pq\lrcorner\ \ulcorner{-}\phi\Big(q\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)^{\urcorner}$ [T253, T270]

T272. $\llcorner pq\lrcorner\ \ulcorner\phi{-}\Big(\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)q\Big)^{\urcorner}$ [T260, T271]

T273. $\llcorner gp\lrcorner\ \ulcorner\phi{-}\Big(\phi\big(p\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\phi\Big(g(p)g\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)\Big)^{\urcorner}$

$\Big[\mathrm{T272};\ \phi\Big(g(p)g\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)/q\Big]$

D6. $\llcorner pq\lrcorner\ \ulcorner\phi\Big(\phi{-}\big(p\phi{-}(qp)\big){-}\phi{-}(qp)\Big)^{\urcorner}$

T274. $\llcorner pq\lrcorner\ \ulcorner\phi\Big(\phi\Big(\phi{-}\big(p\phi{-}(qp)\big){-}\phi{-}(qp)\Big)\phi\Big({-}\phi{-}(qp)\phi{-}\big(p\phi{-}(qp)\big)\Big)$

$\Big)^{\urcorner}$ $\Big[\mathrm{T8};\ \phi{-}\big(p\phi{-}(qp)\big)/q;\ {-}\phi{-}(qp)/r\Big]$

T275. $\phi\Big(\llcorner pq\lrcorner\ \ulcorner\phi\Big(\phi{-}\big(p\phi{-}(qp)\big){-}\phi{-}(qp)\Big)^{\urcorner}\ \llcorner pq\lrcorner\ \ulcorner\phi\Big({-}\phi{-}(qp)\phi{-}\big(p\phi{-}$

$(qp)\big)\Big)^{\urcorner}\Big)$ [T274]

T276. $$\phi\Big({}_{\llcorner}fp_{\lrcorner}\ \ulcorner\phi\big({}_{\llcorner}r_{\lrcorner}\ \ulcorner\phi(f(rr)\mathbin{-\!\phi\!-}(pp))\urcorner\ {}_{\llcorner}r_{\lrcorner}\ \ulcorner\phi(f(rr)\mathbin{-\!\phi\!-}(\phi(p\ {}_{\llcorner}q_{\lrcorner}\ \ulcorner q\urcorner)p))\urcorner\big)\urcorner\ {}_{\llcorner}fp_{\lrcorner}\ \ulcorner\phi\big(\mathbin{-\!\phi\!-}(pp)\phi\big(\mathbin{-\!\phi\!-}(pp)\phi\big({}_{\llcorner}r_{\lrcorner}\ \ulcorner\phi(f(rr)\mathbin{-\!\phi\!-}(pp))\urcorner\ {}_{\llcorner}r_{\lrcorner}\ \ulcorner\phi(f(rr)\mathbin{-\!\phi\!-}(\phi(p\ {}_{\llcorner}q_{\lrcorner}\ \ulcorner q\urcorner)p))\urcorner\big)\big)\big)\urcorner\Big)$$ [T45; $\mathbin{-\!\phi\!-}/g$]

T277. $$\phi\Big({}_{\llcorner}p_{\lrcorner}\ \ulcorner\mathbin{-\!\phi\!-}(pp)\urcorner\ {}_{\llcorner}fpr_{\lrcorner}\ \ulcorner\phi\big(\mathbin{-\!\phi\!-}(\phi(p\ {}_{\llcorner}q_{\lrcorner}\ \ulcorner q\urcorner)p)\phi\big(\phi(f(rr)\mathbin{-\!\phi\!-}(pp))\phi(f(rr)\mathbin{-\!\phi\!-}(\phi(p\ {}_{\llcorner}q_{\lrcorner}\ \ulcorner q\urcorner)p))\big)\big)\urcorner\Big)$$ [T74; $\mathbin{-\!\phi\!-}/g$]

T278. $$\phi\Big({}_{\llcorner}fp_{\lrcorner}\ \ulcorner\phi\big(\mathbin{-\!\phi\!-}(pp)\phi\big({}_{\llcorner}r_{\lrcorner}\ \ulcorner\phi(f(rr)\mathbin{-\!\phi\!-}(pp))\urcorner\ {}_{\llcorner}r_{\lrcorner}\ \ulcorner\phi(f(rr)\mathbin{-\!\phi\!-}(\phi(p\ {}_{\llcorner}q_{\lrcorner}\ \ulcorner q\urcorner)p))\urcorner\big)\big)\urcorner\ {}_{\llcorner}pq_{\lrcorner}\ \ulcorner\mathbin{-\!\phi\!-}(pp)\urcorner\Big)$$ [T89; $\mathbin{-\!\phi\!-}/g$]

T279. $\llcorner p \lrcorner \ulcorner \phi\Big(\phi\text{-}\big(p\phi\text{-}(pp)\big)\text{-}\phi\text{-}(pp)\Big)\urcorner$ [D6; p/q]

T280. $\phi\Big(\llcorner p \lrcorner \ulcorner \phi\text{-}\big(p\phi\text{-}(pp)\big)\urcorner \llcorner p \lrcorner \ulcorner \text{-}\phi\text{-}(pp)\urcorner$ [T279]

T281. $\llcorner p \lrcorner \ulcorner \phi\Bigg(\phi\text{-}\Big(p\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\text{-}\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Bigg)\urcorner$
[D6; $\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/q$]

T282. $\phi\Bigg(\llcorner p \lrcorner \ulcorner \phi\text{-}\Big(p\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner \llcorner p \lrcorner \ulcorner \text{-}\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)$
$\urcorner\Bigg)$ [T281]

T283. $\llcorner pq \lrcorner \ulcorner \phi\Big(\text{-}\phi\text{-}(qp)\phi\text{-}\big(p\phi\text{-}(qp)\big)\Big)\urcorner$ [T275, D6]

T284. $\phi\Big(\llcorner pq \lrcorner \ulcorner \text{-}\phi\text{-}(qp)\urcorner \llcorner pq \lrcorner \ulcorner \phi\text{-}\big(p\phi\text{-}(qp)\big)\urcorner\Big)$ [T283]

T285. $\llcorner p \lrcorner \ulcorner \text{-}\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\urcorner$ [T282, T197]

T286. $\llcorner p \lrcorner \ulcorner \text{-}\phi\text{-}(pp)\urcorner$ [T280, T240]

T287. $\llcorner fpr \lrcorner \ulcorner \phi\Bigg(\text{-}\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\Bigg(\phi\big(f(rr)\text{-}\phi\text{-}(pp)\big)\phi\Bigg(f(rr)$
$\text{-}\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Bigg)\Bigg)\Bigg)\urcorner$ [T277, T286]

T288. $$\varphi\left(\llcorner p \lrcorner \ulcorner -\!\varphi\!-\left(\varphi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\urcorner \llcorner fpr \lrcorner \ulcorner \varphi\left(\varphi\left(f(rr) -\!\varphi\!- (pp)\right)\varphi\left(f(rr) -\!\varphi\!- \left(\varphi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\urcorner\right)$$ [T287]

T289. $$\llcorner fpr \lrcorner \ulcorner \varphi\left(\varphi\left(f(rr) -\!\varphi\!- (pp)\right)\varphi\left(f(rr) -\!\varphi\!- \left(\varphi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\urcorner$$ [T288, T285]

T290. $$\llcorner fp \lrcorner \ulcorner \varphi\left(\llcorner r \lrcorner \ulcorner \varphi\left(f(rr) -\!\varphi\!- (pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \varphi\left(f(rr) -\!\varphi\!- \left(\varphi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\urcorner$$ [T289]

T291. $$\llcorner fp \lrcorner \ulcorner \varphi\left(-\!\varphi\!-(pp)\varphi\left(-\!\varphi\!-(pp)\varphi\left(\llcorner r \lrcorner \ulcorner \varphi\left(f(rr) -\!\varphi\!- (pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \varphi\left(f(rr) -\!\varphi\!- \left(\varphi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\right)\right)\urcorner$$ [T276, T290]

T292. $\phi\left(\llcorner p \lrcorner\ \ulcorner\text{-}\phi\text{-}(pp)\urcorner\ \llcorner fp \lrcorner\ \ulcorner\phi\left(\text{-}\phi\text{-}(pp)\phi\left(\llcorner r \lrcorner\ \ulcorner\phi\left(f(rr)\text{-}\phi\text{-}(pp)\right)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi\left(f(rr)\text{-}\phi\text{-}\left(\phi(p\,\llcorner q \lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\right)\urcorner\right)$
[T291]

T293. $\llcorner fp \lrcorner\ \ulcorner\phi\left(\text{-}\phi\text{-}(pp)\phi\left(\llcorner r \lrcorner\ \ulcorner\phi\left(f(rr)\text{-}\phi\text{-}(pp)\right)\urcorner\ \llcorner r \lrcorner\ \ulcorner\phi\left(f(rr)\text{-}\phi\text{-}\left(\phi(p\,\llcorner q \lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\right)\urcorner$ [T292, 286]

T294. $\llcorner pq \lrcorner\ \ulcorner\text{-}\phi\text{-}(qp)\urcorner$ [T278, T293]

T295. $\llcorner pq \lrcorner\ \ulcorner\phi\text{-}\left(p\phi\text{-}(qp)\right)\urcorner$[25] [T284, T294]

T296. $\llcorner ps \lrcorner\ \ulcorner\phi\text{-}\left(\phi\text{-}(sp)\phi\text{-}\left(\phi\text{-}(pp)\phi\text{-}(sp)\right)\right)\urcorner$ [T295; $\phi\text{-}(sp)/p$; $\text{-}\phi\text{-}(pp)/q$]

D7. $\llcorner pq \lrcorner\ \ulcorner\phi\left(\phi\text{-}\left(\phi\text{-}(qp)p\right)\phi\text{-}\{p\}(q)\right)\urcorner$[26]

T297. $\llcorner pq \lrcorner\ \ulcorner\phi\left(\phi\left(\phi\text{-}\left(\phi\text{-}(qp)p\right)\text{-}\varphi\text{-}\{p\}(q)\phi\left(\text{-}\varphi\text{-}\{p\}(q)\phi\text{-}\left(\phi\text{-}(qp)p\right)\right)\right)\right)\urcorner$ [T8; $\phi\text{-}\left(\phi\text{-}(qp)p\right)/q$; $\text{-}\varphi\text{-}\{p\}(q)/r$]

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25 Cf. *loc. cit.*

26 Cf. above the footnote to TE XVIII in §11. Cf. also Schönfinkel [1], pp. 307–315, Sobociński [1], p. 159 [17], and Leśniewski [2], p. 44 [44].

T298. $\phi\left(\llcorner pq\lrcorner\ulcorner\phi\left(\phi\text{-}\left(\phi\text{-}(qp)p\right)\text{-}\phi\text{-}\{p\}(q)\right)\urcorner\llcorner pq\lrcorner\ulcorner\phi\left(\text{-}\phi\text{-}\{p\}(q)\phi\text{-}\left(\phi\text{-}(qp)p\right)\right)\urcorner\right)$ [T297]

T299. $\llcorner p\lrcorner\ulcorner\phi\left(\phi\text{-}\left(\phi\text{-}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)p\right)p\right)\text{-}\phi\text{-}\{p\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)\right)\urcorner$ [D7; $\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)/q$]

T300. $\phi\left(\llcorner p\lrcorner\phi\text{-}\left(\phi\text{-}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)p\right)p\right)\urcorner\llcorner p\lrcorner\ulcorner\text{-}\phi\text{-}\{p\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)\urcorner\right)$ [T299]

T301. $\llcorner pq\lrcorner\ulcorner\phi\left(\text{-}\phi\text{-}\{p\}(q)\phi\text{-}\left(\phi\text{-}(qp)p\right)\right)\urcorner$ [T298, D7]

T302. $\llcorner p\lrcorner\ulcorner\phi\left(\text{-}\phi\text{-}\{p\}\left(\vdash(p)\right)\phi\text{-}\left(\phi\text{-}\left(\vdash(p)p\right)p\right)\right)\urcorner$ [T301; $\vdash(p)/q$]

T303. $\phi\left(\llcorner p\lrcorner\ulcorner\text{-}\phi\text{-}\{p\}\left(\vdash(p)\right)\urcorner\llcorner p\lrcorner\ulcorner\phi\text{-}\left(\phi\text{-}\left(\vdash(p)p\right)p\right)\urcorner\right)$ [T302]

T304. $\llcorner p\lrcorner\ulcorner\text{-}\phi\text{-}\{p\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)\urcorner$ [T300, T198]

D8. $\llcorner pqr\lrcorner\ulcorner\phi\left(\phi\text{-}\left(p\phi\text{-}(rq)\right)fa\{pq\}(r)\right)\urcorner$

T305. $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\Big(\phi\text{-}\big(p\phi\text{-}(rq)\big)fa\{pq\}(r)\Big)\phi\Big(fa\{pq\}(r)\phi\text{-}\big(p\phi\text{-}(rq)\big)\Big)\Big)\urcorner$ $\Big[T8; \phi\text{-}\big(p\phi\text{-}(rq)\big)/q; fa\{pq\}(r)/r\Big]$

T306. $\phi\Big(\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\text{-}\big(p\phi\text{-}(rq)\big)fa\{pq\}(r)\Big)\urcorner \llcorner pqr \lrcorner \ulcorner \phi\Big(fa\{pq\}(r)\phi\text{-}\big(p\phi\text{-}(rq)\big)\Big)\urcorner\Big)$ [T305]

T307. $\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\text{-}\Big(p\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)q\big)\Big)fa\{pq\}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\Big)\urcorner$ $[D8; \phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/r]$

T308. $\phi\Big(\llcorner pq \lrcorner \ulcorner \phi\text{-}\Big(p\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)q\big)\Big)\urcorner \llcorner pq \lrcorner \ulcorner fa\{pq\}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big)\urcorner\Big)$ [T307]

T309. $\llcorner pqr \lrcorner \ulcorner \phi\Big(fa\{pq\}(r)\phi\text{-}\big(p\phi\text{-}(rq)\big)\Big)\urcorner$ [T306, D8]

T310. $\llcorner pq \lrcorner \ulcorner \phi\Big(fa\{pq\}\big(\vdash (p)\big)\phi\text{-}\Big(p\phi\text{-}\big(\vdash (p)q\big)\Big)\Big)\urcorner$ $[T309; \vdash (p)/r]$

T311. $\phi\Big(\llcorner pq \lrcorner \ulcorner fa\{pq\}\big(\vdash (p)\big)\urcorner \llcorner pq \lrcorner \ulcorner \phi\text{-}\Big(p\phi\text{-}\big(\vdash (p)q\big)\Big)\urcorner\Big)$ [T310]

D9. $\llcorner pq \lrcorner \ulcorner \phi\Big(\text{-}\phi\text{-}\Big(p\text{-}\phi\text{-}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)q\big)\Big)\underset{1}{\text{-}\phi\text{-}}(qp)\Big)\urcorner$

T312. $\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\Big(\text{-}\phi\text{-}\Big(p\text{-}\phi\text{-}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)q\big)\Big)\underset{1}{\text{-}\phi\text{-}}(qp)\Big)\phi\Big(\underset{1}{\text{-}\phi\text{-}}(qp)\text{-}\phi\text{-}$
$\Big(p\text{-}\phi\text{-}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)q\big)\Big)\Big)\Big)\urcorner$ $\Big[$T8; $\text{-}\phi\text{-}\Big(p\text{-}\phi\text{-}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)q\big)$
$\Big)\Big/q;\ \underset{1}{\text{-}\phi\text{-}}(qp)/r\Big]$

T313. $\phi\Big(\llcorner pq \lrcorner \ulcorner \phi\Big(\text{-}\phi\text{-}\Big(p\text{-}\phi\text{-}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)q\big)\Big)\underset{1}{\text{-}\phi\text{-}}(qp)\Big)\urcorner\ \llcorner pq \lrcorner \ulcorner \phi$
$\Big(\underset{1}{\text{-}\phi\text{-}}(qp)\text{-}\phi\text{-}\Big(p\text{-}\phi\text{-}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)q\big)\Big)\Big)\urcorner\Big)$ [T312]

T314. $\phi\Big(\llcorner fp \lrcorner \ulcorner \phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\underset{1}{\text{-}\phi\text{-}}(pp)\big)\urcorner\ \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{1}{\text{-}\phi\text{-}}\big(\phi(p$
$\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\urcorner\ \llcorner fp \lrcorner \ulcorner \phi\Big(\underset{1}{\text{-}\phi\text{-}}(pp)\phi\Big(\underset{1}{\text{-}\phi\text{-}}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f$
$(rr)\underset{1}{\text{-}\phi\text{-}}(pp)\big)\urcorner\ \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{1}{\text{-}\phi\text{-}}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\Big)\urcorner$
$\Big)$ [T45; $\underset{1}{\text{-}\phi\text{-}}/g$]

T315. $\phi\left(\llcorner p \lrcorner \ulcorner \underset{1}{\phi}(pp) \urcorner \llcorner fpr \lrcorner \ulcorner \phi\left(\underset{1}{\phi}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\left(\phi\left(f(rr)\underset{1}{\phi}(pp)\right)\phi\left(f(rr)\underset{1}{\phi}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\right)\urcorner\right)$ [T74; $\underset{1}{\phi}$/g]

T316. $\phi\left(\llcorner fp \lrcorner \ulcorner \phi\left(\underset{1}{\phi}(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{1}{\phi}(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{1}{\phi}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\right)\urcorner \llcorner pq \lrcorner \ulcorner \underset{1}{\phi}(qp) \urcorner\right).$

[T89; $\underset{1}{\phi}$/g]

T317. $\llcorner p \lrcorner \ulcorner \phi\left(\phi\!-\left(p\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\underset{1}{\phi}(pp)\right)\urcorner$ [D9; p/q]

T318. $\phi\left(\llcorner p \lrcorner \ulcorner \phi\!-\left(p\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner \llcorner p \lrcorner \ulcorner \underset{1}{\phi}(pp) \urcorner\right)$

[T317]

T319. $\llcorner p \lrcorner \ulcorner \phi\left(\phi\!-\left(p\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\right)\right)\underset{1}{\phi}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner$ [D9; $\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/q$]

T320. $\phi\Big(\llcorner p\lrcorner\ \ulcorner\phi\!\!-\Big(p\phi\!\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)\Big)\urcorner\ \llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!\!-}\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\urcorner\Big)$ [T319]

T321. $\llcorner pq\lrcorner\ \ulcorner\phi\Big(\underset{1}{\phi\!\!-}(qp)\phi\!\!-\Big(p\phi\!\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)q\big)\Big)\Big)\urcorner$ [T313, D9]

T322. $\phi\Big(\llcorner pq\lrcorner\ \ulcorner\underset{1}{\phi\!\!-}(qp)\urcorner\ \llcorner pq\lrcorner\ \ulcorner\phi\!\!-\Big(p\phi\!\!-\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)q\big)\Big)\urcorner\Big)$ [T321]

T323. $\llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!\!-}(pp)\urcorner$ [T318, T197]

T324. $\llcorner fpr\lrcorner\ \ulcorner\phi\Big(\underset{1}{\phi\!\!-}\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\phi\Big(\phi\big(f(rr)\underset{1}{\phi\!\!-}(pp)\big)\phi\Big(f(rr)\underset{1}{\phi\!\!-}\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\Big)\Big)\urcorner$ [T315, T323]

T325. $\phi\Big(\llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!\!-}\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\urcorner\ \llcorner fpr\lrcorner\ \ulcorner\phi\Big(\phi\big(f(rr)\underset{1}{\phi\!\!-}(pp)\big)\phi\Big(f(rr)\underset{1}{\phi\!\!-}\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\Big)\Big)\urcorner\Big)$ [T324]

T326. $\llcorner p\lrcorner\ \ulcorner\underset{1}{\phi\!\!-}\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\urcorner$ [T320, T242]

T327. $$\llcorner fpr \lrcorner \ulcorner \phi\Big(\phi\big(f(rr)\underset{1}{\phi}(pp)\big)\phi\Big(f(rr)\underset{1}{\phi}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\Big)\urcorner$$ [T325, T326]

T328. $$\llcorner fp \lrcorner \ulcorner \phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\underset{1}{\phi}(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{1}{\phi}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\urcorner$$ [T327]

T329. $$\llcorner fp \lrcorner \ulcorner \phi\Bigg(\underset{1}{\phi}(pp)\phi\Bigg(\underset{1}{\phi}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\underset{1}{\phi}(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{1}{\phi}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Bigg)\Bigg)\urcorner$$ [T314, T328]

T330. $$\phi\Bigg(\llcorner p \lrcorner \ulcorner \underset{1}{\phi}(pp)\llcorner fp \lrcorner \ulcorner \phi\Bigg(\underset{1}{\phi}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\underset{1}{\phi}(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{1}{\phi}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Bigg)\urcorner\Bigg)$$ [T329]

T331. $$\llcorner fp \lrcorner \ulcorner \phi\Big(\underset{1}{\phi}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\underset{1}{\phi}(pp)\big)\urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{1}{\phi}\big(\phi(p\llcorner q\lrcorner \ulcorner q\urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner$$ [T330, T323]

T332. $\llcorner pq \lrcorner \ulcorner \underset{1}{\phi\text{-}}(qp) \urcorner$ [T316, T331]

T333. $\llcorner pq \lrcorner \ulcorner \phi\text{-}\Big(p\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)q\big)\Big) \urcorner$ [T322, T332]

T334. $\llcorner pq \lrcorner \ulcorner fa\{pq\}(r)\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\big) \urcorner$ [T308, T333]

D10. $\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\text{-}(qp)\Big)\underset{2}{\phi\text{-}}(qp)\Big) \urcorner$

T335. $\llcorner pq \lrcorner \ulcorner \phi\Bigg(\phi\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\text{-}(qp)\Big)\underset{2}{\phi\text{-}}(qp)\Big)\phi\Big(\underset{2}{\phi\text{-}}$

$(qp)\phi\text{-}\Big(\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\text{-}(qp)\Big)\Big)\Bigg) \urcorner$ $\Big[$T8; $\phi\text{-}\Big(\phi\text{-}\big(\phi(p$

$\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\text{-}(qp)\Big)\Big/ q;\ \underset{2}{\phi\text{-}}(qp)/r\Big]$

T336. $\phi\Bigg(\llcorner pq \lrcorner \ulcorner \phi\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\text{-}(qp)\Big)\underset{2}{\phi\text{-}}(qp)\Big) \urcorner \llcorner pq \lrcorner$

$\ulcorner \phi\Big(\underset{2}{\phi\text{-}}(qp)\phi\text{-}\Big(\phi\text{-}\big(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)\phi\text{-}(qp)\Big)\Big) \urcorner \Bigg)$ [T335]

T337. $\phi\Bigg(\llcorner fp \lrcorner \ulcorner \phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)\underset{2}{\phi\text{-}}(pp)\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)\underset{2}{\phi\text{-}}\big(\phi(p$

$\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big) \urcorner \Big) \urcorner \llcorner fp \lrcorner \ulcorner \phi\Bigg(\underset{2}{\phi\text{-}}(pp)\phi\Bigg(\underset{2}{\phi\text{-}}(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f$

$$(rr)\underset{2}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi\Big(f(rr)\underset{2}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)\Big)^{\urcorner}$$

$$\Bigg)\quad [\text{T45}; \underset{2}{\text{-}\phi\text{-}}/g]$$

T338. $\phi\Bigg(\llcorner p\lrcorner\ {}^{\ulcorner}\underset{2}{\text{-}\phi\text{-}}(pp)^{\urcorner}\ \llcorner fpr\lrcorner\ {}^{\ulcorner}\phi\Bigg(\underset{2}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\phi\Bigg(\phi\Big(f(rr)$

$\underset{2}{\text{-}\phi\text{-}}(pp)\Big)\phi\Big(f(rr)\underset{2}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\Big)\Bigg)\Bigg)^{\urcorner}\Bigg)$ [T74; $\underset{2}{\text{-}\phi\text{-}}/$ g]

T339. $\phi\Bigg(\llcorner fp\lrcorner\ {}^{\ulcorner}\phi\Bigg(\underset{2}{\text{-}\phi\text{-}}(pp)\phi\Bigg(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\Big(f(rr)\underset{2}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi$

$\Big(f(rr)\underset{2}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Bigg)\Bigg)^{\urcorner}\ \llcorner pq\lrcorner\ {}^{\ulcorner}\underset{2}{\text{-}\phi\text{-}}(qp)^{\urcorner}\Bigg)$

[T89; $\underset{2}{\text{-}\phi\text{-}}/g$]

T340. $\llcorner p\lrcorner\ {}^{\ulcorner}\phi\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi(p\,\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\big)\phi\text{-}(pp)\Big)\underset{2}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}$

[D10; p/q]

T341. $\phi\left(\llcorner p\lrcorner\ \ulcorner\phi\!-\!\left(\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-\!(pp)\right)\urcorner\ \llcorner p\lrcorner\ \ulcorner\underset{2}{-\!\phi\!-}(pp)\urcorner\right)$
[T340]

T342. $\llcorner p\lrcorner\ \ulcorner\phi\left(\phi\!-\!\left(\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\underset{2}{-\!\phi\!-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner$ [D10; $\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)/q$]

T343. $\phi\left(\llcorner p\lrcorner\ \ulcorner\phi\!-\!\left(\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\ \llcorner p\lrcorner\ \ulcorner\underset{2}{-\!\phi\!-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\urcorner\right)$ [T342]

T344. $\llcorner pq\lrcorner\ \ulcorner\phi\left(\underset{2}{-\!\phi\!-}(qp)\phi\!-\!\left(\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-\!(qp)\right)\right)\urcorner$
[T336, D10]

T345. $\phi\left(\llcorner pq\lrcorner\ \ulcorner\underset{2}{-\!\phi\!-}(qp)\urcorner\ \llcorner pq\lrcorner\ \ulcorner\phi\!-\!\left(\phi\!-\!\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-\!(qp)\right)\urcorner\right)$
[T344]

T346. $\llcorner p\lrcorner\ \ulcorner\underset{2}{-\!\phi\!-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\urcorner$ [T343, T181]

T347. $\llcorner p\lrcorner\ \ulcorner\underset{2}{-\!\phi\!-}(pp)\urcorner$ [T341, T241]

T348. $\llcorner fpr\lrcorner\ \ulcorner\phi\left(\underset{2}{-\!\phi\!-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\left(\phi\left(f(rr)\underset{2}{-\!\phi\!-}(pp)\right)\phi\left(f(rr)\underset{2}{-\!\phi\!-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\right)\right)\urcorner$ [T338, T347]

T349. $\phi\left(\llcorner p\lrcorner\ \ulcorner\underset{2}{\phi}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\urcorner\ \llcorner fpr\lrcorner\ \ulcorner\phi\left(\phi\left(f(rr)\underset{2}{\phi}(pp)\right)\phi\left(f(rr)\underset{2}{\phi}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\right)\urcorner\right)$ [T348]

T350. $\llcorner fpr\lrcorner\ \ulcorner\phi\left(\phi\left(f(rr)\underset{2}{\phi}(pp)\right)\phi\left(f(rr)\underset{2}{\phi}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\right)\urcorner$ [T349, T346]

T351. $\llcorner fp\lrcorner\ \ulcorner\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{\phi}(pp)\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{\phi}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\urcorner$ [T350]

T352. $\llcorner fp\lrcorner\ \ulcorner\phi\left(\underset{2}{\phi}(pp)\phi\left(\underset{2}{\phi}(pp)\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{\phi}(pp)\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{\phi}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\right)\right)\urcorner$ [T337, T351]

T353. $$\phi\left(\llcorner p\lrcorner\ \ulcorner\underset{2}{-\phi-}(pp)\urcorner\ \llcorner fp\lrcorner\ \ulcorner\phi\left(\underset{2}{-\phi-}(pp)\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{-\phi-}(pp)\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{-\phi-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\right)\urcorner\right)$$

[T352]

T354. $$\llcorner fp\lrcorner\ \ulcorner\phi\left(\underset{2}{-\phi-}(pp)\phi\left(\llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{-\phi-}(pp)\right)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\left(f(rr)\underset{2}{-\phi-}\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\right)\urcorner\right)\right)\urcorner$$ [T353, T347]

T355. $\llcorner pq\lrcorner\ \ulcorner\underset{2}{-\phi-}(qp)\urcorner$ [T339, T354]

T356. $\llcorner pq\lrcorner\ \ulcorner\phi\!-\left(\phi\!-\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-(qp)\right)\urcorner$ [T345, T355]

T357. $\llcorner ps\lrcorner\ \ulcorner\phi\!-\left(\phi\!-\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-(sp)\right)\urcorner$ [T356; s/q]

T358. $\llcorner pqs\lrcorner\ \ulcorner\phi\!-\left(q\phi\!-\left(\phi\!-\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-(sp)\right)\right)\urcorner$

[T238, T357]

T359. $$\llcorner ps\lrcorner\ \ulcorner\phi\!-\left(\phi\!-\left(s\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)\phi\!-\left(\phi\!-\left(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)p\right)\phi\!-(sp)\right)\right)\urcorner$$ $\left[\mathrm{T358};\ \phi\!-\left(s\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\right)/q\right]$

D11. $\llcorner pq \lrcorner \ulcorner \phi \left(\llcorner g \lrcorner \ulcorner \phi\!- \left(\phi(pq) \phi \left(g(p) g(q) \right) \right) \urcorner \underset{3}{-\!\phi\!-}(qp) \right) \urcorner$

T360. $\llcorner pq \lrcorner \ulcorner \phi \left(\phi \left(\llcorner g \lrcorner \ulcorner \phi\!- \left(\phi(pq) \phi \left(g(p) g(q) \right) \right) \urcorner \underset{3}{-\!\phi\!-}(qp) \right) \phi \left(\underset{3}{-\!\phi\!-} \right.\right.$

$\left.\left. (qp) \llcorner g \lrcorner \ulcorner \phi\!- \left(\phi(pq) \phi \left(g(p) g(q) \right) \right) \urcorner \right) \right) \urcorner$ $\left[\text{T8; } \llcorner g \lrcorner \ulcorner \phi\!- \left(\phi \right.\right.$

$\left.\left. (pq) \phi \left(g(p) g(q) \right) \right) \urcorner / q; \underset{3}{-\!\phi\!-}(qp)/r \right]$

T361. $\phi \left(\llcorner pq \lrcorner \ulcorner \phi \left(\llcorner g \lrcorner \ulcorner \phi\!- \left(\phi(pq) \phi \left(g(p) g(q) \right) \right) \urcorner \underset{3}{-\!\phi\!-}(qp) \right) \urcorner \right.$

$\left. \llcorner pq \lrcorner \ulcorner \phi \left(\underset{3}{-\!\phi\!-}(qp) \llcorner g \lrcorner \ulcorner \phi\!- \left(\phi(pq) \phi \left(g(p) g(q) \right) \right) \urcorner \right) \urcorner \right)$

[T360]

T362. $\phi \left(\llcorner fp \lrcorner \ulcorner \phi \left(\llcorner r \lrcorner \ulcorner \phi \left(f(rr) \underset{3}{-\!\phi\!-}(pp) \right) \urcorner \llcorner r \lrcorner \ulcorner \phi \left(f(rr) \underset{3}{-\!\phi\!-} \left(\phi (p \right.\right.\right.\right.$

$\left.\left.\left. \llcorner q \lrcorner \ulcorner q \urcorner) p \right) \right) \urcorner \right) \urcorner \llcorner fp \lrcorner \ulcorner \phi \left(\underset{3}{-\!\phi\!-}(pp) \phi \left(\underset{3}{-\!\phi\!-}(pp) \phi \left(\llcorner r \lrcorner \ulcorner \phi \left(f \right.\right.\right.\right.$

$$(rr)\underset{3}{\text{-}\phi\text{-}}(pp))^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi\Big(f(rr)\underset{3}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)\Big)^{\urcorner}$$

$$\Big)\quad [\text{T45}; \underset{3}{\text{-}\phi\text{-}}/g]$$

T363. $$\phi\Big(\llcorner p\lrcorner\ ^{\ulcorner}\underset{3}{\text{-}\phi\text{-}}(pp)^{\urcorner}\ \llcorner fpr\lrcorner\ ^{\ulcorner}\phi\Big(\underset{3}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\big)\phi\Big(\phi\Big(f(rr)$$

$$\underset{3}{\text{-}\phi\text{-}}(pp)\Big)\phi\Big(f(rr)\underset{3}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\big)\Big)\Big)\Big)^{\urcorner}\Big)\quad [\text{T74}; \underset{3}{\text{-}\phi\text{-}}/$$

$g]$

T364. $$\phi\Big(\llcorner fp\lrcorner\ ^{\ulcorner}\phi\Big(\underset{3}{\text{-}\phi\text{-}}(pp)\phi\Big(\llcorner r\lrcorner\ ^{\ulcorner}\phi\big(f(rr)\underset{3}{\text{-}\phi\text{-}}(pp)\big)^{\urcorner}\ \llcorner r\lrcorner\ ^{\ulcorner}\phi$$

$$\Big(f(rr)\Big(\underset{3}{\text{-}\phi\text{-}}\big(\phi(p\,\llcorner q\lrcorner\ ^{\ulcorner}q^{\urcorner})p\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\ \llcorner pq\lrcorner\ ^{\ulcorner}\underset{3}{\text{-}\phi\text{-}}(qp)^{\urcorner}\Big)$$

$[\text{T89}; \underset{3}{\text{-}\phi\text{-}}/g]$

T365. $$\llcorner p\lrcorner\ ^{\ulcorner}\phi\Big(\llcorner g\lrcorner\ ^{\ulcorner}\text{-}\phi\text{-}\Big(\phi(pp)\phi\big(g(p)g(p)\big)\Big)\Big)^{\urcorner}\underset{3}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}$$

$[\text{D11}; p/q]$

T366. $\phi\left(\llcorner gp \lrcorner\ \ulcorner \phi\text{-}\left(\phi(pp)\phi\left(g(p)g(p)\right)\right)\urcorner\ \llcorner p \lrcorner\ \ulcorner \underset{3}{\phi\text{-}}(pp)\urcorner\right)$
[T365]

T367. $\llcorner p \lrcorner\ \ulcorner \phi\left(\llcorner g \lrcorner\ \ulcorner \phi\text{-}\left(\phi\left(p\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\phi\left(g(p)g\left(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\right.\right.\right.\right.$
$\left.\left.\left.)))\urcorner\ \underset{3}{\phi\text{-}}\left(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)p\right)\right)\urcorner$ [D11; $\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)/q$]

T368. $\phi\left(\llcorner gp \lrcorner\ \ulcorner \phi\text{-}\left(\phi\left(p\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\phi\left(g(p)g\left(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\right)\right)\right)\right.$
$\left.\urcorner\ \llcorner p \lrcorner\ \ulcorner \underset{3}{\phi\text{-}}\left(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)p\right)\urcorner\right)$ [T367]

T369. $\llcorner pq \lrcorner\ \ulcorner \phi\left(\underset{3}{\phi\text{-}}(qp)\ \llcorner g \lrcorner\ \ulcorner \phi\text{-}\left(\phi(pq)\phi\left(g(p)g(q)\right)\right)\urcorner\right)\urcorner$
[T361, D11]

T370. $\phi\left(\llcorner pq \lrcorner\ \ulcorner \underset{3}{\phi\text{-}}(qp)\urcorner\ \llcorner gpq \lrcorner\ \ulcorner \phi\text{-}\left(\phi(pq)\phi\left(g(p)g(q)\right)\right)\urcorner\right)$
[T369]

T371. $\llcorner p \lrcorner\ \ulcorner \underset{3}{\phi\text{-}}(pp)\urcorner$ [T366, T244]

T372. $\llcorner fpr \lrcorner\ \ulcorner \phi\left(\underset{3}{\phi\text{-}}\left(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)p\right)\phi\left(\phi\left(f(rr)\underset{3}{\phi\text{-}}(pp)\right)\phi\left(f(rr)\right.\right.\right.$
$\left.\left.\left.\underset{3}{\phi\text{-}}\left(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)p\right)\right)\right)\right)\urcorner$ [T363, T371]

T373. $\phi\left(\llcorner p\lrcorner\ \ulcorner\underset{3}{\phi}\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\urcorner\ \llcorner fpr\lrcorner\ \ulcorner\phi\Big(\phi\big(f(rr)\underset{3}{\phi}(pp)\big)\phi\big(f(rr)\underset{3}{\phi}\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\Big)\urcorner\right)$ [T372]

T374. $\llcorner p\lrcorner\ \ulcorner\underset{3}{\phi}\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\urcorner$ [T368, T273]

T375. $\llcorner fpr\lrcorner\ \ulcorner\phi\Big(\phi\big(f(rr)\underset{3}{\phi}(pp)\big)\phi\big(f(rr)\underset{3}{\phi}\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\Big)\urcorner$ [T373, T374]

T376. $\llcorner fp\lrcorner\ \ulcorner\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\underset{3}{\phi}(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\underset{3}{\phi}\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\urcorner\Big)\urcorner$ [T375]

T377. $\llcorner fp\lrcorner\ \ulcorner\phi\bigg(\underset{3}{\phi}(pp)\phi\Big(\underset{3}{\phi}(pp)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\underset{3}{\phi}(pp)\big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\big(f(rr)\underset{3}{\phi}\big(\phi(p\,\llcorner q\lrcorner\ \ulcorner q\urcorner)p\big)\big)\urcorner\Big)\Big)\bigg)\urcorner$ [T362, T376]

T378. $$\phi(\llcorner p \lrcorner \ulcorner \underset{3}{-\!\phi\!-}(pp)\urcorner \llcorner fp \lrcorner \ulcorner \phi(\underset{3}{-\!\phi\!-}(pp)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)\underset{3}{-\!\phi\!-}(pp))\urcorner \llcorner r \lrcorner \phi(f(rr)\underset{3}{-\!\phi\!-}(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p))\urcorner))\urcorner)$$ [T377]

T379. $$\llcorner fp \lrcorner \ulcorner \phi(\underset{3}{-\!\phi\!-}(pp)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)\underset{3}{-\!\phi\!-}(pp))\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)\underset{3}{-\!\phi\!-}(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p))\urcorner))\urcorner$$ [T378, T371]

T380. $\llcorner pq \lrcorner \ulcorner \underset{3}{-\!\phi\!-}(qp)\urcorner$ [T364, T379]

T381. $\llcorner gpq \lrcorner \ulcorner \phi\text{-}(\phi(pq)\phi(g(p)g(q)))\urcorner$ [T370, T380]

T382. $$\llcorner gpq \lrcorner \ulcorner \phi(\phi(pq)\, \llcorner f \lrcorner \ulcorner \phi(\phi(pq)\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)\phi(pq))\urcorner \llcorner r \lrcorner \ulcorner \phi(f(rr)\phi(g(p)g(q)))\urcorner))\urcorner)\urcorner$$ [T176, T381]

T383. $$\llcorner gp \lrcorner \ulcorner \phi(\phi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p))\, \llcorner f \lrcorner \ulcorner \phi(\phi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p))\phi(\llcorner r \lrcorner \ulcorner \phi(f(rr)\phi(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner) \vdash (p)))\urcorner \llcorner r \lrcorner \ulcorner \phi(f$$

$$(rr)\phi\Big(g\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)g\big(\vdash(p)\big)\Big)\urcorner\Big)\Big)\urcorner\Big)\urcorner$$ [T382; ϕ $(p\llcorner q\lrcorner\ \ulcorner q\urcorner)/p; \vdash(p)/q$]

T384. $$\phi\Big(\llcorner p\lrcorner\ \ulcorner\phi\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\vdash(p)\big)\urcorner\ \llcorner fgp\lrcorner\ \ulcorner\phi\Big(\phi\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\vdash(p)\big)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\vdash(p)\big)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\Big(g\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)g\big(\vdash(p)\big)\Big)\Big)\urcorner\Big)\Big)\urcorner\Big)$$

[T383]

T385. $$\llcorner fgp\lrcorner\ \ulcorner\phi\Big(\phi\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\vdash(p)\big)\phi\Big(\llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\vdash(p)\big)\Big)\urcorner\ \llcorner r\lrcorner\ \ulcorner\phi\Big(f(rr)\phi\Big(g\big(\phi(p\llcorner q\lrcorner\ \ulcorner q\urcorner)\big)g\big(\vdash(p)\big)\Big)\Big)\urcorner\Big)\Big)\urcorner$$ [T384, D1]

T386. $\phi\Big(\llcorner p \lrcorner\ \ulcorner \phi\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\big) \urcorner\ \llcorner fgp \lrcorner\ \ulcorner \phi\Big(\llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\phi\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\big)\Big) \urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\phi\Big(g\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\big) g\big(\vdash (p)\big)\Big)\Big) \urcorner\Big) \urcorner\Big)$ [T385]

T387. $\llcorner fgp \lrcorner\ \ulcorner \phi\Big(\llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\phi\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\big)\Big) \urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f(rr)\phi\Big(g\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\big) g\big(\vdash (p)\big)\Big)\Big) \urcorner\Big) \urcorner$
[T386, D1]

T388. $\llcorner gp \lrcorner\ \ulcorner \phi\Big(\llcorner r \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\big)\Big) \urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\Big(g\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\big) g\big(\vdash (p)\big)\Big)\Big) \urcorner\Big) \urcorner$ [T387; ϕ/ f]

T389. $\phi\Big(\llcorner pr \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner) \vdash (p)\big)\Big) \urcorner\ \llcorner gpr \lrcorner\ \ulcorner \phi\Big(\phi(rr)\phi\Big(g\big(\phi(p \llcorner q \lrcorner\ \ulcorner q \urcorner)\big) g\big(\vdash (p)\big)\Big)\Big) \urcorner\Big)$ [T388]

T390. $\llcorner gpr\lrcorner\ \ulcorner\phi\left(\phi(rr)\phi\left(g\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)g\left(\vdash(p)\right)\right)\right)\urcorner$

[T389, T96]

T391. $\phi\left(\llcorner r\lrcorner\ \ulcorner\phi(rr)\urcorner\ \llcorner gp\lrcorner\ \ulcorner\phi\left(g\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)g\left(\vdash(p)\right)\right)\urcorner\right)$

[T390]

T392. $\llcorner gp\lrcorner\ \ulcorner\phi\left(g\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)g\left(\vdash(p)\right)\right)\urcorner$ [T391, T18]

T393. $\llcorner p\lrcorner\ \ulcorner\phi\left(\text{-}\phi\text{-}\{p\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)\text{-}\phi\text{-}\{p\}\left(\vdash(p)\right)\right)\urcorner$

[T392; $\text{-}\phi\text{-}\{p\}/g$]

T394. $\phi\left(\llcorner p\lrcorner\ \ulcorner\text{-}\phi\text{-}\{p\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)\urcorner\ \llcorner p\lrcorner\ \ulcorner\text{-}\phi\text{-}\{p\}\left(\vdash(p)\right)\urcorner\right)$

[T393]

T395. $\llcorner pq\lrcorner\ \ulcorner\phi\left(fa\{pq\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)fa\{pq\}\left(\vdash(p)\right)\right)\urcorner$

[T392; $fa\{pq\}/g$]

T396. $\phi\left(\llcorner pq\lrcorner\ \ulcorner fa\{pq\}\left(\phi(p\llcorner q\lrcorner\ulcorner q\urcorner)\right)\urcorner\ \llcorner pq\lrcorner\ \ulcorner fa\{pq\}\left(\vdash(p)\right)\urcorner\right)$ [T395]

T397. $\llcorner p\lrcorner\ \ulcorner\text{-}\phi\text{-}\{p\}\left(\vdash(p)\right)\urcorner$ [T394, T304]

T398. $\llcorner p\lrcorner\ \ulcorner\phi\text{-}\left(\phi\text{-}\left(\vdash(p)p\right)p\right)\urcorner$ [T303, T397]

T399. $\llcorner pq\lrcorner\ \ulcorner fa\{pq\}\left(\vdash(p)\right)\urcorner$ [T396, T334]

T400. $\llcorner pq \lrcorner \ulcorner \phi\!\!-\!\left(p\phi\!\!-\!\left(\vdash (p)q\right)\right)\urcorner$ [T311, T399]

D12. $\llcorner pq \lrcorner \ulcorner \phi\left(\llcorner s \lrcorner \ulcorner \phi\!\!-\!\left(\phi\!\!-\!(sq)\phi\!\!-\!\left(\phi\!\!-\!(qp)\phi\!\!-\!(sp)\right)\right)\urcorner \underset{4}{-\!\phi\!-}(qp)\right)\urcorner$

T401. $\llcorner pq \lrcorner \ulcorner \phi\left(\phi\left(\llcorner s \lrcorner \phi\!\!-\!\left(\phi\!\!-\!(sq)\phi\!\!-\!\left(\phi\!\!-\!(qp)\phi\!\!-\!(sp)\right)\right)\urcorner \underset{4}{-\!\phi\!-}(qp)\right)\phi\right.$
$\left.\left(\underset{4}{-\!\phi\!-}(qp)\,\llcorner s \lrcorner \ulcorner \phi\!\!-\!\left(\phi\!\!-\!(sq)\phi\!\!-\!\left(\phi\!\!-\!(qp)\phi\!\!-\!(sp)\right)\right)\urcorner\right)\right)\urcorner$ [T8;
$\llcorner s \lrcorner \ulcorner \phi\!\!-\!\left(\phi\!\!-\!(sq)\phi\!\!-\!\left(\phi\!\!-\!(qp)\phi\!\!-\!(sp)\right)\right)\urcorner / q;\ \underset{4}{-\!\phi\!-}(qp)/r$]

T402. $\phi\left(\llcorner pq \lrcorner \ulcorner \phi\left(\llcorner s \lrcorner \ulcorner \phi\!\!-\!\left(\phi\!\!-\!(sq)\phi\!\!-\!\left(\phi\!\!-\!(qp)\phi\!\!-\!(sp)\right)\right)\urcorner \underset{4}{-\!\phi\!-}(qp)\right)\urcorner\right.$
$\left.\llcorner pq \lrcorner \ulcorner \phi\left(\underset{4}{-\!\phi\!-}(qp)\,\llcorner s \lrcorner \ulcorner \phi\!\!-\!\left(\phi\!\!-\!(sq)\phi\!\!-\!\left(\phi\!\!-\!(qp)\phi\!\!-\!(sp)\right)\right)\urcorner\right)\urcorner\right)$
[T401]

T403. $\phi\left(\llcorner fp \lrcorner \ulcorner \phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{4}{-\!\phi\!-}(pp)\right)\urcorner\ \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{4}{-\!\phi\!-}\left(\phi(p\right.\right.\right.\right.$
$\left.\left.\left.\llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\urcorner\ \llcorner fp \lrcorner \ulcorner \phi\left(\underset{4}{-\!\phi\!-}(pp)\phi\left(\underset{4}{-\!\phi\!-}(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f\right.\right.\right.\right.$

$$(rr)\underset{4}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}\ {}^{\ulcorner}\phi\Big(f(rr)\underset{4}{\text{-}\phi\text{-}}\Big(\phi(p\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\Big)\Big)^{\urcorner}\Big)\Big)\Big)\Big)^{\urcorner}\Big)$$

$[T45; \underset{4}{\text{-}\phi\text{-}}/g]$

T404. $$\phi\Big(\llcorner p\lrcorner\ {}^{\ulcorner}\underset{4}{\text{-}\phi\text{-}}(pp)^{\urcorner}\ \llcorner fpr\lrcorner\ {}^{\ulcorner}\phi\Big(\underset{4}{\text{-}\phi\text{-}}\Big(\phi(p\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\Big)\phi\Big(\phi\Big(f(rr)$$

$$\underset{4}{\text{-}\phi\text{-}}(pp)\Big)\phi\Big(f(rr)\underset{4}{\text{-}\phi\text{-}}\Big(\phi(p\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\Big)\Big)\Big)\Big)^{\urcorner}\Big)$$ $[T74; \underset{4}{\text{-}\phi\text{-}}/$

$g]$

T405. $$\phi\Big(\llcorner fp\lrcorner\ {}^{\ulcorner}\phi\Big(\underset{4}{\text{-}\phi\text{-}}(pp)\phi\Big(\llcorner r\lrcorner\ {}^{\ulcorner}\phi\Big(f(rr)\underset{4}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}\ \llcorner r\lrcorner\ {}^{\ulcorner}\phi$$

$$\Big(f(rr)\underset{4}{\text{-}\phi\text{-}}\Big(\phi(p\llcorner q\lrcorner\ {}^{\ulcorner}q^{\urcorner})p\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\ \llcorner pq\lrcorner\ {}^{\ulcorner}\underset{4}{\text{-}\phi\text{-}}(pp)^{\urcorner}\Big)$$

$[T89; \underset{4}{\text{-}\phi\text{-}}/g]$

T406. $$\llcorner p\lrcorner\ {}^{\ulcorner}\phi\Big(\llcorner s\lrcorner\ {}^{\ulcorner}\phi\text{-}\Big(\phi\text{-}(sp)\phi\text{-}\Big(\phi\text{-}(pp)\phi\text{-}(sp)\Big)\Big)^{\urcorner}\underset{4}{\text{-}\phi\text{-}}(pp)\Big)^{\urcorner}$$

$[D12; p/q]$

T407. $$\phi\Big(\llcorner ps\lrcorner\ {}^{\ulcorner}\phi\text{-}\Big(\phi\text{-}(sp)\phi\text{-}\Big(\phi\text{-}(pp)\phi\text{-}(sp)\Big)\Big)^{\urcorner}\ \llcorner p\lrcorner\ {}^{\ulcorner}\underset{4}{\text{-}\phi\text{-}}(pp)^{\urcorner}\Big)$$

$[T406]$

T408. $\llcorner p \lrcorner \ulcorner \phi\left(\llcorner s \lrcorner \ulcorner \phi\!-\left(\phi\!-\left(s\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\right)\phi\!-\left(\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!-(sp)\right)\right)\urcorner \underset{4}{-\phi-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner$ [D12; $\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)/q$]

T409. $\phi\left(\llcorner ps \lrcorner \ulcorner \phi\!-\left(\phi\!-\left(s\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)\right)\phi\!-\left(\phi\!-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\!-(sp)\right)\right)\urcorner \llcorner p \lrcorner \ulcorner \underset{4}{-\phi-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner$ [T408]

T410. $\llcorner pq \lrcorner \ulcorner \phi\left(\underset{4}{-\phi-}(qp) \llcorner s \lrcorner \ulcorner \phi\!-\left(\phi\!-(sq)\phi\!-\left(\phi\!-(qp)\phi\!-(sp)\right)\right)\urcorner\right)\urcorner$ [T402, D12]

T411. $\phi\left(\llcorner pq \lrcorner \ulcorner \underset{4}{-\phi-}(qp)\urcorner \llcorner pqs \lrcorner \ulcorner \phi\!-\left(\phi\!-(sq)\phi\!-\left(\phi\!-(qp)\phi\!-(sp)\right)\right)\urcorner\right)$ [T410]

T412. $\llcorner p \lrcorner \ulcorner \underset{4}{-\phi-}(pp)\urcorner$ [T407, T296]

T413. $\llcorner fpr \lrcorner \ulcorner \phi\left(\underset{4}{-\phi-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\phi\left(\phi\left(f(rr)\underset{4}{-\phi-}(pp)\right)\phi\left(f(rr)\underset{4}{-\phi-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\right)\urcorner$ [T404, T412]

T414. $\phi\left(\llcorner p \lrcorner \ulcorner \underset{4}{-\!\phi\!-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\urcorner \llcorner fpr \lrcorner \ulcorner \phi\left(\phi\left(f(rr)\underset{4}{-\!\phi\!-}(pp)\right)\phi\left(f(rr)\underset{4}{-\!\phi\!-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\urcorner\right)$ [T413]

T415. $\llcorner p \lrcorner \ulcorner \underset{4}{-\!\phi\!-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\urcorner$ [T409, T359]

T416. $\llcorner fpr \lrcorner \ulcorner \phi\left(\phi\left(f(rr)\underset{4}{-\!\phi\!-}(pp)\right)\phi\left(f(rr)\underset{4}{-\!\phi\!-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\right)\urcorner$ [T414, T415]

T417. $\llcorner fp \lrcorner \ulcorner \phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{4}{-\!\phi\!-}(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{4}{-\!\phi\!-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\urcorner$ [T416]

T418. $\llcorner fp \lrcorner \ulcorner \phi\left(\underset{4}{-\!\phi\!-}(pp)\phi\left(\underset{4}{-\!\phi\!-}(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{4}{-\!\phi\!-}(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)\underset{4}{-\!\phi\!-}\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\right)\right)\urcorner$ [T403, T417]

T419. $\phi\left(\llcorner p \lrcorner \ulcorner -\underset{4}{\phi}-(pp)\urcorner \llcorner fp \lrcorner \ulcorner \phi\left(-\underset{4}{\phi}-(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)-\underset{4}{\phi}-(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr)-\underset{4}{\phi}-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\right)\urcorner\right)$

[T418]

T420. $\llcorner fp \lrcorner \ulcorner \phi\left(-\underset{4}{\phi}-(pp)\phi\left(\llcorner r \lrcorner \ulcorner \phi\left(f(rr)-\underset{4}{\phi}-(pp)\right)\urcorner \llcorner r \lrcorner \ulcorner \phi\left(f(rr) -\underset{4}{\phi}-\left(\phi(p \llcorner q \lrcorner \ulcorner q \urcorner)p\right)\right)\urcorner\right)\right)\urcorner$ [T419, T412]

T421. $\llcorner pq \lrcorner \ulcorner -\underset{4}{\phi}-(qp)\urcorner$ [T405, T420]

T422. $\llcorner pqs \lrcorner \ulcorner \phi-\left(\phi-(sq)\phi-\left(\phi-(qp)\phi-(sp)\right)\right)\urcorner$ [T411, T421]

In carrying out the deductions of this section I set myself the task of deriving explicitly theses T422, T398, T400, and T381. All the other theses are here only auxiliary definitions and theorems which I used in order to prove these four. They represent a starting-point for further considerations of this communication regarding the method for obtaining various additional theses in Protothetic, and in particular, all theses of the usual theory of deduction.

Theses T422, T398, and T400 correspond to the three propositions which Łukasiewicz proved in 1924 can together suffice as an axiomatization for the usual theory of deduction using implication and negation as the two primitive functions.[27] (In order to

[27] Cf. Łukasiewicz [2]; Łukasiewicz [3], pp. 37, 38, 45–49, and 66–98; Łukasiewicz [4], pp. 610–612; Łukasiewicz–Tarski [1], p. 6.

preclude possible chronological misunderstanding, I mention here that when in 1922 I derived within Protothetic the usual theory of deduction, I did this by means of other well known axiomatizations of it. For this article I chose Łukasiewicz's axiomatization from amongst a variety of alternatives on editorial grounds.)

Thesis T381 corresponds to $[p, q, f] .\therefore p \equiv q . \supset : f(p) . \equiv . f(q)$, of which there is an account in §4.

BIBLIOGRAPHY

Besides the abbreviations already given earlier, the following further abbreviations will be used in bibliographical references.

Leśniewski [2]: Stanisław Leśniewski, 'Einleitende Bemerkugen zur Forsetzung meiner Mitteilung u. d. T. 'Grundzüge eines neuen Systems der Grundlagen der Mathematik'', *Collectanea Logica*, Wydawnictwo poświęcone logice, jej historii i zastosowaniom, 1, 1938.

Łukasiewicz [3]: Jan Łukasiewicz, 'Démonstration de la compatibilité des axiomes de la théorie de la déduction', *Rocznik Polskiego Tow. Matematycznego. Annales de la Société Polonaise de Mathematique* III, 1924–25.

Łukasiewicz [4]: Jan Łukasiewicz, 'Über Bedeutung und Bedürfnisse der mathematischen Logik', *Nauka Polska. Jej potrzeby, organizacja i rozwój* X, 1929. (in Polish)

Łukasiewicz–Tarski [1]: Jan Łukasiewicz and A. Tarski, 'Untersuchungen über den Aussagenkalkül', *Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego* XXIII, 1930, Wydział III, (*Comptes Rendus des séances de là Societé des Sciences et des Lettres de Varsovie* XXIII, 1930, Clase III). Preliminary communication submitted by J. Łukasiewicz on March 27, 1930.

Schönfinkel [1]: M. Schönfinkel, 'Über die Bausteine der mathematischen Logik', *Mathematische Annalen*, separate printing of vol. 92. no. 34, Berlin 1924.

Sobociński [1]: Bolesław Sobociński, 'Über die aufeinanderfolgenden Vereinfachungen der Axiomatik der 'Ontologie' von Prof. St. Leśniewski', an offprint of the commemorative book honoring the jubilee of Prof. Tadeusz Kotarbiński's 15th year of teaching at the University of Warsaw, Warsaw 1934. (in Polish)

ON THE FOUNDATIONS OF ONTOLOGY

In *Przegląd Filozoficzny*, 1929, I began the publication of a larger work (in Polish) entitled 'On the Foundations of Mathematics'. Up till now the following parts have appeared (142 pages altogether):

(1) Introduction. Section I: On several questions concerning the meaning of 'logistic' theses. Section II: On Russell's 'antimony' concerning the 'class of classes which are not their own elements'. Section III: On various ways of understanding the words 'class' and 'set'.

(2) Section IV: On 'The Foundations of the General Theory of Collective Sets, I'.[1]

(3) Section V: Further theorems and definitions of the general theory of collective sets, dating from the period up to and including 1920.

(4) Section VI: The axiomatization of the general theory of collective sets, 1918. Section VII: The axiomatization of the general theory of collective sets, 1920. Section VIII: On certain necessary and sufficient conditions for P to be the class of a, established by Kuratowski and Tarski. Section IX: Further theorems of the general theory of collective sets, dating from 1921–1923.[2]

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[1] This is an account of my work (in Polish) published under this title in Moscow, 1916. [Translator's note: even though Leśniewski used the same German term '*Menge*' both in the title of Section III and in the titles of Sections IV–VII and IX, I have translated it differently as 'set' and 'collective set' to correspond with the different original Polish terms used, '*zbiór*' and '*mnogi*'; Leśniewski later introduced the term 'Mereology' for 'the theory of collective sets', which is quite different from what is known as 'set theory'.]

[2] *Przegląd Filozoficzny*: (1) – vol. 30, nos. 1–3, 1927. (2) – vol. 31, no. 3, 1928. (3) – vol. 32, nos. 1–2, 1929. (4) – vol. 33, nos. 1–2, 1930.

Despite the fact that the editorial staff of *Przegląd Filozoficzny* have put their columns at my disposal with a constantly hospitable and patient complaisance, the publication of my treatise on the foundations of mathematics will certainly still take quite some time. Taking this state of affairs into consideration (which, incidentally, I had anticipated from the beginning, and which is technically unavoidable), I decided in 1928 to publish a shorter paper which would, in a somewhat different arrangement, summarize the various results of my research for over ten years in the area of the foundations of mathematics, – results which I more thoroughly discussed in the work published in *Przegląd Filozoficzny*. In 1929 I began the publication (in German) of the shorter paper in *Fundamenta Mathematicae*.[3] In the same year I had already submitted its continuation to the same journal, and it had been accepted for publication by the editorial staff. However, in 1930, for reasons of a personal nature, I withdrew this part. In this situation it is difficult for me to foresee whether, where, and when I might find a place for its publication.

Thus it has turned out that the sections of my work on the foundations of mathematics previously published in Polish are completely different in content from those parts published in German in *Fundamenta Mathematicae*: in my Polish treatise I dealt almost exclusively with problems in Mereology (which I have elsewhere called the 'general theory of collective sets'). In the German paper I worked on various questions in Protothetic, which constitutes the first of the theories belonging to my system of the foundations of mathematics. In particular, I published in this paper the shortest axiom of Protothetic, formulated with the help of the equivalence sign as the sole primitive term, then known to me.[4] I also there determined a combination of directives for a carefully

[3] Cf. Stanisław Leśniewski, 'Fundamentals of a New System of the Foundations of Mathematics', Introduction and §§1–11, *Fundamenta Mathematicae* XIV, Warsaw (1929), 81 pages. (Separate printing with unchanged pagination.)

[4] *Op. cit.*, p. 59.

formalized system of Protothetic which could be constructed from the designated axiom.[5]

The aim of my present paper is to set forth a single axiom and a combination of directives for a formalized system of Ontology based upon Protothetic.[6] I did not want to delay the publication of this system any longer, even if this meant presenting the system in a 'potential' manner, i.e., without including any theorems.[7] From the single axioms of Protothetic and Ontology it is possible, using their directives, to derive the whole formalized system of the foundations of mathematics. With respect to content, this system is roughly analogous to Whitehead and Russell's *Principia Mathematica*.[8]

For practical and terminological reasons it must be assumed that the reader is already familiar with at least §11 of my paper in *Fundamenta Mathematicae* mentioned above.

Except for functions which already occur in Protothetic, the system of Ontology I have constructed operates with only one special primitive function '$\varepsilon\{Aa\}$', in which the term 'ε' is a constant function sign,[9] while the expressions 'A' and 'a' appear as

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[5] *Op. cit.*, p. 76.

[6] Cf. *Op. cit.*, p. 5.

[7] A general characteristic of my Ontology. This theory has for some time now become known to a wider circle of my colleagues and students through copies of my university lectures. Its axiom, given further below, and a selection of its basic definitions and theorems (all with systematic reference to those results of mine which are relevant) can be found in the Polish work of my friend Tadeusz Kotarbiński entitled *Elemente der Erkenntnistheorie, der formalen Logik und der Methodologie der Wissenschaften*, Lwów 1929 (cf. pp. 227–254 and 459 of this instructive work). Cf. also A. Tarski, 'Communication sur les recherches de la Théorie des Ensembles', *Comptes rendus de séances de la Société des Sciences et de lettres de Varsovie* XIX, Classe III, 1926, pp. 229, 312, 322, 323, 326.

[8] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, 2nd ed., Cambridge, vol. I, 1925, vol. II, 1927, vol. III, 1927.

[9] Cf. G. Peano, 'Logique mathematique', *Formulaire de Mathematiques*, vol. II, §1, II–VII, 1897.

two name arguments. Expressions of the type '$\varepsilon\{Aa\}$' should be considered to be equivalent in meaning to the corresponding Latin sentences about individuals of the type '*A est a*'. (I have appealed here to Latin; from the start however, I shall, amongst other things, independently proceed from numerous and various well-known problems one has to deal with in several other languages (such as, e.g., German, English, or French) in connection with the definite and indefinite articles. I did not wish to have to discuss the logical pseudo-problems allied to the genuine problems mentioned above, concerning the 'substantival' or 'adjectival' form of sentences predicates.)

The single axiom of my Ontology (dating from 1920) can be written out in the well known Peano-Russell symbolism as follows:[10]

$$(A,a) :: \varepsilon\{Aa\} . \equiv .\because \sim \Big((B) . \sim (\varepsilon\{BA\})\Big) \because (B,C) : \varepsilon\{BA\} . \varepsilon\{CA\} . \supset . \varepsilon\{BC\} \because (B) : \varepsilon\{BA\} . \supset . \varepsilon\{Ba\}.$$

In my own symbolism, this axiom has the following form (expressions of the form '$\vdash(p)$', '$\text{♀}(pq)$', '$\text{♀}(pqr)$', and '$\text{ϕ-}(pq)$', which here must be assumed already to have been defined, shall correspond to expressions of the type '$\sim p$', '$p \,.\, q$', '$p \,.\, q \,.\, r$', and '$p \supset q$' in the Peano-Russell symbolism):

$$\text{Axiom 0.}\quad \lfloor Aa \rfloor \ulcorner \phi\Big(\varepsilon\{Aa\}\text{♀}\Big(\vdash\big(\lfloor B \rfloor \ulcorner\vdash(\varepsilon\{BA\})\urcorner\big)\lfloor BC \rfloor \ulcorner\text{ϕ-}\big(\text{♀}(\varepsilon\{BA\}\,\varepsilon\{CA\})\,\varepsilon\{BC\}\big)\urcorner \lfloor B \rfloor \ulcorner\text{ϕ-}(\varepsilon\{BA\}\,\varepsilon\{Ba\})\urcorner\Big)\Big)\urcorner .$$

[10] When the 'particular' quantifier is at one's disposal, which is not the case in my system (cf. Leśniewski, *op. cit.*, p. 77, D), one can simply write '$(\exists B) . \varepsilon\{BA\}$' in the axiom instead of '$\sim \Big((B) . \sim (\varepsilon\{BA\})\Big)$'. In connection with the content of my axiom, cf. the analysis of the sentence 'the author of *Waverly* was Scotch' in Russell's *Introduction to Mathematical Philosophy*, 2nd ed., London-New York 1920, p. 177. Cf. also Kotarbiński, *op. cit.*, pp. 227–229.

I now come to the problem concerning the method of constructing the system of Ontology from Axiom 0. I have already determined the meaning of some of the expressions occurring in the instructions concerning this construction method in Section 11 of my article published in *Fundamenta Mathematicae*. I shall formulate appropriate meanings for the remaining expressions in a series of new terminological explanations given below.

I should point out that in the proper formulation of my system of the foundations of mathematics, the system of Ontology ensues from the system of Protothetic. In this formulation the system of Protothetic consists of the axiom of Protothetic and a finite number of concrete theses which have been *effectively* added to the system. In the construction of my system of Ontology I shall take into consideration only those theses which already effectively belong to Protothetic: I shall not appeal to various other theses which *could* be added to Protothetic according to its directives. In this paper I shall use the expression 'efthp' to designate those theses effectively belonging to Protothetic at the moment I begin constructing Ontology. (It is assumed that expressions of the type '⊢ (p)', '⫯(pq)', '⫯(pqr)', and '⫯(pq)' found in Axiom 0 are also already effectively admitted in Protothetic. It is further assumed that the parentheses '{' and '}' are not the same as any parentheses effectively admitted in Protothetic.) The expressions 'A0' and 'tho'[11] will count as abbreviations of the expressions 'Axiom 0' and 'thesis of Ontology', respectively. In accordance with the explanations above, whenever $A \,\varepsilon$ efthp and $\beta \,\varepsilon$ tho, I can say that $A \,\varepsilon$ prcd(β).

Terminological Explanation XXXII^0.[12] $[A, B] \therefore A \,\varepsilon\, \text{tho}(B) \,.\, = \colon$
$A \,\varepsilon\, \text{efthp} \,.\, \vee \,.\, A \,\varepsilon\, \text{tho} \colon$

[11] Cf. Leśniewski, *op. cit.*, pp. 68, 69.

[12] In numbering the terminological explanations in this paper, I shall constantly take into consideration their extensive analogies with the corresponding terminological explanations in Section 11 of my *op. cit.*

$B \varepsilon \text{tho}:$
$A \varepsilon \text{prcd}(B) . \vee . A \varepsilon \text{Id}(B)$[13]

Term. Exp. XXXIII^0. $[A, B] \therefore A \varepsilon \text{fro}(B) . = : A \varepsilon \text{tho}(B) . \vee . [\exists C, D] . C \varepsilon \text{tho}(B) . D \varepsilon \text{ingr}(C) . A \varepsilon \text{Arg1}(D) . \vee . [\exists C, D] . C \varepsilon \text{tho}(B) . D \varepsilon \text{ingr}(C) . A \varepsilon \text{Arg2}(D) . \vee . [\exists C, D] . C \varepsilon \text{tho}(B) . D \varepsilon \text{sbqntf} . D \varepsilon \text{ingr}(C) . A \varepsilon \text{Cmpl}\left(\text{int}(D)\right)$

T. E. XXXIV^0 $[A, B, C] \therefore A \varepsilon \text{1homosemo}(B, C) . = : A \varepsilon \text{fro}(C) . B \varepsilon \text{fro}(C) . \vee . [\exists D, E] . D \varepsilon \text{tho}(C) . E \varepsilon \text{ingr}(D) . A \varepsilon \text{cnvar}(B, E) . \vee . [\exists D, E, F, G] . D \varepsilon \text{tho}(C) . E \varepsilon \text{ingr}(D) . F \varepsilon \text{tho}(C) . G \varepsilon \text{ingr}(F) . A \varepsilon \text{An}(B, E, G)$

T. E. XXXV^0 $[A, B, C] :: A \varepsilon \text{homosemo}(B, C) . = :\cdot: A \varepsilon \text{1homosemo}(A, C) . B \varepsilon \text{1homosemo}(B, C) :\cdot: [a] :: [D] : D \varepsilon a . \supset . D \varepsilon \text{1homosemo}(D, C) \therefore [D, E] : D \varepsilon a . E \varepsilon \text{1homosemo}(D, C) . \supset . E \varepsilon a \therefore B \varepsilon a \therefore \supset . A \varepsilon a$[14]

T. E. XXXVI^0 $[A, B, C, D, E] :: A \varepsilon \text{consto}(B, C, D, E) . = \therefore D \varepsilon \text{homosemo}(E, B) \therefore [F, G] : G \varepsilon \text{tho}(B) . F \varepsilon \text{ingr}(G) . \supset . \varepsilon \sim \left(\text{cnvar}(D, F)\right) \therefore A \varepsilon \text{cnf}(D) : [\exists F, G, H] . F \varepsilon \text{ingr}(C) . G \varepsilon \text{tho}(B) . H \varepsilon \text{ingr}(G) . A \varepsilon \text{An}(E, F, H)$

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13 Refer to *op. cit.*, p. 63, fn., in connection with the terminological explanations of this paper.

14 Refer, *mutatis mutandis*, to *op. cit.*, p. 68, fn., in connection with T.E. XXXV. Speaking freely, expressions of the type '$A \varepsilon$ homosemo(B, C)' can be read off, by means of corresponding phrases of the type 'A is an expression of the same semantic category as B, with respect to thesis C, which already belongs to Ontology'.

T. E. XXXVII^0 $[A, B, C] : A \, \varepsilon \, \text{consto}(B, C) . = . [\exists D, E] .$
$A \, \varepsilon \, \text{consto}(B, C, D, E)$

T. E. XXXVIII^0 $[A, B, C, D, E, F] .\because . A \, \varepsilon$
$\text{quasihomosemo}(B, C, D, E, F) . = : E \, \varepsilon \, \text{homosemo}(F, C) :$
$[\exists G, H, I] . G \, \varepsilon \, \text{ingr}(D) . H \, \varepsilon \, \text{tho}(C) . I \, \varepsilon \, \text{ingr}(H) .$
$A \, \varepsilon \, \text{An}(E, G, I) :$
$[\exists G, H, I] . G \, \varepsilon \, \text{ingr}(D) . H \, \varepsilon \, \text{tho}(C) . I \, \varepsilon \, \text{ingr}(H) .$
$B \, \varepsilon \, \text{An}(F, G, I)$

T. E. XXXIX^0 $[A, B, C, D, E] .\because . A \, \varepsilon \, \text{fncto}(B, C, D, E) . = :$
$D \, \varepsilon \, \text{homosemo}(E, B) .$
$A \, \varepsilon \, \text{genfnct}(D) :$
$[\exists F, G, H] . F \, \varepsilon \, \text{ingr}(C) . G \, \varepsilon \, \text{tho}(B) . H \, \varepsilon \, \text{ingr}(G) .$
$A \, \varepsilon \, \text{An}(E, F, H)$

T. E. XL^0 $[A, B, C, D, E, F] .\because . A \, \varepsilon \, \text{varo}(B, C, D, E, F) . = :$
$E \, \varepsilon \, \text{homosemo}(B, C) :$
$[\exists G, H, I] . G \, \varepsilon \, \text{ingr}(D) . H \, \varepsilon \, \text{tho}(C) . I \, \varepsilon \, \text{ingr}(H) . F \, \varepsilon \, \text{An}$
$(E, G, I) :$
$F \, \varepsilon \, \text{ingr}\Big(\text{Eqvl1}\big(\text{Essnt}(D)\big)\Big) .$
$A \, \varepsilon \, \text{cnvar}(F, D)$

T. E. XLI^0 $[A, B, C, D, E] :: A \, \varepsilon \, \text{propprntmo}(B, C, D, E) . = .\because$
$D \, \varepsilon \, \text{homosemo}(B, B) .$
$E \, \varepsilon \, \text{prntm}(D) .$
$A \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\big(\text{Essnt}(C)\big)\Big) .$
$\text{arg}(A) \infty \text{arg}(E) .\because$
$[F, G] : F \, \varepsilon \, \text{arg}(A) . G \, \varepsilon \, \text{arg}(E) . \big(\text{arg}(A) \cap \text{prcd}(F)\big) \infty \big(\text{arg}$
$(E) \cap \text{prcd}(G)\big) . \supset . [\exists H, I] . F \, \varepsilon \, \text{varo}(G, B, C, H, I)$

T. E. XLII^0 $[A, B, C, D, E] : A \, \varepsilon \, \text{1propprntmo}(B, C, D, E) . = .$
$A \, \varepsilon \, \text{propprntmo}(B, C, D, E) .$
$\text{Uingr}(D) \, \varepsilon \, \text{ingr}(E)$

T. E. XLIII^0 $[A, B, C, D, E, F, G] : A \varepsilon$
$2\text{propprntmo}(B, C, D, E, F, G) \,.\, = \,.\, A \varepsilon$
$\text{propprntmo}(B, C, D, E) \,.$
$F \varepsilon \text{prntm}(D) \,.$
$\text{Uprcd}(F) \varepsilon \text{ingr}(E) \,.$
$G \varepsilon \text{simprntm}(F)$

T. E. XLIV^0 $[A, B] \therefore\!\cdot A \varepsilon \text{1defo}(B)^{15} \,.\, = :: \text{1ingr}\big(\text{Essnt}(A)\big)$
$\varepsilon \sim \Big(\text{cnvar}\Big(\text{1ingr}\big(\text{Essnt}(A)\big), A\Big)\Big) \,.$
$\text{1ingr}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\varepsilon \sim \Big(\text{cnvar}\Big(\text{1ingr}\Big(\text{Eqvl2}\big(\text{Essnt}$
$(A)\big)\Big), A\Big)\Big) \,.$
$\text{1ingr}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\varepsilon \sim \big(\text{consto}(B, A)\big) ::$
$[C] \therefore C \varepsilon \text{trm} \,.\, C \varepsilon \text{ingr}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big) \,.\, \supset : [\exists D] \,.$
$D \varepsilon \text{qntf} \,.\, D \varepsilon \text{ingr}(A) \,.\, C \varepsilon \text{int}(D) \,.\, \vee \,.\, [\exists D, E] \,.\, D \varepsilon \text{ingr}(A)$
$.\, C \varepsilon \text{var}(E, D) \,.\, \vee \,.\, C \varepsilon \text{consto}(B, A) ::$
$[C, D] : D \varepsilon \text{qntf} \,.\, D \varepsilon \text{ingr}(A) \,.\, C \varepsilon \text{int}(D) \,.\, \supset \,.\, [\exists E, F] \,.$
$E \varepsilon \text{ingr}(A) \,.\, F \varepsilon \text{var}(C, E) \therefore$
$[C, D, E] : C \varepsilon \text{int}\big(\text{Qntf}(A)\big) \,.\, E \varepsilon \text{prntm}\big(\text{Essnt}(A)\big) \,.$
$D \varepsilon \text{arg}(E) \,.\, \supset \,.\, [\exists F] \,.\, F \varepsilon \text{ingr}(D) \,.\, F \varepsilon \text{var}(C, A) ::$
$[C, D, E] \therefore C \varepsilon \text{ingr}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big) \,.\, E \varepsilon \text{ingr}(A) \,.$
$D \varepsilon \text{cnvar}(C, E) \,.\, D \varepsilon \text{ingr}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big) \,.\, \supset : D \varepsilon \text{Id}$

[15] Speaking freely, expressions of the type '$A \varepsilon$ 1defo(B)' can be read off by means of corresponding phrases of the type 'A is an expression which can count in Ontology as a definition of the first type, directly after thesis B'.

$(C) . \vee . [\exists F, G] . D \, \varepsilon \, \text{quasihomosemo}(C, B, A, F, G) : :$
$[C] : C \, \varepsilon \, \text{gnrl} . C \, \varepsilon \, \text{ingr}(A) . C \varepsilon \sim \Big(\text{Id}(A) . \supset . [\exists D, E, F, G]$
$. D \, \varepsilon \, \text{homosemo}(B, B) . E \, \varepsilon \, \text{tho}(B) . F \, \varepsilon \, \text{ingr}(E) . G \, \varepsilon \, \text{ingr}$
$(A) . D \, \varepsilon \, \text{Anarg}(C, F, G) : :$
$[C, D] \therefore C \, \varepsilon \, \text{gnrl} . C \, \varepsilon \, \text{ingr}(A) . D \, \varepsilon \, \text{Essnt}(C) . \supset : D \, \varepsilon \, \text{vrb} .$
$\vee . [\exists E] . E \, \varepsilon \, \text{fro}(B) . D \, \varepsilon \, \text{genfnct}(E) : :$
$[C] \therefore C \, \varepsilon \, \text{fnct} . C \, \varepsilon \, \text{ingr}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big) . \supset : [\exists D] .$
$D \, \varepsilon \, \text{gnrl} . D \, \varepsilon \, \text{ingr}(A) . C \, \varepsilon \, \text{Essnt}(D) . \vee . [\exists D, E] . C \, \varepsilon \, \text{fncto}$
$(B, A, D, E) : :$
$[C] : C \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) . \supset . [\exists D] . D \text{arg}(C) \therefore$
$[C, D] : C \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) . D \, \varepsilon \, \text{arg}(C) . \supset .$
$[\exists E] . D \, \varepsilon \, \text{var}(E, A) \therefore$
$[C, D] : C \, \varepsilon \, \text{trm} . C \, \varepsilon \, \text{ingr}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) . D \, \varepsilon \, \text{trm} .$
$D \, \varepsilon \, \text{ingr}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) . C \, \varepsilon \, \text{cnf}(D) . \supset . C \, \varepsilon \, \text{Id}(D) \therefore$
$[C, D] : C \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) . D \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}$
$\big(\text{Essnt}(A)\big)\Big) . C \, \varepsilon \, \text{simprntm}(D) . \supset . C \, \varepsilon \, \text{Id}(D) \therefore$
$[C, D, E] : C \, \varepsilon \, \text{1propprntmo}(B, A, D, E) . \text{Uingr}\Big(\text{Eqvl2}$
$\big(\text{Essnt}(A)\big)\Big) \, \varepsilon \, \text{ingr}(C) . \supset . C \, \varepsilon \, \text{simprntm}(E) \therefore$
$[C, D, E, F, G] : C \, \varepsilon \, \text{2propprntmo}(B, A, D, E, F, G) .$
$G \, \varepsilon \, \text{ingr}(A) . \text{Uprcd}(G) \, \varepsilon \, \text{ingr}(C) . \supset . C \, \varepsilon \, \text{simprntm}(E) \therefore$
$[C, D, E] : C \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) . \text{Uingr}$
$\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big) \, \varepsilon \, \text{ingr}(C) . D \, \varepsilon \, \text{tho}(B) . E \, \varepsilon \, \text{ingr}$

$(D) . C \varepsilon$ simprntm$(E) . \supset . [\exists F, G] . C \varepsilon$ 1propprntmo $(B, A, F, G) \therefore$

$[C, D, E, F] : C \varepsilon$ prntm$\Big($Eqvl2$\big($Essnt$(A)\big)\Big) . D \varepsilon$ prntm $. D \varepsilon$ ingr(A) . Uprcd$(D) \varepsilon$ ingr$(C) . E \varepsilon$ tho$(B) . F \varepsilon$ ingr $(E) . C \varepsilon$ simprntm$(F) . \supset . [\exists G, H, I] . C \varepsilon$ 2propprntmo (B, A, G, H, I, D)

T. E. XLVII^0 $[A, a, B, C] \because A \varepsilon$ cnsqsbsto(B, C, a)[16] $. = ::$
Essnt$(A) \varepsilon$ Cmpl(a) .
$a \infty$int$\big($Sbqntf$(C)\big) ::$
$[D, E] \therefore D \varepsilon$ int$\big($Sbqntf$(C)\big) . E \varepsilon a . \big(a \cap$ prcd$(E)\big) \infty \Big($int $\big($Sbqntf$(C)\big) \cap$ prcd$(D)\Big) . \supset : [\exists F] . D \varepsilon$ var$(F, C) . \vee .$ $D \varepsilon$ cnf$(E) ::$
$[D, E] \therefore D \varepsilon$ int$\big($Sbqntf$(C)\big) . E \varepsilon a . \big(a \cap$ prcd$(E)\big) \infty \Big($int $\big($Sbqntf$(C)\big) \cap$ prcd$(D)\Big) . \supset : E \varepsilon$ trm $. \vee . E \varepsilon$ gnrl $. \vee .$ $E \varepsilon$ fnct $. \vee . E \varepsilon$ cnf$(D) ::$
$[D, E, F, G] : D \varepsilon$ cnvar$(E, C) . F \varepsilon a . G \varepsilon a . \big(a \cap$ prcd(F) $\big) \infty \Big($int$\big($Sbqntf$(C)\big) \cap$ prcd$(D)\Big) . \big(a \cap$ prcd$(G)\big) \infty \Big($int $\big($Sbqntf$(C)\big) \cap$ prcd$(E)\Big) . \supset . F \varepsilon$ cnf$(G) \therefore$
$[D, E, F, G, H, I, K, L] : D \varepsilon$ ingr$\big($Essnt$(C)\big) . E \varepsilon$ int$\big($Qntf $(D)\big) . F \varepsilon$ var$(K, C) . F \varepsilon$ ingr$(D) . G \varepsilon a . H \varepsilon a . \big(a \cap$ prcd

[16] Speaking freely, expressions of the type '$A \varepsilon$ cnsqsbsto(B, C, a)' can be read off by means of corresponding phrases of the type 'A is derivable from C with the help of the expression a, by means of an insertion correct in Ontology with regard to B'.

$(G)\Big)\infty\Big(\mathrm{int}\Big(\mathrm{Sbqntf}(C)\Big)\cap\mathrm{prcd}(E)\Big)\,.\,\Big(a\cap\mathrm{prcd}(H)\Big)\infty$
$\Big(\mathrm{int}\Big(\mathrm{Sbqntf}(C)\Big)\cap\mathrm{prcd}(F)\Big)\,.\,L\,\varepsilon\,\mathrm{ingr}(A)\,.\,I\,\varepsilon\,\mathrm{var}(G,L)\,.$
$\supset .\, I\varepsilon \sim \Big(\mathrm{ingr}(H)\Big) ::$
$[D,E] \therefore D\,\varepsilon\,\mathrm{int}\Big(\mathrm{Qntf}(A)\Big)\,.\,E\,\varepsilon\,\mathrm{cnf}(D)\,.\,E\,\varepsilon\,\mathrm{ingr}(C)\,.$
$\supset : [\exists F]\,.\,F\,\varepsilon\,\mathrm{qntf}\,.\,F\,\varepsilon\,\mathrm{ingr}(C)\,.\,E\,\varepsilon\,\mathrm{int}(F)\,.\vee.\,[\exists F,G]\,.$
$F\,\varepsilon\,\mathrm{ingr}(C)\,.\,E\,\varepsilon\,\mathrm{var}(G,F) ::$
$B\,\varepsilon\,\mathrm{expr} ::$
$[D] \therefore D\,\varepsilon\,\mathrm{trm}\,.\,D\,\varepsilon\,\mathrm{ingr}(A)\,.\supset : [\exists E]\,.\,E\,\varepsilon\,\mathrm{qntf}\,.\,E\,\varepsilon\,\mathrm{ingr}$
$(A)\,.\,D\,\varepsilon\,\mathrm{int}(E)\,.\vee.\,[\exists E,F]\,.\,E\,\varepsilon\,\mathrm{ingr}(A)\,.\,D\,\varepsilon\,\mathrm{var}(F,E)\,.\vee$
$.\,D\,\varepsilon\,\mathrm{consto}(B,A) ::$
$[D,E] : E\,\varepsilon\,\mathrm{qntf}\,.\,E\,\varepsilon\,\mathrm{ingr}(A)\,.\,D\,\varepsilon\,\mathrm{int}(E)\,.\supset.\,[\exists F,G]\,.$
$F\,\varepsilon\,\mathrm{ingr}(A)\,.\,G\,\varepsilon\,\mathrm{var}(D,F) ::$
$[D,E,F] \therefore E\,\varepsilon\,\mathrm{ingr}(A)\,.\,F\,\varepsilon\,\mathrm{cnvar}(D,E)\,.\supset : F\,\varepsilon\,\mathrm{Id}(D)\,.\vee$
$.\,[\exists G,H]\,.\,F\,\varepsilon\,\mathrm{quasihomosemo}(D,B,A,G,H) ::$
$[D] : D\,\varepsilon\,\mathrm{gnrl}\,.\,D\,\varepsilon\,\mathrm{ingr}(A)\,.\,D\varepsilon \sim \Big(\mathrm{Id}(A)\Big)\,.\supset.$
$[\exists E,F,G,H]\,.\,E\,\varepsilon\,\mathrm{homosemo}(B,B)\,.\,F\,\varepsilon\,\mathrm{tho}(B)\,.\,G\,\varepsilon\,\mathrm{ingr}$
$(F)\,.\,H\,\varepsilon\,\mathrm{ingr}(A)\,.\,E\,\varepsilon\,\mathrm{Anarg}(D,G,H) ::$
$[D,E] \therefore D\,\varepsilon\,\mathrm{gnrl}\,.\,D\,\varepsilon\,\mathrm{ingr}(A)\,.\,E\,\varepsilon\,\mathrm{Essnt}(D)\,.\supset : E\,\varepsilon\,\mathrm{vrb}\,.$
$\vee.\,[\exists F]\,.\,F\,\varepsilon\,\mathrm{fro}(B)\,.\,E\,\varepsilon\,\mathrm{genfnct}(F) ::$
$[D] \therefore D\,\varepsilon\,\mathrm{fnct}\,.\,D\,\varepsilon\,\mathrm{ingr}(A)\,.\supset : D\,\varepsilon\,\mathrm{Id}(A)\,.\vee.\,[\exists E]\,.$
$E\,\varepsilon\,\mathrm{gnrl}\,.\,E\,\varepsilon\,\mathrm{ingr}(A)\,.\,D\,\varepsilon\,\mathrm{Essnt}(E)\,.\vee.\,[\exists E,F]\,.\,D\,\varepsilon\,\mathrm{fncto}$
(B,A,E,F)

T. E. XLVIII^0 $[A,B,C] : A\,\varepsilon\,\mathrm{cnsqsbsto}(B,C)\,.=.\,[\exists a]\,.$
$A\,\varepsilon\,\mathrm{cnsqsbsto}(B,C,a)$

T. E. IL^0 $[A,B] \because A\,\varepsilon\,\mathrm{1extnslo}(B)\,.= :: [\exists C,D]\,.\,C\,\varepsilon\,\mathrm{int}\Big(\mathrm{Qntf}$
$(A)\Big)\,.\,D\,\varepsilon\,\mathrm{int}\Big(\mathrm{Qntf}(A)\Big)\,.\,C\,\varepsilon\,\mathrm{prcd}(D) \therefore$
$[C,D] : D\,\varepsilon\,\mathrm{qntf}\,.\,D\,\varepsilon\,\mathrm{ingr}(A)\,.\,C\,\varepsilon\,\mathrm{int}(D)\,.\supset.\,[\exists E,F]\,.$
$E\,\varepsilon\,\mathrm{ingr}(A)\,.\,F\,\varepsilon\,\mathrm{var}(C,E)\,.\,F\varepsilon \sim \Bigg(\mathrm{cnf}\Bigg(\mathrm{1ingr}\Big(\mathrm{Essnt}(A)\Big)\Bigg)\Bigg)$

$\Big) \therefore$

$[\exists C] . C \, \varepsilon \, \text{prntm}\Big(\text{Eqvl1}\Big(\text{Essnt}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\Big)\Big) .$

$\text{lingr}\Big(\text{Eqvl1}\Big(\text{Cmpl}\Big(\text{int}\Big(\text{Sbqntf}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)\Big)\Big)$

$\Big)\Big) \, \varepsilon \, \text{cnvar}\Big(\text{Cmpl}\big(\text{int}(C)\big), A\Big) :$

$[\exists C] . C \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\Big(\text{Essnt}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\Big)\Big) .$

$\text{lingr}\Big(\text{Eqvl2}\Big(\text{Essnt}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)\Big)\Big) \, \varepsilon \, \text{cnvar}\Big(\text{Cmpl}$

$\big(\text{int}(C)\big) . A\Big) : :$

$[C] \therefore C \, \varepsilon \, \text{fnct} . C \, \varepsilon \, \text{ingr}(A) . \supset : [\exists D] . D \, \varepsilon \, \text{gnrl} . D \, \varepsilon \, \text{ingr}(A)$
$. C \, \varepsilon \, \text{Essnt}(D) . \lor . [\exists D, E] . C \, \varepsilon \, \text{fncto}(B, A, D, E) : :$

$[C, D, E, F] : D \, \varepsilon \, \text{prntm}\Big(\text{Eqvl1}\Big(\text{Essnt}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)$

$\Big)\Big)\Big) . E \, \varepsilon \, \text{prntm}\Big(\text{Eqvl2}\Big(\text{Essnt}\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)\Big)\Big) .$

$F \, \varepsilon \, \text{Anarg}(C, D, E) . \supset . F \, \varepsilon \, \text{cnvar}\Big(C, \text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)$

$\therefore$

$[C, D, E] : D \, \varepsilon \, \text{ingr}(A) . E \, \varepsilon \, \text{cnvar}(C, D) . \supset . [\exists F, G] .$
$E \, \varepsilon \, \text{quasihomosemo}(C, B, A, F, G) \therefore$

$[C,D]:D\,\varepsilon\,\mathrm{cnvar}\Big(C,\mathrm{Eqvl1}\big(\mathrm{Essnt}(A)\big)\Big).\supset.[\exists E,F].$
$E\,\varepsilon\,\mathrm{ingr}(A).F\,\varepsilon\,\mathrm{ingr}(A).D\,\varepsilon\,\mathrm{Anarg}(C,E,F)\therefore$
$[C,D,E]:C\,\varepsilon\,\mathrm{prntm}\Bigg(\mathrm{Essnt}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big)\Bigg).D\,\varepsilon\,\mathrm{arg}$
$(C).E\,\varepsilon\,\mathrm{Sgnfnct}(D).\supset.E\,\varepsilon\,\mathrm{var}\Bigg(\mathrm{Cmpl}\Bigg(\mathrm{int}\Bigg(\mathrm{Qnft}\Bigg(\mathrm{Eqvl2}$
$\big(\mathrm{Essnt}(A)\big)\Bigg)\Bigg)\Bigg),\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Bigg)$

T. E. L^0 $[A,B]::A\,\varepsilon\,\mathrm{cnjnct}(B).=\therefore\mathrm{Sgnfnct}(B)\,\varepsilon\,\mathrm{cnf}\Big(13\mathrm{ingr}$
$(A0)\Big)\therefore$
$[\exists C]:C\,\varepsilon\,\mathrm{prntm}(B):A\,\varepsilon\,\mathrm{Arg1}(C).\vee.A\,\varepsilon\,\mathrm{Arg2}(C)$

T. E. LI^0 $[A,B]\therefore A\,\varepsilon\,\mathrm{Sbjct}(B).=:\mathrm{Sgnfnct}(B)\,\varepsilon\,\mathrm{cnf}\Big(8\mathrm{ingr}$
$(A0)\Big):$
$[\exists C,D].C\,\varepsilon\,\mathrm{prntm}(B).D\,\varepsilon\,\mathrm{ingr}(A0).A\,\varepsilon\,\mathrm{Anarg}\Big(10\mathrm{ingr}$
$(A0),C,D\Big)$

T. E. LII^0 $[A,B]\therefore A\,\varepsilon\,\mathrm{Prdct}(B).=:\mathrm{Sgnfnct}(B)\,\varepsilon\,\mathrm{cnf}\Big(8\mathrm{ingr}$
$(A0)\Big):$
$[\exists C,D].C\,\varepsilon\,\mathrm{prntm}(B).D\,\varepsilon\,\mathrm{ingr}(A0).A\,\varepsilon\,\mathrm{Anarg}\Big(11\mathrm{ingr}$
$(A0),C,D\Big)$

T. E. LIII^0 $[A,B,C,D,E]::A\,\varepsilon\,\mathrm{nomprntmo}(B,C,D,E).=\therefore$
$D\,\varepsilon\,\mathrm{homosemo}\Big(10\mathrm{ingr}(A0),B\Big).$
$E\,\varepsilon\,\mathrm{prntm}(D).$

$A \,\varepsilon\, \mathrm{prntm}\left(\mathrm{Prdct}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}(C)\right)\right)\right)$.

$\mathrm{arg}(A) \infty \mathrm{arg}(E)$.·.

$[F, G] : F \,\varepsilon\, \mathrm{arg}(A) \,.\, G \,\varepsilon\, \mathrm{arg}(E) \,.\, \left(\mathrm{arg}(A) \cap \mathrm{prcd}(F)\right) \infty \left(\mathrm{arg}(E) \cap \mathrm{prcd}(G)\right) \,.\supset .\, [\exists H, I] \,.\, F \,\varepsilon\, \mathrm{varo}(G, B, C, H, I)$

T. E. LIV^0 $[A, B, C, D, E] : A \,\varepsilon\, 1\mathrm{nomprntmo}(B, C, D, E) \,.=.$
$A \,\varepsilon\, \mathrm{nomprntmo}(B, C, D, E)$.
$\mathrm{Uingr}(D) \,\varepsilon\, \mathrm{ingr}(E)$

T. E. LV^0 $[A, B, C, D, E, F, G] : A \,\varepsilon\, 2\mathrm{nomprntmo}(B, C, D, E, F, G) \,.=.\, A \,\varepsilon\, \mathrm{nomprntmo}(B, C, D, E)$.
$F \,\varepsilon\, \mathrm{prntm}(D)$.
$\mathrm{Uprcd}(F) \,\varepsilon\, \mathrm{ingr}(E)$.
$G \,\varepsilon\, \mathrm{simprntm}(F)$

T. E. LVI^0 $[A, B]$ ∴ $A \,\varepsilon\, 2\mathrm{defo}(B)$[17] $.=::\, 1\mathrm{ingr}\left(\mathrm{Essnt}(A)\right) \varepsilon \sim \left(\mathrm{cnvar}\left(1\mathrm{ingr}\left(\mathrm{Essnt}(A)\right), A\right)\right)$.

$1\mathrm{ingr}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right) \varepsilon \sim \left(\mathrm{cnvar}\left(1\mathrm{ingr}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right), A\right)\right)$.

$1\mathrm{ingr}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}(A)\right)\right) \varepsilon \sim \left(\mathrm{cnvar}\left(1\mathrm{ingr}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}\right.\right.\right.\right.$

[17] Speaking freely, expressions of the type '$A \,\varepsilon\, 2\mathrm{defo}(B)$' can be read off by means of corresponding phrases of the type 'A is an expression which can count in Ontology as a definition of the second type, directly after thesis B'.

$(A))\Big),A\Big)\Big)$.

$\text{1ingr}\Big(\text{Prdct}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\Big)\varepsilon \sim \Big(\text{cnvar}\Big(\text{1ingr}$

$\Big(\text{Prdct}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\Big),A\Big)\Big)$.

$\text{1ingr}\Big(\text{Prdct}\Big(\text{Eqvl2}\big(\text{Essnt}(A)\big)\Big)\Big)\varepsilon \sim \big(\text{consto}(B,A)\big)$: :

$[C]$.˙. $C\ \varepsilon$ trm . $C\ \varepsilon$ ingr$\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)$. ⊃ : $[\exists D]$.
$D\ \varepsilon$ qntf . $D\ \varepsilon$ ingr(A) . $C\ \varepsilon$ int(D) . ∨ . $[\exists D,E]$. $D\ \varepsilon$ ingr(A)
. $C\ \varepsilon$ var(E,D) . ∨ . $C\ \varepsilon$ consto(B,A) : :
$[C,D]$: $D\ \varepsilon$ qntf . $D\ \varepsilon$ ingr(A) . $C\ \varepsilon$ int(D) . ⊃ . $[\exists E,F]$.
$E\ \varepsilon$ ingr(A) . $F\ \varepsilon$ var(C,E) .˙.
$[C,D,E]$: $C\ \varepsilon$ int$\big(\text{Qntf}(A)\big)$. $E\ \varepsilon$ prntm$\big(\text{Essnt}(A)\big)$.
$D\ \varepsilon$ arg(E) . ⊃ . $[\exists F]$. $F\ \varepsilon$ ingr(D) . $F\ \varepsilon$ var(C,A) : :
$[C,D,E]$.˙. $C\ \varepsilon$ ingr$\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)$. $E\ \varepsilon$ ingr(A) .
$D\ \varepsilon$ cnvar(C,E) . $D\ \varepsilon$ ingr$\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)$. ⊃ : $D\ \varepsilon$ Id
(C) . ∨ . $[\exists F,G]$. $D\ \varepsilon$ quasihomosemo(C,B,A,F,G) : :
$[C]$: $C\ \varepsilon$ gnrl . $C\ \varepsilon$ ingr(A) . $C\varepsilon \sim \big(\text{Id}(A)\big)$. ⊃ .
$[\exists D,E,F,G]$. $D\ \varepsilon$ homosemo(B,B) . $E\ \varepsilon$ tho(B) . $F\ \varepsilon$ ingr
(E) . $G\ \varepsilon$ ingr(A) . $D\ \varepsilon$ Anarg(C,F,G) : :
$[C,D]$.˙. $C\ \varepsilon$ gnrl . $C\ \varepsilon$ ingr(A) . $D\ \varepsilon$ Essnt(C) . ⊃ : $D\ \varepsilon$ vrb .
∨ . $[\exists E]$. $E\ \varepsilon$ fro(B) . $D\ \varepsilon$ genfnct(E) : :
$[C]$.˙. $C\ \varepsilon$ fnct . $C\ \varepsilon$ ingr$\Big(\text{Eqvl1}\big(\text{Essnt}(A)\big)\Big)$. ⊃ : $[\exists D]$.

$D \varepsilon$ gnrl . $D \varepsilon$ ingr(A) . $C \varepsilon$ Essnt(D) . $\vee$. $[\exists D, E]$. $C \varepsilon$ fncto (B, A, D, E) : :

$[\exists C] : C \varepsilon$ Eqvl1$\Big($Essnt$(A)\Big)$. $\vee$. $C \varepsilon$ cnjnct$\Big($Eqvl1$\Big($Essnt $(A)\Big)\Big)$: Sbjct(C) ε cnvar$\Big($Sbjct$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big), A\Big)$ $\therefore$

$[C] : C \varepsilon$ prntm$\Big($Prdct$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$. $\supset$. $[\exists D]$. $D \varepsilon$ arg(C) $\therefore$

$[C, D] : C \varepsilon$ prntm$\Big($Prdct$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$. $D \varepsilon$ arg (C) . $\supset$. $[\exists E]$. $D \varepsilon$ var(E, A) $\therefore$

$[C, D] : C \varepsilon$ trm . $C \varepsilon$ ingr$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)$. $C \varepsilon \sim \Big($1ingr $\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$. $D \varepsilon$ trm . $D \varepsilon$ ingr$\Big($Eqvl2$\Big($Essnt(A) $\Big)\Big)$. $D \varepsilon \sim \Big($1ingr$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$. $C \varepsilon$ cnf(D) . $\supset$. $C \varepsilon$ Id(D) $\therefore$

$[C, D] : C \varepsilon$ prntm$\Big($Prdct$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$. $D \varepsilon$ prntm $\Big($Prdct$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$. $C \varepsilon$ simprntm(D) . $\supset$. $C \varepsilon$ Id(D) $\therefore$

$[C, D, E] : C \varepsilon$ 1nomprntmo(B, A, D, E) . Uingr$\Big($Prdct $\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Big)$ ε ingr(C) . $\supset$. $C \varepsilon$ simprntm(E) $\therefore$

$[C, D, E, F, G] : C \,\varepsilon\, \text{2nomprntmo}(B, A, D, E, F, G) \,.\, G \,\varepsilon\, \text{ingr}(A) \,.\, \text{Uprcd}(G) \,\varepsilon\, \text{ingr}(C) \,.\supset.\, C \,\varepsilon\, \text{simprntm}(E) \,.\!\cdot\!.$

$[C, D, E] : C \,\varepsilon\, \text{prntm}\left(\text{Prdct}\left(\text{Eqvl2}\left(\text{Essnt}(A)\right)\right)\right) \,.$

$\text{Uingr}\left(\text{Prdct}\left(\text{Eqvl2}\left(\text{Essnt}(A)\right)\right)\right) \,\varepsilon\, \text{ingr}(C) \,.\, D \,\varepsilon\, \text{tho}$

$(B) \,.\, E \,\varepsilon\, \text{ingr}(D) \,.\, C \,\varepsilon\, \text{simprntm}(E) \,.\supset.\, [\exists F, G] \,.$
$C \,\varepsilon\, \text{1nomprntmo}(B, A, F, G) \,.\!\cdot\!.$

$[C, D, E, F] : C \,\varepsilon\, \text{prntm}\left(\text{Prdct}\left(\text{Eqvl2}\left(\text{Essnt}(A)\right)\right)\right)$

$.\, D \,\varepsilon\, \text{prntm} \,.\, D \,\varepsilon\, \text{ingr}(A) \,.\, \text{Uprcd}(D) \,\varepsilon\, \text{ingr}(C) \,.\, E \,\varepsilon\, \text{tho}$
$(B) \,.\, F \,\varepsilon\, \text{ingr}(E) \,.\, C \,\varepsilon\, \text{simprntm}(F) \,.\supset.\, [\exists G, H, I] \,.$
$C \,\varepsilon\, \text{2nomprntmo}(B, A, G, H, I, D)$

T. E. LVII^0 $[A, B] \therefore A \,\varepsilon\, \text{2extnsnlo}(B) \,.=::\, [\exists C, D] \,.\, C \,\varepsilon\, \text{int}$
$\left(\text{Qntf}(A)\right) \,.\, D \,\varepsilon\, \text{int}\left(\text{Qntf}(A)\right) \,.\, C \,\varepsilon\, \text{prcd}(D) \,.\!\cdot\!.$
$[C, D] : D \,\varepsilon\, \text{qntf} \,.\, D \,\varepsilon\, \text{ingr}(A) \,.\, C \,\varepsilon\, \text{int}(D) \,.\supset.\, [\exists E, F] \,.$

$E \,\varepsilon\, \text{ingr}(A) \,.\, F \,\varepsilon\, \text{var}(C, E) \,.\, F \varepsilon \sim \left(\text{cnf}\left(\text{1ingr}\left(\text{Essnt}(A)\right)\right)\right.$

$\left.\right) \,.\!\cdot\!.$

$\text{1ingr}\left(\text{Eqvl1}\left(\text{Essnt}\left(\text{Eqvl1}\left(\text{Essnt}(A)\right)\right)\right)\right) \varepsilon \sim \left(\text{cnvar}\right.$

$\left.\left(\text{1ingr}\left(\text{Eqvl1}\left(\text{Essnt}\left(\text{Eqvl1}\left(\text{Essnt}(A)\right)\right)\right)\right), A\right)\right) :$

$[\exists C] \,.\, C \,\varepsilon\, \text{prntm}\left(\text{Eqvl1}\left(\text{Essnt}\left(\text{Eqvl2}\left(\text{Essnt}(A)\right)\right)\right)\right)$

$$.\, \mathrm{1ingr}\left(\mathrm{Prdct}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right)\right)\right)\right)$$

$$\varepsilon\, \mathrm{cnvar}\left(\mathrm{Cmpl}\left(\mathrm{int}(C)\right), A\right) :$$

$$[\exists C] \,.\, C \,\varepsilon\, \mathrm{prntm}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}(A)\right)\right)\right)\right)$$

$$.\, \mathrm{1ingr}\left(\mathrm{Prdct}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right)\right)\right)\right)$$

$$\varepsilon\, \mathrm{cnvar}\left(\mathrm{Cmpl}\left(\mathrm{int}(C)\right), A\right) ::$$

$$[C] \therefore C \,\varepsilon\, \mathrm{fnct} \,.\, C \,\varepsilon\, \mathrm{ingr}(A) \,.\, \supset : [\exists D] \,.\, D \,\varepsilon\, \mathrm{gnrl} \,.\, D \,\varepsilon\, \mathrm{ingr}(A)$$
$$.\, C \,\varepsilon\, \mathrm{Essnt}(D) \,.\, \vee \,.\, [\exists D, E] \,.\, C \,\varepsilon\, \mathrm{fncto}(B, A, D, E) \because$$

$$\mathrm{Sbjct}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right)\right)\right) \varepsilon\, \mathrm{cnvar}\left(\mathrm{Sbjct}\right.$$

$$\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right)\right)\right), \mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)$$

$$\left.\right) \therefore$$

$$[C, D, E, F] : D \,\varepsilon\, \mathrm{prntm}\left(\mathrm{Prdct}\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}\left(\mathrm{Eqvl1}\right.\right.\right.\right.$$

$$\left(\mathrm{Essnt}(A)\right)\left.\left.\left.\left.\right)\right)\right)\right) \,.\, E \,\varepsilon\, \mathrm{prntm}\left(\mathrm{Prdct}\left(\mathrm{Eqvl2}\left(\mathrm{Essnt}\right.\right.\right.$$

$$\left(\mathrm{Eqvl1}\left(\mathrm{Essnt}(A)\right)\right)\Bigg)\Bigg)\Bigg) . F \,\varepsilon\, \mathrm{Anarg}(C, D, E) . \supset .$$

$$F \,\varepsilon\, \mathrm{cnvar}\Big(C, \mathrm{Eqvl1}\big(\mathrm{Essnt}(A)\big)\Big) \therefore$$

$$[C, D, E] : D \,\varepsilon\, \mathrm{ingr}(A) . E \,\varepsilon\, \mathrm{cnvar}(C, D) . \supset . [\exists F, G] .$$
$$E \,\varepsilon\, \mathrm{quasihomosemo}(C, B, A, F, G) \therefore$$

$$[C, D] : D \,\varepsilon\, \mathrm{cnvar}\Big(C, \mathrm{Eqvl1}\big(\mathrm{Essnt}(A)\big)\Big) . \supset . [\exists E, F] .$$

$$E \,\varepsilon\, \mathrm{ingr}(A) . F \,\varepsilon\, \mathrm{ingr}(A) . D \,\varepsilon\, \mathrm{Anarg}(C, E, F) \therefore$$

$$[C, D, E] : C \,\varepsilon\, \mathrm{prntm}\Bigg(\mathrm{Essnt}\bigg(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\bigg)\Bigg) . D \,\varepsilon\, \mathrm{arg}$$

$$(C) . E \,\varepsilon\, \mathrm{Sgnfnct}(D) . \supset . E \,\varepsilon\, \mathrm{var}\Bigg(\mathrm{Cmpl}\bigg(\mathrm{int}\Big(\mathrm{Qntf}\Big(\mathrm{Eqvl2}$$

$$\left(\mathrm{Essnt}(A)\right)\Big)\Big)\bigg), \mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Bigg)$$

The formulation of the directives of my system of Ontology based on Axiom 0 could be reduced, with the help of the terms whose meanings I have discussed in the terminological explanations, to fixing the following instruction:

On the premise that a thesis A is the last thesis already belonging to the system, an expression B may be added to the system as a new thesis only in case at least one of the seven following conditions is fulfilled:

1) $B \,\varepsilon\, \mathrm{1defo}(A)$
2) $B \,\varepsilon\, \mathrm{2defo}(A)$
3) $[\exists C] . C \,\varepsilon\, \mathrm{tho}(A) . B \,\varepsilon\, \mathrm{cnsqrprtqntf}(C)$
4) $[\exists C, D] . C \,\varepsilon\, \mathrm{tho}(A) . D \,\varepsilon\, \mathrm{tho}(A) . B \,\varepsilon\, \mathrm{cnsqeqvl}(C, D)$
5) $[\exists C] . C \,\varepsilon\, \mathrm{tho}(A) . B \,\varepsilon\, \mathrm{cnsqsbsto}(A, C)$

6) $B \varepsilon$ 1extnsnlo(A)

7) $B \varepsilon$ 2extnsnlo(A)

Conditions 1) and 3)–6) are here the corresponding analogues of conditions 1)–5) in the instruction concerning the method of construction of Protothetic,[18] while conditions 2) and 7) are peculiar to Ontology. Condition 2) concerns definitions in which the definiendum will be an expression of the type '$\varepsilon\{Aa\}$'. Condition 7) guarantees, in conjunction with condition 6), the extensionality of every type of function occurring in Ontology.

Actually, the essentials of the system of conditions 1)–7) date from 1922.[19] It does not contain a condition corresponding to the directive (analogous to directive (ξ) of the system SS1 discussed in the *op. cit.*) which I had adopted in earlier developmental studies of Ontology, and which Tarski proved to be superfluous in 1922.[20] Until 1922 I made no use of the extensionality directives corresponding to conditions 6) and 7). Consequently, my Ontology represented at that time a considerably weaker system. As for the systematic arrangement of the terminological directives and the conditions 1)–7) of my present paper, it dates essentially from 1926, except for several completely minor alterations.

I have styled the instruction concerning the construction method for Ontology in such a way that in any system built in accordance with it, one can also obtain as a thesis of Ontology any sentence which corresponds to some sentence already obtainable in Protothetic on the basis of its own directives. In exactly this sense, Ontology contains the whole of Protothetic. If, however, one only wished to strengthen particular conditions in a certain way, very sleight transformations could be made in the instruction in order to make it impossible to obtain in Ontology various sentences already obtainable in Protothetic. In such an event, Protothetic and Ontology would be left always lying outside one

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[18] Cf. *op. cit.*, p. 76.

[19] Cf. *op. cit.*, p. 14.

[20] Cf. *op. cit.*, p. 41.

another. Consequently, the instructions for constructing Ontology could be formulated in at least a double fashion, making it dependent upon the relation one might fix between some earlier and some later theory in the whole of mathematical science.

I have derived from Axiom 0 its four natural factors; they comprise an axiom system which can be taken as equivalent to Ontology. I here state these four natural factors explicitly:

I. $\llcorner Aa \lrcorner \ulcorner \phi\!\!-\Big(\varepsilon\{Aa\} \vdash \big(\llcorner B \lrcorner \ulcorner \vdash (\varepsilon\{BA\})\urcorner\big)\Big)\urcorner$

II. $\llcorner AaBC \lrcorner \ulcorner \phi\!\!-\Big(\varphi(\varepsilon\{Aa\}\ \varepsilon\ \{BA\}\ \varepsilon\ \{CA\})\ \varepsilon\ \{BC\}\Big)\urcorner$

III. $\llcorner AaB \lrcorner \ulcorner \phi\!\!-\Big(\varphi(\varepsilon\{Aa\}\ \varepsilon\ \{BA\})\ \varepsilon\ \{Ba\}\Big)\urcorner$

IV. $\llcorner AaB \lrcorner \ulcorner \phi\!\!-\Bigg(\varphi\bigg(\varepsilon\{BA\}\ \llcorner BC \lrcorner \ulcorner \phi\!\!-\Big(\varphi(\varepsilon\{BA\}\ \varepsilon\ \{CA\})$

$\varepsilon\ \{BC\}\Big)\urcorner\ \llcorner B \lrcorner \ulcorner \phi\!\!-(\varepsilon\{BA\}\ \varepsilon\ \{Ba\})\urcorner\bigg)\ \varepsilon\ \{Aa\}\Bigg)\urcorner.$

Comment on I: thesis I can be strengthened in Ontology by means of the easily proved sentence which says that:

$\llcorner Aa \lrcorner \ulcorner \phi\!\!-(\varepsilon\{Aa\}\ \varepsilon\ \{AA\})\urcorner.$

(I call this the 'ontological identity-sentence';[21] it should be noticed that the yet stronger thesis '$\llcorner A \lrcorner \ulcorner \varepsilon\{AA\}\urcorner$' is not provable in Ontology — indeed, its negation is provable.) In connection with this sentence, I want to emphasize expressly that in Ontology there is always a very good possibility of proving theses having a single component of the type '$\varepsilon\{AA\}$' or (what is indifferently the same in Ontology) of the type '$\varepsilon\{aa\}$'. This does not, however, lead to a contradiction via the well-known schema

[21] One can consider the meaning I have adopted for the expressions 'ontology' and 'ontological' to have originated as the result of the generalization of the relevant terminology of Jan Łukasiewicz. Cf. Łukasiewicz, 'Über den Satz des Widerspruchs bei Aristoteles', *Bulletin de l'Academie des Sciences de Cracovie*, Novembre–Decembre, 1909, pp. 16, 17. Cf. further, Kotarbiński, *op. cit.*, p. 254.

of *Principia Mathematica*,[22] because the definition directives of Ontology have been appropriately formulated so that no thesis of the type

$$\llcorner A \lrcorner \ulcorner \phi\Big(\varepsilon\{Ax\} \vdash (\varepsilon\{AA\})\Big)\urcorner$$

can be obtained.

Comment on II: by means of the definition

$$\llcorner AB \lrcorner \ulcorner \phi\Big(\varphi(\varepsilon\{AB\}\,\varepsilon\,\{BA\}) = \{AB\}\Big)\urcorner$$

I introduced into Ontology the identity-sign '=', for which the sentence saying that

$$\llcorner AB\varphi \lrcorner \ulcorner \phi\!-\Big(= \{AB\}\phi(\varphi\{A\}\varphi\{B\})\Big)\urcorner$$

is also easily proved by appealing only to the extensionality directives. As soon as I had '=' at my disposal I was able to derive from thesis II the more symmetrical thesis[23]

$$\llcorner AaBC \lrcorner \ulcorner \phi\!-\Big(\varphi(\varepsilon\{Aa\}\,\varepsilon\,\{BA\}\,\varepsilon\,\{CA\}) = \{BC\}\Big)\urcorner.$$

Comment on III: it can be seen from thesis III that my ε-relation (as one would say today) is transitive; while this fully accords with traditional logic, it contrasts with the Peano-Russell logic.

In connection with theses I–IV notice expressly that in my system of Ontology there are no means for deriving the existential sentence

$$\vdash \Big(\llcorner Aa \lrcorner \ulcorner\vdash (\varepsilon\{Aa\})\urcorner\Big).$$

A fortiori there is no way to derive sentences of the type '$\varepsilon\{Aa\}$'.

I shall now give some historical dates concerning the axiomatization of Ontology:

In 1921 Tarski derived II from III in accordance with the directives of Ontology (the extensionality directives did not yet exist in 1921, as I have already mentioned). Reflecting on Tarski's result, I proved in the same year that without changing the directives,

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[22] Cf. Whitehead and Russell, *Principia Mathematica* I, 2nd ed., p. 77.

[23] Cf. Russell, *loc. cit.*

one could replace Axiom 0 with a shorter, yet equivalent, axiom which says that

$$\llcorner Aa \lrcorner \ulcorner \phi\Big(\varepsilon\{Aa\}\varphi\Big(\vdash \Big(\llcorner B \lrcorner \ulcorner \vdash \big(\varphi(\varepsilon\{BA\}\,\varepsilon\,\{Ba\})\big)\urcorner\Big)$$

$$\llcorner BC \lrcorner \ulcorner \phi\!-\!\big(\varphi(\varepsilon\{BA\}\,\varepsilon\,\{CA\})\,\varepsilon\,\{BC\}\big)\urcorner\Big)\Big)\urcorner .$$

In 1929 Bolesław Sobociński (a student at the University of Warsaw), relying on Tarski's result and using an extensionality directive corresponding to condition 7) of my instruction for constructing Ontology, was able to replace this shorter axiom with the still simpler axiom

$$\llcorner Aa \lrcorner \ulcorner \phi\Big(\varepsilon\{Aa\}\varphi\Big(\vdash \Big(\llcorner B \lrcorner \ulcorner \vdash \big(\varphi(\varepsilon\{BA\}\,\varepsilon\,\{Ba\})\big)\urcorner\Big) \llcorner B \lrcorner$$

$$\ulcorner \phi\!-\!(\varepsilon\{BA\}\,\varepsilon\,\{AB\})\urcorner\Big)\Big)\urcorner .$$

At the same time he showed that my axiom of Ontology could be replaced by theses I and III if they were adopted as two axioms.

Using Sobociński's and Tarski's results, I proved in 1929 that without any changes in the directives, the fairly simple thesis which says that

$$\llcorner Aa \lrcorner \ulcorner \phi\Big(\varepsilon\{Aa\} \vdash \Big(\llcorner B \lrcorner \ulcorner \vdash \big(\varphi(\varepsilon\{AB\}\,\varepsilon\,\{Ba\})\big)\urcorner\Big)\Big)\urcorner$$

can suffice as the sole axiom of Ontology.[24]

[24] In connection with the form of the axioms considered in this paper, cf. Leśniewski, 'On Functions whose Fields, with Respect to These Functions, are Groups', *Fundamenta Mathematicae* XIII (1929), p. 332.

ON DEFINITIONS IN THE SO-CALLED THEORY OF DEDUCTION†

This paper is a résumé of the course of lectures (in Polish) 'On foundations of the 'theory of deduction'' that I delivered in Warsaw University in the academic year 1930–31. My main task here is to formulate a directive permitting addition, to the system of the 'theory of deduction', of theses of the special kind that I call *definitions*, as distinguished from *axioms* and *theorems*, and codifying as precisely as possible conditions to be satisfied by such definitions.

The problem of definition in the theory of deduction lies quite outside my system of foundations of mathematics, which I have begun publishing in the last few years.[1] What interested me in this problem, if I may so express myself, was its own constructive appeal — in view of the still rather stepmotherly treatment of it even in the current scientific trend in theory of deduction and theory of theory of deduction.

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† This paper originally appeared under the title 'Über Definitionen in der sogenannten Theorie der Deduktion' in *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Cl. iii, 24 (1931), pp. 289–309, and was presented to the Society by Jan Łukasiewicz on 21 November 1931. Tranlated by E. C. Luschei. Reprinted from "Polish Logic 1920–1939",

[1] See Stanisław Leśniewski: (1) 'O podstawach matematyki' (On the foundations of mathematics), *Przegląd Filozoficzny* 30 (1927), pp. 164–206; 31 (1928) pp. 261–91; 32 (1929), pp. 60–101; 33 (1930), pp. 77–105; 34 (1931), pp. 142-70. (2) 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', *Fundamenta Mathematicae* 14 (1929), pp. 1–81. (3) 'Über die Grundlagen der Ontologie', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Cl. iii, 23 (1930), pp. 111–32. Presented by Jan Łukasiewicz on 22 May 1930.

I base my directive for definition formulated below on the well-known bracketless and dotless notation devised for mathematical logic by Jan Łukasiewicz in 1924[2] and since adopted by several others.[3] In terms of this notation — the simplest (though by no means the clearest) symbolism I know for the theory of deduction — the problems of introducing definitions, which if brackets are retained could be resolved by simply adapting my directive for definition in 'protothetic',[4] lose much theoretical banality.

Although my chief problem, in the foundations of the theory of deduction, concerned the directive for definition, naturally I could not carry out my investigations in complete abstraction from other directives of the theory; so, for example, introducing definitions into the theory, I felt compelled to give the 'directive for substitution' too a form permitting replacement of variables even by formulae containing defined terms of the theory. All these considerations have led me to present here a complete system of directives for the theory of deduction.

I base this system of the theory of deduction with definitions on the familiar 33-word axiom quoted below, formulated by Łukasiewicz in terms of negation and implication, which, as

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[2] See (1) Jan Łukasiewicz, *Elementy logiki matematycznej* (Elements of Mathematical Logic), Wydawnictwa Koła Matematyczno-Fizycznego Słuchaczów Uniwersytetu Warszawskiego, vol. 18 (1929), pp. 37–40, 45, 154–6, 158–9, 171–2. [English translation, Warsaw 1963 — Ed.] (2) Jan Łukasiewicz, 'O znaczeniu i potrzebach logiki matematycznej', *Nauka Polska* 10 (1929), pp. 610–12. (3) Jan Łukasiewicz and Alfred Tarski, 'Untersuchungen über den Aussagenkalkül', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Cl. iii, 23 (1930), pp. 31–32.

[3] See (1) Leon Chwistek, 'Neue Grundlagen der Logik und Mathematik', *Mathematische Zeitschrift*, 30 (1929), p. 713. (2) M. Presburger, Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt', *Sprawozdanie z I Kongresu Matematyków Krajów Słowiańskich (Comptes rendus du I^er^ Congrès des Mathématiciens des pays slaves)*, Warsaw, 1929, pp. 92–93.

[4] See Leśniewski, 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', pp. 70–72, 76.

he has shown, forms, together with the directives for detachment and substitution, an axiomatic foundation adequate for the ordinary theory of deduction. The directives I give here for a system based on these two primitive terms can very easily be transposed to a system based on others, in particular to the familiar system of Nicod.[5]

Łukasiewicz's[6] Axiom (*L*):*

$CCCαCβαCCCNγCδNεCCγCδζCCεδCεζηCθη$.

Before proceeding to formulate the directives of this system of the theory of deduction based on Axiom (*L*), I give the following series of *terminological explanations* of the technical expressions peculiar to these directives.[7]

Terminological explanation I. I say of object *A* that it is (the) complex of (the) *a*[8] if and only if the following conditions are fulfilled:

(1) *A* is an expression;

(2) if any object is a word that belongs to *A*, then it belongs to a certain *a*;

(3) if any object *B* is *a*, any object *C* is *a*, and some word that belongs to *B* belongs to *C*, then *B* is the same object as *C*;

(4) if any object is *a*, then it is an expression that belongs to *A*.[9]

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[5] See Jean G. P. Nicod, 'A reduction in the number of the primitive propositions of logic', *Proceedings of the Cambridge Philosophical Society* 19 (1917), pp. 32–41.

[6] See Łukasiewicz and Tarski, *op. cit.*, pp. 36–37.

* [*Translator's note*: Henceforth in this translation called *Axiom* (*L*) Leśniewski used the designation '*Axiom*' in italics.]

[7] To understand the significance of such technical expressions and to preclude possible misinterpretations of these terminological explanations and of the directives, see Leśniewski, *op. cit.*, pp. 59–62.

[8] The uncapitalized variable was here used in the plural.

[9] See Leśniewski, *op. cit.*, p. 63, *T.E.VII*.

Examples (I have composed pertinent examples to show the mutual independence of individual conditions of the relevant terminological explanations). (1) Axiom (*L*) is the complex of words that belong to Axiom (*L*).

(2) The first word of Axiom (*L*) is not the complex of words that belong to Axiom (*L*) [Conditions (1–3) are here fulfilled, Condition (4) is not fulfilled (the 2nd word of Axiom (*L*) is a word that belongs to Axiom (*L*), but it is not an expression that belongs to the first word of Axiom (*L*))].

(3) Axiom (*L*) is not a complex of expressions that belong to Axiom (*L*) [Conditions(1), (2), (4) fulfilled (f.), Condition (3) not fulfilled (n.f) (Axiom (*L*) is an expression that belongs to Axiom (*L*), the first word of Axiom (*L*) is an expression that belongs to Axiom (*L*), some word that belongs to Axiom (*L*) belongs to the first word of Axiom (*L*), but Axiom (*L*) is not the same object as the first word of Axiom (*L*))].)

(4) Axiom (*L*) is not the complex of expressions that belong to Axiom (*L*) and are equiform to the first word of Axiom (*L*) [C.(1), (3), (4)f., C.(2)n.f. (the 4th word of Axiom (*L*) is a word that belongs to Axiom (*L*), but it belongs to no expression that both belongs to Axiom (*L*) and is equiform to the first word of Axiom (*L*))].

(5) The class of expressions[10] that belong to Axiom (*L*) and are equiform to the first word of Axiom (*L*) is not the complex of expressions that belong to Axiom (*L*) and are equiform to the first word of Axiom (*L*) [C.(2)–(4)f., C.(1)n.f.].

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[10] Expressions of the form 'class of *a*' as used here always mean class (i.e. totality) in the *collective* sense of my 'general set theory', which I have come to call mereology. (See Leśniewski: (1) 'O podstawach matematyki', *Przegląd Filozoficzny* 30, pp. 185–206, and 31, pp. 261–5; (2) 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', p. 5) So, for example, the class of expressions that belong to Axiom (*L*) and are equiform to the first word of Axiom (*L*) is an object that *consists* of all expressions that belong to Axiom (*L*) and are equiform to the first word of Axiom (*L*), just as an orchestra consists of all its members. Expressions of the form 'class of *a*' occur here only in examples.

Terminological explanation II. I say of object A that it is (the) negate of B if and only if the following conditions are fulfilled:

(1) A is an expression;

(2) B is the complex of objects that are either A or the first of the words that belong to B;

(3) B is not a word;

(4) the first of the words that belong to B is an expression equiform to the 11th word of Axiom (L).

Examples. (1) The 12th word of Axiom (L) is the negate of the class of objects that are either the 11th or the 12th word of Axiom (L).

(2) The class of words of Axiom (L) that follow the first word of Axiom (L) is not the negate of Axiom (L) [C.(1)–(3)f., C.(4)n.f.].

(3) The 11th word of Axiom (L) is not the negate of the 11th word of Axiom (L) [C.(1), (2), (4)f., C.(3)n.f.].

(4) Axiom (L) is not the negate of the class of words of Axiom (L) that follow the 10th word of Axiom (L) [C.(1),(3),(4)f., C.(2)n.f.].

(5) It-is-not-true-that [11] the word of Axiom (L) following the 11th word of Axiom (L) is the negate of the class of words of Axiom (L) that follow the 10th word of Axiom (L)[12] [C.(2)–(4)f., C.(1)n.f.].

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[11] I use this phrase hyphenated as a 'colloquial' representative of the ordinary propositional negation sign of mathematical logic.

[12] In my terminological explanations and examples 'singular propositions' of the form 'A is b' are used in accordance with the axiom of my 'ontology'. (See Leśniewski, 'Über die Grundlagen der Ontologie', pp. 114–15, 129–31.) It follows, since there is more than one word of Axiom (L) following the 11th word of Axiom (L), that for no a can it be true that the word of Axiom (L) following the 11th word of Axiom (L) is a; consequently it cannot be that the word of Axiom (L) following the 11th word of Axiom (L) is the negate of the class of words of Axiom (L) that follow the 10th word of Axiom (L), nor even an expression. (See Leśniewski, 'O podstawach matematyki', *Przegląd Filozoficzny* 31, pp. 263–4.)

Terminological explanation III. I say of object A that it is (the) implicant of B in C if and only if the following conditions are fulfilled:

(1) C is the complex of objects that are either A, B, or the first of the words that belong to C;

(2) the first of the words that belong to C is an expression equiform to the first word of Axiom (L);

(3) A follows the first of the words that belong to C;

(4) B follows A.

Examples. (1) The 2nd word of Axiom (L) is the implicant in Axiom (L) of the class of words of Axiom (L) that follow the 2nd word of Axiom (L).

(2) The class of words of Axiom (L) that follow the first word of Axiom (L) is not the implicant in Axiom (L) of the class of words of Axiom (L) that follow the first word of Axiom (L) [C.(1)–(3)f., C.(4)n.f.].

(3) The first word of Axiom (L) is not the implicant in Axiom (L) of the class of words of Axiom (L) that follow the first word of Axiom (L) [C.(1),(2),(4)f., C.(3)n.f.].

(4) The 5th word of Axiom (L) is not the implicant, in the class of words of Axiom (L) that follow the 3rd word of Axiom (L), of the class of words of Axiom (L) that follow the 5th word of Axiom (L) [C.(1),(3),(4)f., C.(2)n.f.].

(5) The 2nd word of Axiom (L) is not the implicant in Axiom (L) of the 3rd word of Axiom (L) [C.(2)–(4)f., C.(1)n.f.].

Terminological explanation IV. I say of object A that it is subordinate to B with respect to (the) a,[13] in particular to (the) b,[13] relative to C if and only if the following conditions are fulfilled:

(1) B is an expression that belongs to C;

(2) B is not a variable;[14]

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[13] The uncapitalized variable was here used in the plural.

[14] I give here no special terminological explanation of the word 'variable'. The details of fixing the denotation of this word are relatively immaterial. But I have to presuppose here that (1) the 4th, 6th, 12th, 14th, 16th, 22nd,

(3) if any object is a word that belongs to C and follows B, then it is a variable;
(4) if any object is a word that belongs to some thesis of this system which thesis precedes C,[15] then it is not an expression equiform to B;
(5) A is the complex of objects that are either b or the first of the words that belong to A;
(6) the first of the words that belong to A is an expression equiform to B;
(7) if any object is b, then it is a;
(8) if any object is b, then it follows the first of the words that belong to A;
(9) there are exactly as many b as words that belong to C and follow B.

Examples. (1) The class of words of Axiom (L) that follow the 30th word of Axiom (L) is subordinate to the 31st word of Axiom (L) with respect to variables, in particular to objects that

30th, and 32nd words of Axiom (L) are variables; (2) neither the 1st nor the 11th word of Axiom (L) is a variable; (3) if A is an expression equiform to B, then A is a variable if and only if B is a variable; (4) any variable is a word; and (5) it is always possible to form new variables (i.e. variables equiform to no variable already used) in the same general sense as it is to form new expressions. A concrete definition, if I had to give one here, might well be to the effect that any object is a variable if and only if it is a word consisting solely of small Greek letters. I could not accept a convention that confined variables to letters of this or that alphabet, since such a convention would preclude forming a proposition containing more nonequiform variables than there are nonequiform letters of the alphabet. Cf. Łukasiewicz and Tarski, *op. cit.*, p. 31.)

[15] As theses of this system in addition to Axiom (L) I count only those 'definitions' and 'theorems' *effectively* added to the system, not various other expressions that might be added according to its directives. So the extent of the expression 'thesis of this system' is by no means univocally determined in advance, but rather is conceived as 'growing' by stages. Axiom (L) is the only expression already a thesis of this system.

are either the 32nd or the 33rd word of Axiom (*L*) relative to Axiom (*L*).

(2) The class of words of Axiom (*L*) that follow the 30th word of Axiom (*L*) is not subordinate to the 31st word of Axiom (*L*) with respect to expressions, in particular to the class of objects that are either the 32nd or the 33rd word of Axiom (*L*),[16] relative to Axiom (*L*) [C.(1)–(8)f., C.(9)n.f.].

(3) The class of objects that are either the 31st or the 32nd word of Axiom (*L*) is not subordinate to the 31st word of Axiom (*L*) with respect to words, in particular to objects that are either the 31st or the 32nd word of Axiom (*L*), relative to Axiom (*L*) [C.(1)–(7),(9)f., C.(8)n.f. (the 31st word of Axiom (*L*) is an object that is either the 31st or the 32nd word of Axiom (*L*), but it does not follow the first of the words that belong to the class of objects that are either the 31st or the 32nd word of Axiom (*L*))].

(4) The class of words of Axiom (*L*) that follow the 30th word of Axiom (*L*) is not subordinate to the 31st word of Axiom (*L*) with respect to Axiom (*L*), in particular to objects that are either the 32nd or the 33rd word of Axiom (*L*), relative to Axiom (*L*) [C.(1)–(6),(8),(9)f., C.(7)n.f. (the 32nd word of Axiom (*L*) is an object that is either the 32nd or the 33rd word of Axiom (*L*), but it is not Axiom (*L*))].

(5) The class of words of Axiom (*L*) that follow the 29th and precede the 33rd word of Axiom (*L*) is not subordinate to the 31st word of Axiom (*L*) with respect to words, in particular to objects that are either the 31st or the 32nd word of Axiom (*L*), relative to Axiom (*L*) [C.(1)–(5),(7)–(9)f., C.(6)n.f.].

(6) Axiom (*L*) is not subordinate to the 31st word of Axiom (*L*) with respect to words, in particular to objects that are either the 32nd or the 33rd word of Axiom (*L*), relative to Axiom (*L*) [C.(1)–(4),(6)–(9)f., C.(5)n.f.].

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[16] Of course there is only one such class, as there can be at most one class of *a*, whatever *a* may be. See Leśniewski, *loc. cit.*, p. 265, Axiom III.

(7) The first of the words that follow Axiom (*L*) and are expressions equiform to the first word of Axiom (*L*) is not subordinate to the first of the words that follow Axiom (*L*) and are expressions equiform to the first word of Axiom (*L*) with respect to nonquadrangular quadrangles, in particular to nonquadrangular quadrangles, relative to the first of the words that follow Axiom (*L*) and are expressions equiform to the first word of Axiom (*L*) [C.(1)–(3),(5)–(9)f., C.(4)n.f. (the first word of Axiom (*L*) is a word that belongs to some thesis of this system which thesis precedes the first of the words that follow Axiom (*L*) and are expressions equiform to the first word of Axiom (*L*), but it is an expression equiform to the first of the words that follow Axiom (*L*) and are expressions equiform to the first word of Axiom (*L*))].

(8) Axiom (*L*) is not subordinate to the first word of Axiom (*L*) with respect to words, in particular to words of Axiom (*L*) that follow the first word of Axiom (*L*), relative to Axiom (*L*) [C.(1),(2),(4)–(9)f., C.(3)n.f. (the 2nd word of Axiom (*L*) is a word that belongs to Axiom (*L*) and follows the first word of Axiom (*L*), but it is not a variable)].

(9) The class of words of Axiom (*L*) that follow the 31st word of Axiom (*L*) is not subordinate to the 32nd word of Axiom (*L*) with respect to words, in particular to the 33rd word of Axiom (*L*), relative to Axiom (*L*) [C.(1),(3)–(9)f., C.(2)n.f.].

(10) The second word of Axiom (*L*) is not subordinate to the second word of Axiom (*L*) with respect to Axiom(s) (*L*), in particular to nonquadrangular quadrangles, relative to the first word of Axiom (*L*) [C.(2)–(9)f., C.(1)n.f.].

Terminological explanation V. I say of object *A* that it is an expression fundamental for (the) *a*,[17] relative to *B*, if and only if the following conditions are fulfilled:

(1) *A* is an expression;
(2) some expression is *a*;
(3) if any object is *a*, then it is an expression that belongs to *A*;

[17] The uncapitalized variable was here used in the plural.

(4) if any object is a variable that belongs to A, then it is a;

(5) if any object C is an expression that belongs to A, and the negate of C is a, then C is a;

(6) if any object C is the same object as B or is a thesis of this system which thesis precedes B, and any object D belongs to A and is subordinate to some expression with respect to a, in particular to any arbitrary objects b, relative to C, then D is a.

Examples. (1) Axiom (L) is an expression fundamental for expressions that belong to Axiom (L), relative to Axiom (L).

(2) The class of words of Axiom (L) that follow the 30th word of Axiom (L) is not an expression fundamental for objects that are either the 32nd or the 33rd word of Axiom (L), relative to Axiom (L) [C.(1)–(5)f., C.(6)n.f. (Axiom (L) is the same object as Axiom (L) or is a thesis of this system which thesis precedes Axiom (L), the class of words of Axiom (L) that follow the 30th word of Axiom (L) belongs to the class of words of Axiom (L) that follow the 30th word of Axiom (L) and is subordinate to some expression with respect to objects that are either the 32nd or the 33rd word of Axiom (L), in particular to objects that are either the 32nd or the 33rd word of Axiom (L), relative to Axiom (L), but the class of words of Axiom (L) that follow the 30th word of Axiom (L) is not an object that is either the 32nd or the 33rd word of Axiom (L))].

(3) The class of objects that are either the 11th or the 12th word of Axiom (L) is not an expression fundamental for the 12th word(s) of Axiom (L), relative to Axiom (L) [C.(1)–(4),(6)f., C.(5)n.f. (the class of objects that are either the 11th or the 12th word of Axiom (L) is an expression that belongs to the class of objects that are either the 11th or the 12th word of Axiom (L), the negate of the class of objects that are either the 11th or the 12th word of Axiom (L) is 12th word of Axiom (L), but the class of objects that are either the 11th or the 12th word of Axiom (L) is not 12th word of Axiom (L))].

(4) Axiom (L) is not an expression fundamental for Axiom(s) (L), relative to Axiom (L) [C.(1)–(3),(5),(6)f., C.(4)n.f. (the 4th word of Axiom (L) is a variable that belongs to Axiom (L), but it is not Axiom (L))].

(5) Axiom (L) is not an expression fundamental for expressions, relative to Axiom (L) [C.(1),(2),(4)–(6)f., C.(3)n.f. (the title of this paper is an expression, but is not an expression that belongs to Axiom (L))].

(6) The first word of Axiom (L) is not an expression fundamental for nonquadrangular quadrangles, relative to Axiom (L) [C.(1),(3)–(6)f., C.(2)n.f.].

(7) The class of objects that are either the 1st or the 4th word of Axiom (L) is not an expression fundamental for the 4th word(s) of Axiom (L), relative to Axiom (L) [C.(2)–(6)f., C.(1)n.f.].

Terminological explanation VI. I say of object A that it is propositional (i.e. belongs to the category of propositions) relative to B if and only if the following conditions are fulfilled:

(1) A is an expression;
(2) some variable belongs to A;
(3) if A is an expression fundamental for any arbitrary objects a, relative to B, then A is a.[18]

[18] *Terminological explanation VI* is based on the ideas of 'hereditary class' and 'ancestral relation', well known in mathematical logic. My explanation is a 'generalization', for the theory of deduction including definitions, of the definition of the '*set S of all sentences*' given by Łukasiewicz and Tarski (*op. cit.*, p. 31). [According to Leśniewski's explanation, loosely paraphrased, A is propositional relative to B if and only if expression A belongs to the closure of the one or more variables in A with respect to negation in A or subordination in A to some expression, relative to a thesis relative to B. — *Translator.*] It could easily be proved that the extent of the expression 'propositional relative to B' would not be altered by omitting Condition (5) from the six defining conditions of *Terminological explanation V* above. I have nevertheless retained it because I favour always being able to confirm that a propositional expression is propositional, relative to a given thesis of the system, by a combinatorial decision procedure referring only to a corresponding

Examples. (1) Axiom (*L*) is propositional relative to Axiom (*L*).

(2) The class of words of Axiom (*L*) that follow the 31st word of Axiom (*L*) is not propositional relative to Axiom (*L*)

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specific finite domain of expressions. I know yet another method, quite different from that explained here, for defining propositions of various deductive theories. I first explained this other method, in which 'hereditary classes' and the 'ancestral relation' play no role, and which essentially originated in 1922, in my 'logistic' lectures in the academic year 1924–5 (see Leśniewski, 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', p. 59), in application to my system of protothetic (see *op. cit.*, pp. 9–81). In terms of the 'symbolic' abbreviations used in the terminological explanations and directives for protothetic (see *op. cit.*, pp. 59–76), such a definition for protothetic could be formulated as follows (essentially as I worded it in 1926 — regarding the suffix 'p' of 'propp' see *op. cit.*, pp. 68–69):

$$[A, B] :\cdot: A \in \mathrm{propp}(B) \,.\, = \,:\,: B \in \mathrm{thp} \,:\,:$$

$$[\exists C] \therefore C \in \mathrm{vrb} \,.\, C \in \mathrm{frp}(B) \,.\, A \in \mathrm{cnf}(C) \therefore [D, E] : D \in \mathrm{thp}(B) \,.\, E \in \mathrm{ingr}(D) \,.\, \supset .\, C \in N(\mathrm{cnvar}(C, E)) \therefore \vee \,.\, [\exists C] \,.\, C \in \mathrm{frp}(B) \,.\, A \in \mathrm{genfnct}(C) \,.\, \vee \,.\, A \in \mathrm{gnrl} \,:\,:$$

$$[C] \therefore C \in \mathrm{trm} \,.\, C \in \mathrm{ingr}(A) \,.\, \supset : C \in \mathrm{Id}(A) \,.\, \vee \,.\, [\exists D] \,.\, D \in \mathrm{qntf} D \in \mathrm{ingr}(A) \,.\, C \in \mathrm{int}(D) \,.\, \vee \,.\, [\exists D, E] \,.\, D \in \mathrm{ingr}(A) \,.\, C \in \mathrm{var}(E, D) \,.\, \vee \,.\, C \in \mathrm{constp}(B, A) \,:\,:$$

$$[C, D] : D \in \mathrm{qntf} \,.\, D \in \mathrm{ingr}(A) \,.\, C \in \mathrm{int}(D) \,.\, \supset .\, [\exists E, F] \,.\, E \in \mathrm{ingr}(A) \,.\, F \in \mathrm{var}(C, E) \,:\,:$$

$$[C, D, E] \therefore E \in \mathrm{ingr}(A) \,.\, C \in \mathrm{cnvar}(D, E) \,.\, \supset : C \in \mathrm{Id}(D) \,.\, \vee \,.\, [\exists F, G] \,.\, C \in \mathrm{quasihomosemp}(D, B, A, F, G) \,:\,:$$

$$[C] \therefore C \in \mathrm{gnrl} \,.\, C \in \mathrm{ingr}(A) \,.\, \supset : C \in \mathrm{Id}(A) \,.\, \vee \,.\, [\exists D, E, F, G] \,.\, D \in \mathrm{thp}(B) \,.\, E \in \mathrm{ingr}(D) \,.\, F \in \mathrm{ingr}(A) \,.\, G \in \mathrm{homosemp}(B, B) \,.\, G \in \mathrm{Anarg}(C, E, F) \,:\,:$$

$$[C, D] \therefore C \in \mathrm{gnrl} \,.\, C \in \mathrm{ingr}(A) \,.\, D \in \mathrm{Essnt}(C) \,.\, \supset : D \in \mathrm{vrb} \,.\, \vee \,.\, [\exists E] \,.\, E \in \mathrm{frp}(B) \,.\, D \in \mathrm{genfnct}(E) \,:\,:$$

$$[C] \therefore C \in \mathrm{fnct} \,.\, C \in \mathrm{ingr}(A) \,.\, \supset : C \in \mathrm{Id}(A) \,.\, \vee \,.\, [\exists D] \,.\, D \in \mathrm{gnrl} \,.\, D \in \mathrm{ingr}(A) \,.\, C \in \mathrm{Essnt}(D) \,.\, \vee \,.\, [\exists D, E] \,.\, C \in \mathrm{fnctp}(B, A, D, E)$$

It is not difficult to formulate analogous definitions for further theories belonging to my system of foundations of mathematics. In one of the first

[C.(1),(2)f., C.(3)n.f. (the class of words of Axiom (*L*) that follow the 31st word of Axiom (*L*) is an expression fundamental for objects that are either the 32nd or the 33rd word of Axiom (*L*), relative to Axiom (*L*), but the class of words of Axiom (*L*) that follow the 31st word of Axiom (*L*) is not an object that is either the 32nd or the 33rd word of Axiom (*L*))].

(3) The first word of Axiom (*L*) is not propositional relative to Axiom (*L*) [C.(1),(3)f., C.(2)n.f.].

(4) The class of variables that belong to Axiom (*L*) is not propositional relative to Axiom (*L*) [C.(2),(3)f., C.(1)n.f.].

Terminological explanation VII. I say of object *A* that it is a consequence of *B*, relative to *C*, by substitution of (the) *a*[19] if and only if the following conditions are fulfilled:

(1) *A* is the complex of (the) *a*;
(2) there are exactly as many *a* as words that belong to *B*;
(3) if any object *D* is a word that belongs to *B*, any object *E* is *a*, and there are exactly as many *a* that precede *E* as words that belong to *B* and precede *D*, then *D* is a variable or is an expression equiform to *E*;
(4) if any object *D* is a variable that belongs to *B*, any object *E* is *a*, and there are exactly as many *a* that precede *E* as words that belong to *B* and precede *D*, then *E* is propositional relative to *C*;
(5) if any object *D* is a word that belongs to *B*, any object *E* is an expression that belongs to *B* and is equiform to *D*, any object *F* is *a*, any object *G* is *a*, there are exactly as many *a* that precede *F* as words that belong to *B* and precede *D*,

lectures of my above-mentioned university course 'On foundations of the 'theory of deduction'', I remarked that the same scheme of definition could very easily be adapted to the theory of deduction, if brackets were used in this theory. At the same time I mentioned that, for the bracketless symbolism of Łukasiewicz, I did not know whether or how one could find such a definition equivalent to *Terminological explanation VI* but essentially independent of the idea of 'hereditary class' and that of the 'ancestral relation'.

19 The uncapitalized variable was here used in the plural.

and there are exactly as many a that precede G as words that belong to B and precede E, then G is an expression equiform to F.

Examples. (1) Axiom (L) is a consequence of Axiom (L), relative to Axiom (L), by substitution of the words that belong to Axiom (L).

(2) The class of words between the 18th and the 23rd word of Axiom (L) (i.e. words of Axiom (L) which follow the 18th and precede the 23rd word of Axiom (L)) is not a consequence of the class of words between the 3rd and the 8th word of Axiom (L) (i.e. words of Axiom (L) which follow the 3rd and precede the 8th word of Axiom (L)), relative to Axiom (L), by substitution of the words between the 18th and the 23rd word of Axiom (L) [C.(1)–(4)f., C.(5)n.f. (the 4th word of Axiom (L) is a word that belongs to the class of words between the 3rd and the 8th word of Axiom (L), the 7th word of Axiom (L) is an expression that belongs to the class of words between the 3rd and the 8th word of Axiom (L) and is equiform to the 4th word of Axiom (L), the 19th word of Axiom (L) is a word between the 18th and the 23rd word of Axiom (L), the 22nd word of Axiom (L) is a word between the 18th and the 23rd word of Axiom (L), there are exactly as many words between the 18th and the 23rd word of Axiom (L) that precede the 19th word of Axiom (L) as words that belong to the class of words between the 3rd and the 8th word of Axiom (L) and precede the 4th word of Axiom (L), and there are exactly as many words between the 18th and the 23rd word of Axiom (L) that precede the 22nd word of Axiom (L) as words that belong to the class of words between the 3rd and the 8th word of Axiom (L) and precede the 7th word of Axiom (L), but the 22nd word of Axiom (L) is not an expression equiform to the 19th word of Axiom (L))].

(3) The 2nd word of Axiom (L) is not a consequence of the 4th word of Axiom (L), relative to Axiom (L), by substitution of the 2nd word(s) of Axiom (L) [C.(1)–(3),(5)f., C.(4)n.f. (the 4th word of Axiom (L) is a variable that belongs to the 4th word of

Axiom (L), the 2nd word of Axiom (L) is 2nd word of Axiom (L), there are exactly as many 2nd words of Axiom (L) that precede the 2nd word of Axiom (L) as words that belong to the 4th word of Axiom (L) and precede the 4th word of Axiom (L), but the 2nd word of Axiom (L) is not propositional relative to Axiom (L))].

(4) Axiom (L) is not a consequence of the first word of Axiom (L), relative to Axiom (L), by substitution of Axiom(s)(L) [C.(1),(2),(4),(5)f., C.(3)n.f. (the first word of Axiom (L) is a word that belongs to the first word of Axiom (L), Axiom (L) is Axiom (L), there are exactly as many Axiom(s)(L) that precede Axiom (L) as words that belong to the first word of Axiom (L) and precede the first word of Axiom (L), but the first word of Axiom (L) neither is a variable nor is an expression equiform to Axiom (L))].

(5) The 2nd word of Axiom (L) is not a consequence of the class of words of Axiom (L) that precede the 3rd word of Axiom (L), relative to Axiom (L), by substitution of the 2nd word(s) of Axiom (L) [C.(1),(3)–(5)f., C.(2)n.f.].

(6) The first word of Axiom (L) is not a consequence of Axiom (L), relative to Axiom (L), by substitution of the words that belong to Axiom (L) [C.(2)–(5)f., C.(1)n.f.].

Terminological explanation VIII. I say of object A that it is a consequence of B by substitution, relative to C, if and only if, for some a,[20] A is a consequence of B, relative to C, by substitution of the A.

Examples. (1) Axiom (L) is a consequence of Axiom (L) by substitution, relative to Axiom (L).[21]

(2) The first word of Axiom (L) is not a consequence of Axiom (L) by substitution, relative to Axiom (L).

[20] The expression 'for some a' here corresponds to the *particular* quantifier '[$\exists a$]' of my 'symbolic' language.

[21] See example (1) of the preceding terminological explanation.

Terminological explanation IX. I say of object A that it is a consequence of B by detachment, relative to C, with respect to D and to E if and only if the following conditions are fulfilled:

(1) D is implicant of E in B;

(2) C is an expression equiform to D;

(3) A is an expression equiform to E.

Examples. (1) The 33rd word of Axiom (L) is a consequence, by detachment, of the class of words of Axiom (L) that follow the 30th word of Axiom (L), relative to the 32nd word of Axiom (L), with respect to the 32nd and to 33rd word of Axiom (L).

(2) Axiom (L) is not a consequence, by detachment, of the class of words of Axiom (L) that follow the 30th word of Axiom (L), relative to the 32nd word of Axiom (L), with respect to the 32nd and to the 33rd word of Axiom (L) [C.(1),(2)f., C.(3)n.f.].

(3) The 33rd word of Axiom (L) is not a consequence, by detachment, of the class of words of Axiom (L) that follow the 30th word of Axiom (L), relative to Axiom (L), with respect to the 32nd and to the 33rd word of Axiom (L) [C.(1),(3)f., C.(2)n.f.].

(4) Axiom (L) is not a consequence of Axiom (L) by detachment, relative to Axiom (L), with respect to Axiom (L) and to Axiom (L) [C.(2),(3)f., C.(1)n.f.].

Terminological explanation X. I say of object A that it is a consequence of B by detachment, relative to C, if and only if A is a consequence of B by detachment, relative to C, with respect to some expression and to some expression.

Examples. (1) The 33rd word of Axiom (L) is a consequence, by detachment, of the class of words of Axiom (L) that follow the 30th word of Axiom (L), relative to the 32nd word of Axiom (L).[22]

(2) Axiom (L) is not a consequence of Axiom (L) by detachment, relative to Axiom (L).

[22] See example (1) of the preceding terminological explanation.

Terminological explanation XI. I say of object A that it is a definition of B, relative to C, by means of D, and with respect to E if and only if the following conditions are fulfilled:

(1) D is propositional relative to C;

(2) the first of the words that belong to B is not a variable;

(3) if any object F is the same object as C or is a thesis of this system which thesis precedes C, and any object G is a word that belongs to F, then the first of the words that belong to B is not an expression equiform to G;

(4) if any object F is a word that belongs to B, any object G is a word that belongs to B, and F is an expression equiform to G, then F is the same object as G;

(5) if any object is a variable that belongs to D, then it is an expression equiform to some word that belongs to B;

(6) if any object is a word that belongs to B and follows the first of the words that belong to B, then it is an expression equiform to some variable that belongs to D;

(7) the implicant of B in the negate of E is an expression equiform to D;

(8) the implicant of D in the implicant of E in the negate of A is an expression equiform to B.

Examples. (1) If any object A is one of the equiform expressions '$NCCF\alpha\alpha NC\alpha F\alpha$', then it is a definition of the class of words of A that follow the 9th word of A, relative to Axiom (L), by means of the 6th word of A, and with respect to the class of words of A that follow the 6th word of A.

(2) If any object A is one of the equiform expressions '$NC\alpha F\alpha$', then it is not a definition of the class of words of A that follow the 3rd word of A, relative to Axiom (L), by means of the 3rd word of A, and with respect to A [C.(1)–(7)f., C.(8)n.f.].

(3) If any object A is one of the equiform expressions '$NCCF\alpha\alpha\alpha$', then it is not a definition of the class of objects that are either the 4th or 5th word of A, relative to Axiom (L),

by means of the 6th word of A, and with respect to the 7th word of A [C.(1)–(6),(8)f., C.(7)n.f.].

(4) If any object A is one of the equiform expressions '$NCCFN\alpha\alpha NC\alpha FN\alpha$', then it is not a definition of the class of words of A which follow the 10th word of A, relative to Axiom (L), by means of the 7th word of A, and with respect to the class of words of A which follow the 7th word of A [C.(1)–(5),(7),(8)f., C.(6)n.f. (the 12th word of A is a word that belongs to the class of words of A which follow the 10th word of A and follows the first of the words that belong to the class of words of A which follow the 10th word of A, but it is an expression equiform to no variable that belongs to the 7th word of A)].

(5) If any object A is one of the equiform expressions '$NCCF\alpha NC\alpha F$', then it is not a definition of the 9th word of A, relative to Axiom (L), by means of the 5th word of A, and with respect to the class of words of A that follow the 5th word of A [C.(1)–(4),(6)–(8)f., C.(5)n.f. (the 5th word of A is a variable that belongs to the 5th word of A, but it is an expression equiform to no word that belongs to the 9th word of A)].

(6) If any object A is one of the equiform expressions '$NCCF\alpha\alpha\alpha NC\alpha F\alpha\alpha$', then it is not a definition of the class of words of A which follow the 10th word of A, relative to Axiom (L), by means of the 7th word of A, and with respect to the class of words of A which follow the 7th word of A [C.(1)–(3),(5)–(8)f., C.(4)n.f. (the 12th word of A is a word that belongs to the class of words of A which follow the 10th word of A, the 13th word of A is a word that belongs to the class of words of A which follow the 10th word of A, the 12th word of A is an expression equiform to the 13th word of A, but the 12th word of A is not the same object as the 13th word of A)].

(7) If any object A is one of the equiform expressions '$NCCN\alpha\alpha NC\alpha N\alpha$', then it is not a definition of the class of words of A which follow the 9th word of A, relative to Axiom (L), by means of the 6th word of A, and with respect to the class of words of A which follow the 6th word of A [C.(1),(2),(4)–(8)f.,

C.(3)n.f. (Axiom (L) is the same object as Axiom (L) or is a thesis of this system which thesis precedes Axiom (L), the 11th word of Axiom (L) is a word that belongs to Axiom (L), but the first of the words that belong to the class of words of A which follow the 9th word of A is an expression equiform to the 11th word of Axiom (L))].

(8) If any object A is one of the equiform expressions '$NCC\iota\iota NC\iota\iota$', then it is not a definition of the 9th word of A, relative to Axiom (L), by means of the 5th word of A, and with respect to the class of words of A that follow the 5th word of A [C.(1),(3)–(8)f., C.(2)n.f.].

(9) If any object A is one of the equiform expressions '$NCCFFNCFF$', then it is not a definition of the 9th word of A, relative to Axiom (L), by means of the 5th word of A, and with respect to the class of words of A that follow the 5th word of A [C.(2)–(8)f., C.(1)n.f.].

Terminological explanation XII. I say of object A that it is a definition, relative to C, if and only if A is a definition of some expression, relative to C, by means of some expression, and with respect to some expression.[23]

Examples. (1) If any object A is one of the equiform expressions '$NCCF\alpha\alpha NC\alpha F\alpha$', then it is a definition, relative to Axiom (L).[24]

(2) Axiom (L) is not a definition, relative to Axiom (L).

I add further theses to this system of the theory of deduction, of which the first thesis is Axiom (L), only if at least one of the three following conditions is fulfilled:

(1) the added thesis is a consequence, by substitution, of one of the preceding theses of this system, relative to the last of the preceding theses of this system;

[23] Regarding this definition of definition of *op. cit.*, p. 11.

[24] See example (1) of the preceding terminological explanation.

(2) the added thesis is a consequence, by detachment, of one of the preceding theses of this system, relative to one of the preceding theses of this system;

(3) the added thesis is a definition, relative to the last of the preceding theses of this system.

The directives for constructing this system of the theory of deduction inclusive of definitions are thus complete.

INTRODUCTORY REMARKS TO THE CONTINUATION OF MY ARTICLE: 'GRUNDZÜGE EINES NEUEN SYSTEMS DER GRUNDLAGEN DER MATHEMATIK'†*

In 1929 I published in *Fundamenta Mathematicae* the beginning of the article referred to in the title.[1] The continuation of this article has not yet appeared in print. This fact derives from circumstances about which I wrote in 1930: "The succeeding part of the above-mentioned article in German which I had already submitted in 1929, and which had been accepted by the editors of *Fundamenta Mathematicae*, I withdrew for personal reasons. In the circumstances it is difficult for me to foresee whether, where, and when, I can find place for its publication."[2] The withdrawn manuscript remained for more than seven years in my desk, where it awaited a more auspicious occasion for its publication.

† This paper was to have appeared, under the title 'Einleitende Bemerkungen zur Fortsetzung meiner Mitteilung u.d.T. 'Grundzüge eines neuen Systems der Grundlagen der Mathematik'', in vol. 1 of the periodical *Collectanea Logica* (Warsaw, 1939), pp. 1–60. (For the fate of this journal see the bibliographical footnote to paper 5 of Storrs McCall [1967].) An offprint copy of Leśniewski's paper survives in the Harvard College Library, together with the continuation (§12) of his original article, which was also to have appeared in *Collectanea Logica*. Translated by W. Teichmann and S. McCall.

1 Stanisław Leśniewski, 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', *Fundamenta Mathematicae* 14 (1929), pp. 1–81.

2 Stanisław Leśniewski, 'Über die Grundlagen der Ontologie', *Comptes rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, Cl. iii, 23 (1930), p. 112. Paper presented by J. Łukasiewicz at the meeting of 22 May 1930.

The editors of *Collectanea Logica* have obligingly offered me space for the continuation of my article. I am naturally taking speedy advantage of this kind offer, submitting the abovementioned manuscript for the first volume of the new journal with only slight symbolic improvements and bibliographical additions. Because of the long delay in the printing of my article, and the change of place of publication, I have, for the convenience of the reader of this further part of the intended whole, and in accordance with the wishes of the editors, decided to preface it with a résumé of its beginning published in *Fundamenta Mathematicae*. Since I do not wish to ruin the architectonic structure of my article by including this résumé, I have decided to bring out these 'Introductory Remarks' separately. In them will be found, besides the résumé, various minor observations of an explicative, informative, and polemical nature connected with my article.

The already published part of my *Grundzüge eines neuen Systems der Grundlagen der Mathematik* consists of an introduction and the first eleven paragraphs of Section I, entitled 'The Foundations of Protothetic', The more important items contained in this section are summarized in what follows.

INTRODUCTION TO MY ARTICLE

The object of the paper is a succinct presentation of my system of the foundations of mathematics. This system consists of three deductive theories, whose union forms one of the possible bases of the whole structure of mathematics. The theories in question are the following: (1) What I call *Protothetic*, which is the result of a certain peculiar enlargement of the well-known theory which goes by the name of the 'propositional calculus', or 'theory of deduction'. (2) What I call *Ontology*, which forms a type of modernized 'traditional logic' and which most closely resembles in its content and power Schröder's 'logic of classes', regarded as including the theory of 'individuals'. (3) What I call *Mereology*, whose first outline was published by me in a work of 1916 entitled *Die Grundlagen der allgemeinen Mengenlehre. I.*

§1 OF THE ARTICLE

In 1912 Henry Maurice Sheffer showed that in the theory of deduction of Whitehead and Russell there could be defined two functions of two propositional variables, in terms of either of which as sole primitive the two primitive functions of Whitehead and Russell, namely alternation and negation, could be defined. One of these functions of Sheffer's is equivalent for all values of its variables to the function '$\sim (p \vee q)$'; the other to the function '$\sim p . \vee . \sim q$'. In 1916 J. G. P. Nicod built up the theory of deduction from a single axiom, which apart from variables contained only the sign for the second of Sheffer's functions. For this sign Nicod used the vertical stroke '|'.

In the definition of non-primitive functions in the theory of deduction, both Sheffer and Nicod make use of a special definitional sign of identity, which is not itself defined in terms of the primitive functions of the system. This fact makes it difficult to say that Nicod's theory of deduction is really based upon the sole primitive sign '|'. In 1921 I remarked that if one wishes really to base a system of the theory of deduction which contains definitions upon a single primitive term, one must write definitions using this primitive term without resorting to a special definitional sign of identity. In particular, if one were to make such a reform in Nicod's system, the definitions occurring in the system could be written, for example, in the form of an expression of the type

$$p \mid . q \mid q : \mid : q \mid . p \mid p .\because \mid \because . p \mid . q \mid q : \mid : q \mid . p \mid p,$$

which, as it is easy to verify, is equivalent to the corresponding equivalence '$p \equiv q$'.

In 1922 Alfred Tarski established that, by employing functional variables and quantifiers, all the familiar functions of the theory of deduction could be defined using the equivalence function as the sole primitive function. The central point in Tarski's arguments consists in the proof of a theorem stating that

$$[p, q] :: p . q . \equiv .\therefore [f] .\therefore p \equiv : [r] . p \equiv f(r) . \equiv . [r] . q \equiv f(r),$$

and in the demonstration that in systems of logistic in which the following thesis holds:

$$[p, q, f] : p \equiv q . f(p) . \supset . f(q)$$

the following thesis must also hold:

$$[p, q] .\therefore p . q . \equiv : [f] : p \equiv . f(p) \equiv f(q).$$

§2 OF THE ARTICLE

In 1922 I sketched my conception of 'semantic categories' and constructed for the fundamental mathematical theories, especially for 'Protothetic' and 'Ontology', directives for definition and inference adapted to this conception. In my axiomatic investigations concerning the directives of protothetic I concentrated upon the task of axiomatizing as simply as possible a system based upon the sign of equivalence as the only primitive term. Tarski's above-mentioned work had made such a system possible, but it had not yet been realized in fact.

§3 OF THE ARTICLE

Terminological note. I say of an expression X that it is an 'equivalence proposition' when it satisfies the following conditions: (a) X is a propositional variable or an equivalence; (b) if any Y is an equivalence forming a proper or improper part of the expression X, then each of the arguments of the equivalence Y is either a propositional variable or an equivalence.

I began the construction of an axiomatic system of protothetic, based on the sign of equivalence as the sole primitive term, with the construction of a weaker system. This system was to consist of all the equivalence propositions that can be proved in the ordinary theory of deduction. As the axioms of this weaker system I took the following two propositions:

A1. $p \equiv r \,.\equiv.\, q \equiv p :\equiv.\, r \equiv q$,

A2. $p \equiv .\, q \equiv r :\equiv: p \equiv q \,.\equiv r$.

(The thesis which here appears as axiom A2 had already been proved before the year 1922 by Jan Łukasiewicz.) As for the directives, I made use in this system (i) of the directive for substitution of equivalence propositions for variables in propositions that already belong to the system, and (ii) of the directive for detachment, permitting the addition to the system of a proposition S when the system already has both an equivalence A, whose second argument is equiform with S, and a proposition equiform with the first argument of the equivalence A. This system I call the system $\mathfrak{S}$.

§4 OF THE ARTICLE

I obtained a further stage of development of protothetic by considering the following question: with which axioms and directives would the system $\mathfrak{S}$ have to be strengthened, in order to obtain from it a system of the ordinary propositional calculus, completed by the addition of the thesis

$[p,q,f] \therefore p \equiv q \,.\supset: f(p) \,.\equiv.\, f(q)$?

The axioms of an enriched propositional calculus named $\mathfrak{S}_1$, constructed by me in answering this question, may be written in Peano-Whitehead-Russell style as follows:

Ax. I. $[p,q,r] \therefore p \equiv r \,.\equiv.\, q \equiv p :\equiv.\, r \equiv q$,

Ax. II. $[p,q,r] \therefore p \equiv .\, q \equiv r :\equiv: p \equiv q \,.\equiv r$,

Ax. III. $[g,p] :: [f] :: g(p,p) \,.\equiv \therefore [r] : f(r,r) \,.\equiv.\, g(p,p) :\equiv: [r] : f(r,r) \,.\equiv.\, g(p \equiv .\, [q] \,.\, q, p) :: \equiv .\, [q] \,.\, g(q,p)$.

In the authentic symbolism of protothetic these axioms have the following form (expressions of the type '$\phi(pq)$' here replace the corresponding expressions of the type '$p \equiv q$' in Ax. I–Ax. III):

$$\text{A1.}\quad \llcorner pqr \lrcorner\ \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big)\urcorner,$$

$$\text{A2.}\quad \llcorner pqr \lrcorner\ \ulcorner \phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big)\urcorner,$$

$$\text{A3.}\quad \llcorner gp \lrcorner \phi\Bigg(\llcorner f \lrcorner\ \ulcorner \phi\Bigg(g(pp)\phi\Bigg(\llcorner r \lrcorner\ \ulcorner \phi\big(f(rr)g(pp)\big)\urcorner\ \llcorner r \lrcorner\ \ulcorner \phi\Big(f$$
$$(rr)g\big(\phi(p\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big)\urcorner\Bigg)\Bigg)\urcorner\ \llcorner q \lrcorner\ \ulcorner g(qp)\urcorner\Bigg).$$

I obtained new propositions from propositions already belonging to the system by the use of six directives, which may be informally characterized as follows:

(α) The directive for detachment, as in the system $\mathfrak{S}$.

(β) The directive for substitution.

(γ) The directive for the 'distribution of the quantifier', which, if the system already contains a thesis T consisting of a universal quantifier Q and an equivalence A within the scope of the quantifier, permits the addition of a new thesis formed from the thesis T through the distribution — in a definite and in practice unambiguous way — of all or some variables occurring in the quantifier Q into quantifiers standing before the two arguments of the equivalence A.

(δ) The directive for the writing of definitions in the form of equivalences, the *definiendum* occurring as the first argument.

(ε) The directive for the writing of definitions consisting of a universal generalization of an equivalence, containing the *definiendum* as the first argument.

(ζ) The directive concerning universal quantifiers, which, in conjunction with the other rules, permits in practice the carrying out of all the familiar operations with these quantifiers.

§5 OF THE ARTICLE

The axiom A3 made it possible for me to employ, from a certain point in the system $\mathfrak{S}_1$, a method of proving or disproving propositions beginning with universal quantifiers containing propositional variables. This method makes appeal to certain corresponding propositions, already proved or disproved in the system, which are made up of propositions to be proved or disproved by the substitution for the propositional variables occurring in them of the expressions

'$\llcorner q \lrcorner \ulcorner q \urcorner$' and '$\phi(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner q \lrcorner \ulcorner q \urcorner)$',

which correspond in $\mathfrak{S}_1$ to the 'zero' and 'one' of the traditional propositional calculus. The axiom A3 was thus a sort of axiomatic correlate for the well-known method of verification of formulae of the propositional calculus, i.e., by substitution of 'zero' and 'one' for the variables contained in them. Regarded genetically, this axiom corresponded to the verification rules of Łukasiewicz's 1921 paper 'Two-valued Logic'.

I did not know how to prove, in the system $\mathfrak{S}_1$, a certain sequence of meaningful propositions which I consider as valid as the familiar theses of the ordinary propositional calculus. Because I wanted to construct a system in which this would be possible, which would contain the system $\mathfrak{S}_1$, and at the same time contain no meaningful proposition that I did not know either how to prove or how to disprove, in 1922 I completed the system $\mathfrak{S}_1$ by a new directive (η). This directive was formed on the pattern of one of Łukasiewicz's rules, except that, instead of propositional variables, it concerned all and only those variables occurring in $\mathfrak{S}_1$ which were *not* propositional variables. The directive (η) permitted me to join to the system a new thesis T, beginning with a universal quantifier governing variable function-signs of any 'semantic category', when the system already included all theses which could be obtained from T by substituting for its above-mentioned variable function-signs certain constant signs, the method of definition of the latter having been completely laid down for all semantic categories in advance. The completed system obtained from the system $\mathfrak{S}_1$ by the addition of directive (η) I call the system $\mathfrak{S}_2$. This is one of the many possible mutually equivalent systems of the theory that I name protothetic.

To formulate the directive (η) precisely I needed a complicated apparatus of numerous terminological explanations. This circumstance induced me to look for some other directive that would accomplish the same theoretical effect as the directive (η), and at the same time could be precisely formulated in a simpler way. I concentrated mainly on the question, whether the directive (η) could not be replaced by a directive which would permit a direct determination of the 'extensionality' of the different categories of functions occurring in protothetic.

§6 OF THE ARTICLE

When I formulated the directive (ζ) of the system $\mathfrak{S}_1$ I was aiming above all at the simplest method of guaranteeing the demonstrability in the system of propositions of the type:

$$[f,p] \therefore p \supset . [q] . f(q) : \equiv : [q] : p \supset . f(q),$$
$$[f,p] \therefore p \supset . [q,r] . f(q,r) : \equiv : [q,r] : p \supset . f(q,r),$$
$$[f,p] \therefore p \supset . [q,r,s] . f(q,r,s) : \equiv : [q,r,s] : p \supset . f(q,r,s),$$
etc.,

the signs here appearing in the expressions '$f(q)$', '$f(q,r)$', '$f(q,r,s)$', etc., being of any number or any semantic category. These propositions assured me of the possibility of carrying out all the familiar operations with universal quantifiers. Directive (ζ) was somewhat simplified by Tarski, and his result was further simplified by me. However, the whole question was finally liquidated in 1922 by Tarski, who showed that no special directive at all was necessary, each of the above-mentioned propositions being provable in the system $\mathfrak{S}_1$ with the help of the directives already in that system. Tarski's result was based, among other things, upon an earlier result obtained by me; namely the possibility of proving in the system $\mathfrak{S}_1$, without the help of directive (ζ),

(a) all theses corresponding to the theses of $\mathfrak{S}$;

(b) theses corresponding to the traditional theorems
'$[p] . 0 \supset p$' and '$[p] : 1 \supset p . \equiv p$';

(c) the availability in practice of the method of proving theses beginning with universal quantifiers containing propositional variables by appealing to theses, already proved in the system, which may be constructed out of the theses to be proved by substituting 'zero' and 'one' for the propositional variables of those theses.

In showing that the directive (ζ) is dispensable in the system $\mathfrak{S}_1$, Tarski also noted that analogous directives in the systems based on $\mathfrak{S}_1$ may be dispensed with. In particular this was true of my 'Ontology'. Tarski extended the method of proving the propositions mentioned at the beginning of this section without the help of directive (ζ) to all analogous meaningful propositions in which, in place of the expressions '$f(q)$', '$f(q,r)$', etc., expressions of any construction whatever occur.

§7 OF THE ARTICLE

On the basis of the system $\mathfrak{S}_2$ I could obtain, besides all theses of $\mathfrak{S}_1$ (including all theses of the usual theory of deduction) theses of the following two kinds:

(a) Theses which determined, in a universal form, the 'extensionality' of all functions occurring in the system, independent of the semantic category of expressions occurring in these functions. The following proposition will serve as an example:

$$[f, g] \therefore [p, q] : f(p, q) . \equiv . g(p, q) : \equiv : [\varphi] : \varphi\{f\} . \equiv . \varphi\{g\}.$$

(b) Theses which established, in a universal form, that every propositional function occurring in the system of the type '$\varphi\{f\}$', '$\varphi\{f, g\}$', '$\varphi\{f, g, h\}$', etc., in which at least one argument is not a proposition, is satisfied for all values of its arguments whenever the corresponding 'logical product' of certain propositions is satisfied. This logical product is the product of those propositions which are the values of the function in question for certain values of its arguments, such values for each semantic category being finite in number and specifiable in advance. (As values of arguments of different semantic categories there occur besides the 'zero' and 'one' of the traditional calculus those constant function-signs referred to above in the résumé of §5.) An example of this kind of thesis is:

$$[\varphi] \mathbin{:} [f] \,.\, \varphi\{f\} \,.\equiv.\, \varphi\{vr\} \,.\, \varphi\{as\} \,.\, \varphi\{\sim\} \,.\, \varphi\{fl\},$$

in which the expressions '*vr*', '*as*', '$\sim$' and '*fl*' are constant function-signs of propositional functions of one propositional argument.

The problem I mentioned in the résumé of §5, namely whether the directive (η) of the system $\mathfrak{S}_2$ could not be replaced by an 'extensionality directive', was for me in effect equivalent to the problem, whether the addition to the system $\mathfrak{S}_1$ of all theses of kind (a) made it possible to obtain in this system, without the use of directive (η), all propositions of kind (b). This question was answered in the affirmative by Tarski, who in 1922 sketched a general method of proving in $\mathfrak{S}_1$ individual propositions of kind (b), given corresponding propositions of kind (a). For reasons already given I decided to replace the directive (η) by a new 'extensionality directive (η^*). The system of protothetic based on the axioms A1–A3 and the directives (α), (β), (γ), (δ), (ε), and (η^*) I call the system $\mathfrak{S}_3$.

§8 OF THE ARTICLE

From the beginning I took pains to formulate the directives of protothetic in such a way that they could be easily adapted to systems constructed on the same pattern but on the basis of different primitive terms. In this connexion I took account of the fact that the system $\mathfrak{S}_2$ could be transformed almost automatically into a system of protothetic in which the implication sign occurs as the sole primitive term. The axioms and directives of this system may be given in outline as follows:

A. Directives. (α_1) The directive for detachment, permitting the addition to the system of a proposition S when the system already contains both a conditional K, whose consequent is equiform with S, and a proposition equiform with the antecedent of K.

(β_1) The directive for substitution.

(γ_1) The directive for the distribution of the quantifier, analogous to rule (γ) of the system $\mathfrak{S}_2$ except in that it concerns conditional propositions rather than equivalences.

(δ_1) and (ε_1) The directives which permit the writing of definitions analogous to those allowed for by the directives (δ) and (ε) of the system $\mathfrak{S}_2$. Unlike the definitions of directives (δ) and (ε), however, these definitions are expressed in the form not of an equivalence, nor of an equivalence preceded by a universal quantifier, but in the form of any other function stipulated in advance for all definitions, this function being expressed in terms of the implication sign and being equivalent either to the equivalence in question or to the equivalence preceded by a universal quantifier.

(ζ_1) The directive analogous to and serving the same purpose as the directive (ζ) of $\mathfrak{S}_2$, except that it is framed in terms of conditional propositions instead of equivalences.

(η_1) The directive exactly analogous to (η) of $\mathfrak{S}_2$.

B. Axioms. (I) Any axiom set consisting of theses of the classical theory of deduction which contain no other constants besides the implication sign, and from which, using the directives of the system, the whole of the ordinary theory of deduction can be derived.

(II) An axiom analogous to the axiom A3 of $\mathfrak{S}_2$. This axiom makes possible in the system a method of proving or of disproving propositions beginning with universal quantifiers containing propositional variables. The method makes appeal to corresponding propositions, already proved or disproved in the system, which can be constructed out of the propositions to be proved or disproved by substituting for the propositional variables they contain the expressions '$\llcorner q \lrcorner \ulcorner q \urcorner$' and '$\phi\!-\!(\llcorner q \lrcorner \ulcorner q \urcorner \llcorner q \lrcorner \ulcorner q \urcorner)$', corresponding to the 'zero' and 'one' of the traditional propositional calculus. (The sign '$\phi\!-$' appearing in the second expression plays the role of the implication sign in my symbolism.)

The axiom in question could, for example, take the following form:

$$\llcorner gpq \lrcorner \ulcorner \phi\!-\!\Big(g(pp)\phi\!-\!\Big(g\big(\phi\!-\!(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)g(qp)\Big)\Big) \urcorner .$$

With the discovery of the system $\mathfrak{S}_3$ it became clear that a system of protothetic based on the implication sign could have simpler directives if it were modelled on $\mathfrak{S}_3$ rather than $\mathfrak{S}_2$. This would involve discarding the directive (ζ_1), and replacing (η_1) by (η_1^*), the latter 'extensionality directive' being analogous to the directive (η^*). A certain system of this kind, which I constructed on the basis of axioms of the types (I) and (II) above, and directives of types (α_1),(β_1),(γ_1), (δ_1), (ε_1), and (η_1^*), I named the system $\mathfrak{S}_4$.

In 1922 Tarski established that, however many axioms a given set A sufficient for $\mathfrak{S}_4$ may have, it may be replaced by a set of only two axioms without altering the directives of the system. These axioms are (a) the propositions '$\llcorner pq \lrcorner \ulcorner \phi\!-\!\big(p\phi\!-\!(qp)\big) \urcorner$', and (b) the 'logical product' of all propositions belonging to A which are distinct form (a), the 'logical product' of the propositions 'P' and 'Q' being taken to be a proposition of the type

$$\llcorner r \lrcorner \ulcorner \phi\!-\!\Big(\phi\!-\!\big(P\phi\!-\!(Qr)\big)r\Big) \urcorner .$$

In conversation with Tarski I then conjectured that his result could be improved upon, and that for protothetic two axioms could suffice, one of which would have approximately the same form as the proposition given as an example under (II) above, while the other would be a simple thesis of the ordinary theory of deduction. In the same year (1922) Tarski established that:

(A) For the construction of a system of protothetic with the directives of the system $\mathfrak{S}_4$ the following two axioms suffice:

(1) $\llcorner pq \lrcorner \ulcorner \phi\!-\left(p\phi\!-(qp)\right)\urcorner$,

(2) $\llcorner pqrf \lrcorner \ulcorner \phi\!-\left(f(rp)\phi\!-\left(f\left(r\phi\!-(p\llcorner s\lrcorner \ulcorner s\urcorner)\right)f(rq)\right)\right)\urcorner$.

(B) A system of protothetic could also be based on a single axiom if one took, besides the directives (α_1),(β_1),(γ_1), (η_1^*) of $\mathfrak{S}_4$, new directives (δ_1^*) and (ε_1^*) in place of (δ_1) and (ε_1). These new directives would permit the introduction into the system of definitions in the form of two converse conditionals corresponding to one equivalence (directive (δ_1^*)), or two such conditionals preceded by universal quantifiers (directive (ε_1^*)). The sole axiom of the system might, for example, be axiom (2) of A above.

§9 OF THE ARTICLE

In 1923 Tarski noted that however many axioms a given set A sufficient for $\mathfrak{S}_3$ may have, it may be replaced by a set of only two axioms without altering the directives of the system. Of these one is the thesis

$$\llcorner pq \lrcorner \ulcorner \phi\left(\phi(pq)\phi(qp)\right)\urcorner,$$

while the other is the 'logical product', expressed in terms of the function '$\phi(pq)$', of all propositions belonging to A which are distinct from the above thesis. Tarski also established at the same time that such a 'logical product' of the axioms A1–A3 introduced in the résumé of §4 could take the form of the following expression:

$$\llcorner hs \lrcorner \ulcorner \phi\left(h(Ps)\phi\left(h\left(\llcorner kt \lrcorner \ulcorner \phi\left(k(Qt)\phi\left(k(Rt)Q\right)\right)\urcorner s\right)P\right)\right)\urcorner$$

where the letters 'P', 'Q', and 'R' are to be replaced by the axioms A1, A2, and A3.

§10 OF THE ARTICLE

In the same year 1923 I observed in addition that if I replaced the directives (δ) and (ε) of $\mathfrak{S}_3$ by the corresponding directives (δ^*) and (ε^*), prescribing that the *definiendum* of a definition should occur as the second rather than as the first argument of the stipulated equivalence, I would obtain a system of protothetic equivalent to the system $\mathfrak{S}_3$. The system based upon axioms (A1)–(A3) and directives (α), (β), (γ), (δ^*), (ε^*), and (η^*) I call $\mathfrak{S}_5$. I showed that with the help of its directives a system equivalent to it could be based upon a single axiom. This axiom consists of the 'logical product', after the pattern of the expression

$$\llcorner fp \lrcorner \ulcorner \phi\Big(f(Pp)\phi\big(f(Qp)P\big)\Big)\urcorner,$$

of two propositions; one of these — corresponding to 'P' in the above expression — being the proposition

$$\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big)\urcorner,$$

while the other — corresponding to 'Q' — is again one of those 'logical products' which occur as the second axiom of the axiom sets constructed by the method of Tarski referred to in the résumé of §9. I noted at the same time that it would be simple to produce such an axiom, sufficient as a basis for protothetic, if one retained the directives (β), (γ), (δ), (ε), and (η^*) but replaced (α) by (α^*). This new directive would permit the addition to the system of a proposition S, when there already belonged to the system both an equivalence A, whose first argument was equiform with S, and a proposition equiform with the second argument of A. I know of no proposition which would serve as the sole axiom of a system of protothetic based either on the directives (α), (β), (γ), (δ), (ε), and (η^*), or the directives (α^*), (β), (γ), (δ^*), (ε^*), and (η^*).

In 1925 Tarski gave a method of reducing to a single axiom the axiomatic basis of any system of protothetic with the directives of $\mathfrak{S}_4$ and the implication sign as its primitive term. In connexion with this he also showed how systems of the classical theory of deduction, when they contain the implication sign among their primitive terms, can be based upon a single axiom.

§11 OF THE ARTICLE

The single axiom of the system of protothetic with the directives of $\mathfrak{S}_5$ was, in the years 1923–1926, successively simplified by me and Mordchaj Wajsberg. The following axiom of such a protothetic, devised by me in 1926, has not yet, as far as I know, been further shortened by anybody:*

$$\llcorner fpqrst \lrcorner \ \phi \ulcorner \Bigg(\phi(pq) \llcorner g \lrcorner \ \phi \ulcorner \Bigg(f\Big(pf(p \llcorner u \lrcorner \ulcorner u \urcorner)\Big) \phi \Bigg(\llcorner u \lrcorner \ulcorner f(qu)$$

$$\urcorner \phi \Bigg(g\Big(\phi\big(\phi(rs)t\big)q\Big) g\Big(\phi\big(\phi(st)r\big)p\Big)\Bigg)\Bigg)\Bigg) \urcorner \Bigg) \urcorner .$$

The central position in §11 of my article is occupied by the directives of the system $\mathfrak{S}_5$, which were formulated with all the precision of which I was capable. In the statement of these directives there occurs a sequence of terms whose meaning I first establish with the help of forty-nine 'terminological explanations'. As I see no possibility of summarizing these explanations, I am compelled at this point to refer the reader to my original article (pp. 59–76). In my résumé of the contents of §11 I shall confine myself to a few general remarks.

I first expounded the directives of the system $\mathfrak{S}_5$ in my lectures on 'Logistic' in the academic years 1924–1925 and 1925–1926. These directives were given in considerably simpler form in my lectures on the 'Foundations of Ontology' in the academic year 1926–1927. For some of the simplifications in the exposition

* [*Ed. note.* Sobociński in 1945 reduced protothetic to a single axiom containing only fifty-four signs. See E. C. Luschei, *The Logical Systems of Leśniewski*, Amsterdam 1962, §6.2.2, and the references therein.]

of the directives of protothetic that I introduced at that time I am indebted to Adolf Lindenbaum.

In the final statement of the directives of the system $\mathfrak{S}_5$, I unified in suitably 'organic' fashion directives (δ^*) and (ε^*) in a single directive for writing definitions. After this simplification the whole system of directives could be assembled in the following brief schema:

(1) the directive for writing definitions (union of (δ^*) and (ε^*));

(2) the directive for distribution of the quantifier (γ);

(3) the directive for detachment (α);

(4) the directive for substitution (β);

(5) the extensionality directive (η^*).

The directives of the system $\mathfrak{S}_5$ presuppose no special shape for the constant terms of the system as opposed to the variables. All signs, with the exception of parentheses and the signs '$\llcorner$', '$\lrcorner$', '$\ulcorner$', and '$\urcorner$', can occur in the system either as constants or variables, the character they have in such and such a formula depending on the variety and position of the quantifiers contained in the formula in question.

The directives of the system presuppose no special shape for the constants or variable signs of one semantic category as contrasted with the signs of another.

Function signs may be placed, according to the directives of the system, only in front of the parentheses enclosing their arguments.

The different arguments need not, according to the directives, be separated from each other by commas.

The directives allow no possibility of introducing into the system any kind of quantifier other than the previously mentioned universal quantifier governing any number of variables.

The directives do not allow us to obtain in the system any thesis containing a universal proposition of the type

$$\llcorner ab\ldots\lrcorner \ulcorner \llcorner kl\ldots\lrcorner \ulcorner f(ab\ldots kl\ldots)\urcorner\urcorner$$

In the cases in which we would normally have to deal with propositions of this type we are confronted in my system only with corresponding expressions of the type

$$\llcorner ab \ldots kl \ldots \lrcorner \ulcorner f(ab \ldots kl \ldots) \urcorner,$$

in which, in view of a sufficiently 'liberal' formulation of the directives of the system, the variables occurring in the quantifier may easily be permuted.

The system's directive for substitution, while it permits different substitutions for variables, does not allow anything to be substituted for a whole expression of the type '$f(ab\ldots)$'.

In the last long footnote to §11 of my article I state that, among those works which take as their task the construction of the foundations of mathematics, I do not know of a single one which establishes, in a way that raises no doubt as to interpretation, a combination of rules which is sufficient for the derivation of all theses effectively recognized in the system, and which does not at the same time lead to a contradiction in some way or other not foreseen by the author. I construct explicitly two such contradictions, unforeseen by the authors, in the systems of Leon Chwistek (*The Theory of Constructive Types*) and of J. von Neumann (*Zur Hilbertschen Beweistheorie*).

I pass now to the minor observations, mentioned at the beginning of this paper, with which I would like to supplement the résumé I have just given of the part of my article published in *Fundamenta Mathematicae*.

SUPPLEMENTARY REMARK I

In the course of lectures, entitled 'Introduction to Mathematical Logic', which I delivered in the University of Warsaw in the academic year 1933–1934, I remarked, as it is in fact easy to verify, that

$$p \equiv q \,.\equiv \therefore p \,|\, q \,.\, | \,\colon p \,|\, p \,.\, | \,.\, q \,|\, q,$$

so that in a system of the theory of deduction based upon the primitive term '|' definitions can be written by means of corresponding formulae of the type '$p \mid q . \mid : p \mid p . \mid . q \mid q$'. Such formulae are considerably simpler than those mentioned in §1 of my article of the type

$$p \mid . q \mid q : \mid : q \mid . p \mid p .\cdot. \mid .\cdot. p \mid . q \mid q : \mid : q \mid . p \mid p.$$

SUPPLEMENTARY REMARK II

The sketch of the argument of §3, which aims at proving that the system 𝔖 consists of all equivalence propositions provable in the usual theory of deduction, begins with a direct derivation in the system 𝔖 of the theses T1–T79. Of these only T7, T19–T21, T69, T70, and T79 are necessary for the continuation of my argument. All the others are only auxiliary premises for them.

In 1929 Łukasiewicz considerably shortened my deductions. He deduced in the system 𝔖 the theses T7, T19–T21, T69, T70 and a correlate of my thesis T79 with the help of forty-eight successive applications of the system's directives instead of my seventy-nine applications. The thesis which I call here a correlate of the thesis T79 differs from this thesis in variables only, and, for my proof, this difference is insignificant. I give here Łukasiewicz's derivation copied from the author's original. Łukasiewicz's theses are indicated by the signs 'Ł1', 'Ł2', etc., to 'Ł48'. The thesis Ł48 is the correlate of my thesis T79, mentioned above. A1 and A2 are, of course, the axioms of the system 𝔖 , introduced above.

A1. $p \equiv r . \equiv . q \equiv p : \equiv . r \equiv q,$

A2. $p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r,$

Ł1 (my thesis T1). $q \equiv r . \equiv . r \equiv q : \equiv . r \equiv r$ $\quad$ A1 $\left[\frac{q}{p}, \frac{r}{q}\right]$,

Ł2 (my T2). $r \equiv . q \equiv r : \equiv : r \equiv q . \equiv r .\cdot. \equiv : q \equiv r . \equiv . r \equiv q$

$\left[\text{A1} \frac{r}{p}, \frac{r \equiv q}{q}, \frac{q \equiv r}{r}\right]$,

Ł3 (T4). $r \equiv . q \equiv r : \equiv : r \equiv q . \equiv r$ $\quad$ $\left[\text{A2} \frac{r}{p}\right]$, Ł4 (T7). $q \equiv r . \equiv . r \equiv q$ [Ł2, Ł3],

Ł5 (T12). $p \equiv r . \equiv . q \equiv p : \equiv . r \equiv q \therefore \equiv \therefore r \equiv q . \equiv : p \equiv r . \equiv$
$. q \equiv p$ $\left[Ł4 \frac{p \equiv r . \equiv . q \equiv p}{q}, \frac{r \equiv q}{r}\right]$,

Ł6 (T19). $r \equiv r$ [Ł1, Ł4],

Ł7 (T20). $q \equiv r . \equiv . q \equiv r$ $\left[Ł6 \frac{q \equiv r}{r}\right]$,

Ł8 (T21). $r \equiv r . \equiv . r \equiv r$ $\left[Ł7 \frac{r}{q}\right]$,

Ł9 (T22). $r \equiv q . \equiv : p \equiv r . \equiv . q \equiv p$ [Ł5, A1],

Ł10. $q \equiv r . \equiv . r \equiv q : \equiv \therefore p \equiv . q \equiv r : \equiv : r \equiv q . \equiv p$
$\left[Ł9 . \frac{r \equiv q}{q}, \frac{q \equiv r}{r}\right]$,

Ł11. $p \equiv . q \equiv r : \equiv : r \equiv q . \equiv p$ [Ł10, Ł4],

Ł12. $p \equiv . q \equiv : \equiv : p \equiv q . \equiv r \therefore \equiv \therefore r \equiv . p \equiv q : \equiv : p \equiv .$
$q \equiv r$ $\left[Ł11 \frac{p \equiv . q \equiv r}{p}, \frac{p \equiv q}{q}\right]$,

Ł13. $p \equiv r . \equiv . q \equiv p : \equiv . q \therefore \equiv \therefore q \equiv r . \equiv : p \equiv r . \equiv . q \equiv p$
$\left[Ł11 \frac{p \equiv r . \equiv . q \equiv p}{p}, \frac{r}{q}, \frac{q}{r}\right]$,

Ł14. $r \equiv . p \equiv q : \equiv : p \equiv . q \equiv r$ [Ł12, A2],

Ł15. $q \equiv r . \equiv : p \equiv r . \equiv . q \equiv p \therefore \equiv \therefore p \equiv r . \equiv : q \equiv p . \equiv .$
$q \equiv r$ $\left[Ł14 \frac{p \equiv r}{p}, \frac{q \equiv p}{q}, \frac{q \equiv r}{r}\right]$,

Ł16. $p \equiv r . \equiv : q \equiv p . \equiv . q \equiv r \therefore \equiv \therefore q \equiv p . \equiv : q \equiv r . \equiv .$
$p \equiv r$ $\left[Ł14 \frac{q \equiv p}{p}, \frac{q \equiv r}{q}, \frac{p \equiv r}{r}\right]$,

Ł17. $s \equiv : p \equiv q . r \therefore \equiv : p \equiv q . \equiv . r \equiv s$ $\left[Ł14 \frac{p \equiv q}{p}, \frac{r}{q}, \frac{s}{r}\right]$,

Ł18. $q \equiv r . \equiv : p \equiv r . \equiv . q \equiv p$ [Ł13, A1],

Ł19. $p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r \therefore \equiv :: s \equiv : p \equiv q . \equiv r \therefore \equiv \therefore$
$p \equiv . q \equiv r : s$ $\left[Ł18 \frac{s}{p}, \frac{p \equiv . q \equiv r}{q}, \frac{p \equiv q . \equiv r}{r}\right]$,

Ł20 (T28). $p \equiv r . \equiv : q \equiv p . \equiv . q \equiv r$ [Ł15, Ł18],

Ł21. $s \equiv . p \equiv r : \equiv \therefore q \equiv s . \equiv : q \equiv . p \equiv r$ $\left[Ł20 \frac{s}{p}, \frac{p \equiv r}{r}\right]$,

Ł22. $q \equiv r . \equiv s : \equiv : r \equiv q . \equiv s \therefore \equiv :: p \equiv : q \equiv r . \equiv s \therefore \equiv \therefore$
$p \equiv : r \equiv q . \equiv s$ $\left[Ł20 \frac{q \equiv r . \equiv s}{p}, \frac{p}{q}, \frac{r \equiv q . \equiv s}{r}\right]$,

Ł23. $p \equiv . q \equiv r : \equiv : q \equiv p . \equiv r \therefore \equiv :: s \equiv : p \equiv . q \equiv r \therefore \equiv \therefore s \equiv : q \equiv p . \equiv r$ [Ł20 $\frac{p\equiv.q\equiv r}{p}, \frac{s}{q}, \frac{q\equiv p.\equiv r}{r}$],

Ł24. $q \equiv r . \equiv : p \equiv r . \equiv . q \equiv p \therefore \equiv :: s \equiv . q \equiv r : \equiv \therefore s \equiv : p \equiv r . \equiv . q \equiv p$ [Ł20 $\frac{q\equiv r}{p}, \frac{p\equiv r.\equiv.q\equiv p}{r}$],

Ł25. $s \equiv : p \equiv . q \equiv r \therefore \equiv \therefore s \equiv : q \equiv p . \equiv r :: \equiv :\cdot: t \equiv \therefore s \equiv : p \equiv . q \equiv r :: \equiv :: t \equiv \therefore s \equiv : q \equiv p . \equiv r$ [Ł20 $\frac{s\equiv:p\equiv.q\equiv r}{p}, \frac{t}{q}, \frac{s\equiv:q\equiv p.\equiv r}{r}$],

Ł26. $p \equiv . q \equiv r : s \therefore \equiv : p \equiv q . \equiv . r \equiv s :: \equiv :\cdot: t \equiv \therefore p \equiv . q \equiv r : \equiv s :: \equiv \therefore t \equiv : p \equiv q . \equiv . r \equiv s$ [Ł20 $\frac{p\equiv.q\equiv r:\equiv s}{p}, \frac{t}{q}, \frac{p\equiv q.\equiv.r\equiv s}{r}$],

Ł27 (T44). $q \equiv p . \equiv : q \equiv r . \equiv . p \equiv r$ [Ł16, Ł20],

Ł28. $q \equiv r . \equiv . r \equiv q : \equiv \therefore q \equiv r . \equiv s : \equiv : r \equiv q . \equiv s$ [Ł27 $\frac{r\equiv q}{p}, \frac{q\equiv r}{q}, \frac{s}{r}$],

Ł29. $s \equiv : p \equiv q . \equiv r \therefore \equiv \therefore p \equiv . q \equiv r : \equiv s :: \equiv :\cdot: s \equiv : p \equiv q . \equiv r \therefore \equiv : p \equiv q . \equiv . r \equiv s :: \equiv :: p \equiv . q \equiv r : \equiv s \therefore \equiv : p \equiv q . \equiv . r \equiv s$ [Ł27 $\frac{p\equiv.q\equiv r:\equiv s}{p}, \frac{s\equiv:p\equiv q.\equiv r}{q}, \frac{p\equiv q.\equiv.r\equiv s}{r}$],

Ł30. $q \equiv r . \equiv s : \equiv : r \equiv q . \equiv s$ [Ł28, Ł4],

Ł31. $p \equiv : q \equiv r . \equiv s \therefore \equiv \therefore p \equiv : r \equiv q . \equiv s$ [Ł22, Ł30],

Ł32 (T55). $p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r \therefore \equiv \therefore p \equiv . q \equiv r : \equiv : q \equiv p . \equiv r$ [Ł31 $\frac{p\equiv.q\equiv r}{p}, \frac{p}{q}, \frac{q}{r}, \frac{r}{s}$],

Ł33 (T56). $s \equiv . p \equiv r : \equiv \therefore q \equiv s . \equiv : p \equiv q . \equiv r :: \equiv :: s \equiv . p \equiv r : \equiv \therefore s \equiv q . \equiv : p \equiv q . \equiv r$ [Ł31 $\frac{s\equiv.p\equiv r}{p}, \frac{s}{r}, \frac{p\equiv q.\equiv r}{s}$],

Ł34 (T59). $p \equiv . q \equiv r : \equiv : q \equiv p . \equiv r$ [Ł32, A2],

Ł35. $s \equiv : p \equiv . q \equiv r \therefore \equiv \therefore s \equiv : q \equiv p . \equiv r$ [Ł23, Ł34],

Ł36. $t \equiv \therefore s \equiv : p \equiv . q \equiv r :: \equiv :: t \equiv \therefore s \equiv : q \equiv p . \equiv r$ [Ł25, Ł35],

Ł37. $s \equiv . p \equiv r : \equiv \therefore q \equiv s . \equiv : q \equiv . p \equiv r :: \equiv :: s \equiv . p \equiv r : \equiv \therefore q \equiv s . \equiv : p \equiv q . \equiv r$ [Ł36 $\frac{q}{p}, \frac{p}{q}, \frac{q\equiv s}{s}, \frac{s\equiv.p\equiv r}{t}$],

Ł38 (T69). $s \equiv . p \equiv r : \equiv \therefore q \equiv s . \equiv : p \equiv q . \equiv r$ [Ł37, Ł21], Ł39 (T70). $s \equiv . p \equiv r : \equiv \therefore s \equiv q . \equiv : p \equiv q . \equiv r$ [Ł33, Ł38],

Ł40. $s \equiv : p \equiv q . \equiv r \therefore \equiv \therefore p \equiv . q \equiv r : \equiv s$ [Ł19, A2],

Ł41. $s \equiv : p \equiv q . \equiv r \therefore \equiv : p \equiv q . \equiv . r \equiv s :: \equiv :: p \equiv . q \equiv r : \equiv s \therefore \equiv : p \equiv q . \equiv . r \equiv s$ [Ł29, Ł40],

Ł42. $p \equiv . q \equiv r : \equiv s \therefore \equiv : p \equiv q . \equiv . r \equiv s$ [Ł41, Ł17],

Ł43. $t \equiv \therefore p \equiv . q \equiv r : \equiv s :: \equiv \therefore t \equiv : p \equiv q . \equiv . r \equiv s$ [Ł26, Ł42],

Ł44. $s \equiv . q \equiv r : \equiv :: t \equiv : p \equiv r . \equiv . q \equiv p \therefore \equiv . s \equiv t :: \equiv :: s \equiv . q \equiv r : \equiv \therefore t \equiv . p \equiv r : \equiv : q \equiv p . \equiv . s \equiv t$

$\left[\text{Ł}43 \frac{t}{p}, \frac{p\equiv r}{q}, \frac{q\equiv p}{r}, \frac{s\equiv t}{s}, \frac{s\equiv . q\equiv r}{t}\right]$,

Ł45. $s \equiv . q \equiv r : \equiv \therefore s \equiv : p \equiv r . \equiv . q \equiv p$ [Ł24, Ł18],

Ł46. $s \equiv . q \equiv r : \equiv \therefore s \equiv : p \equiv r . \equiv . q \equiv p :: \equiv \because s \equiv . q \equiv r : \equiv :: t \equiv : p \equiv r . \equiv . q \equiv p \therefore \equiv . s \equiv t$

$\left[\text{Ł}45 \frac{t}{p}, \frac{s}{q}, \frac{p\equiv r . \equiv . q\equiv p}{r}, \frac{s\equiv . q\equiv r}{s}\right]$,

Ł47. $s \equiv . q \equiv r : \equiv :: t \equiv : p \equiv r . \equiv . q \equiv p \therefore \equiv . s \equiv t$ [Ł46, Ł45],

Ł48. $s \equiv . q \equiv r : \equiv \therefore t \equiv . p \equiv r : \equiv : q \equiv p . \equiv . s \equiv t$ [Ł44, Ł47],

SUPPLEMENTARY REMARK III

Here I would like to comment, without any pretence to exactitude, on the form I used for the constant function signs of propositional functions of the type '$f(p)$' and '$f(pq)$' with one and with two propositional arguments. What leads me to make these comments is the fact that I do not set down these function signs at random, but rather construct them according to a general scheme. To avoid possible misunderstandings, I remark that the scheme in question possesses an entirely 'unofficial' character and is in no way a consequence of the directives of any of my systems.

Each one of my constant signs for propositional functions of the type '$f(p)$' or '$f(pq)$' with one or with two propositional arguments is a sign which consists of a 'basic outline', and possibly also of an indicator of the type '1', '2', '3', etc., placed under the basic outline. The basic outline for a function of one argument has always one of the following four forms: '⊢⊣', '⊣', '⊢', '–'; the basic outline for a function of two arguments, one of the following sixteen forms: '-ϕ-', '-ϙ-', '-ϕ', 'ϕ-', '-ȯ-', 'ϙ-', '-ϙ', 'ϕ', '-o-', '-ȯ', 'ȯ-', 'ϙ', '-o', 'o-', 'ȯ', 'o'. In the basic outline for a function of one argument, the perpendicular stroke on the left occurs if and only if the given function with a false argument becomes a true proposition; the perpendicular stroke on the right occurs if and only if the function with a true argument becomes a true proposition. In the basic outline for a function of two arguments, the left-hand horizontal bar occurs if and only if the given function with a true first and false second argument becomes a true proposition. The upper vertical stroke occurs if and only if the function with two false arguments becomes a true proposition. The right-hand horizontal bar occurs if and only if the function with a false first and true second argument becomes a true proposition. The lower vertical stroke occurs if and only if the function with two true arguments becomes a true proposition.

By considering the principles for the construction of function signs of propositional functions of the type '$f(p)$' and '$f(pq)$' with one and with two propositional arguments, we can quite easily construct from the basic outlines the familiar two-valued truth tables of the corresponding functions, and vice versa. We can establish further elementary correlations between the basic outlines and the logical characteristics of the corresponding functions of the following kind:

(1) Two function signs have the same basic outline, as for example the signs '⊢' and '$\underset{1}{⊢}$', or the signs $\underset{1}{\text{-ϕ-}}$ and $\underset{2}{\text{-ϕ-}}$, if and only if the functions corresponding to these signs are equivalent to one another for the same values of corresponding arguments.

(2) The basic outline of the sign of any function F is 'contained' in the basic outline of the sign of a function G, as for example the basic outline of the sign '–' in the basic outline of '$\underset{1}{⊢}$', or the basic outline of '$\underset{2}{\text{ϕ}}$' in the basic outline of '-ϕ-', if and

only if G holds for any given values of its arguments provided F holds for the same values of corresponding arguments.

(3) The basic outline of the sign of the function F 'complements' the basic outline of the sign of any other function G, as for example the outline '⊢' complements the outline '⊣', or the outline 'ϙ' complements the outline '-ϕ-'$^{1}_{2}$, if and only if^{2} the function F is equivalent to the negation of the function G for the same values of corresponding arguments.

(4) The basic outline of the sign of any function F is the 'sum' of the outlines of the signs of two functions G and H, as for example the outline '⊢⊣' is the sum of the outlines '⊢' and '⊣', or the outline '-ϕ-of the outlines 'ϕ-'$^{2}_{1}$ and '-ϙ-', if and only if the function F is equivalent to the logical sum of the functions G and H for the same values of corresponding arguments.

(5) The basic outline of the sign of any function F is the 'intersection' of the outlines of the signs of the functions G and H, as for example the outline '⊢' is the intersection of the outlines '⊢⊣' and '⊢', or the outline 'o' of the outlines 'ϕ' and '-o-', if and only if the function F is equivalent to the logical product of the functions G and H for the same values of corresponding arguments.

SUPPLEMENTARY REMARK IV

The single axiom of protothetic constructed according to the directives of the system $\mathfrak{S}_5$, which I gave in the résumé of §11 of my article, has withstood eleven years of research by me and others without having been shortened by so much as a single word. However, even to this day ideas for the solution of this problem continue to be brought to my attention. In the interests of the reader, who may wish to do independent research on this matter, I shall give a survey of the most important theoretical considerations which contribute to the evolution of a single axiom of protothetic and which illustrate different auxiliary devices. I am impelled to do this without waiting to give a proper discussion, in the continuation of my article, of problems connected with the origin of this axiom and with its further minor alterations.

(1) Reflecting of Tarski's thesis, mentioned above in the résumé of §1 of my article, which states that

$$[p, q] \therefore p \,.\, q \equiv : [f] : p \equiv . f(p) \equiv f(q),$$

I remarked in 1923 that in the propositional calculus, as completed by the thesis

$$[p, q, f] : p \equiv q \,.\, f(p) \,.\, \supset . f(q),$$

and consequently also in the system $\mathfrak{S}_1$, the following related thesis holds true:

$$[p, q, r] \therefore p \,.\, q \equiv r \,.\, \equiv : [f] : p \equiv . f(q) \equiv f(r).$$

From this I concluded that in the theories in question the following thesis also holds:

$$[p, q] \therefore p \,.\, q \,.\, \equiv : [f] : p \equiv . f(q) \equiv f(1).$$

In this thesis any true proposition can replace the sign '1'.

(2) In the year 1923 I established that from the proposition

$$\text{A1*.} \quad \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\big(\phi(rq)\phi(pr)\big)\Big)\urcorner$$

together with the directives of the system $\mathfrak{S}_5$ and in particular with the help of the auxiliary definition

$$\llcorner p \lrcorner \ulcorner \phi\big(p \dashv (p)\big)\urcorner,$$

the propositions which state that

$$\llcorner p \lrcorner \ulcorner \phi(pp)\urcorner \qquad \text{(Law of Identity)}$$

and

$$\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big)\urcorner \qquad \text{(Law of Commutativity)}$$

and the axiom A1 can be successively derived. From this I concluded that the axiom system A1–A3 could be replaced by the axiom system A1*, A2, A3.

(3) After finding the method of construction, stated in the résumé of §10 of my article, for single axioms of protothetic based on the directives of the system $\mathfrak{S}_5$, I based my deductions for some time, in practice, on the following axiom composed of 290 signs:

$$(\mathrm{A}_a)\quad \llcorner fp \lrcorner \ulcorner \phi \Bigg(f \Big(\llcorner pq \lrcorner \ulcorner \phi \big(\phi (pq) \phi (qp) \big) \urcorner p \Big) \phi \Bigg(f \Bigg(\llcorner hs \lrcorner \ulcorner \phi \Bigg(h$$

$$\Big(\llcorner pqr \lrcorner \ulcorner \phi \Big(\phi \big(\phi (pr) \phi (qp) \big) \phi (rq) \Big) \urcorner s \Big) \phi \Bigg(h \Bigg(\llcorner kt \lrcorner \ulcorner \phi \Bigg(k \Big($$

$$\llcorner pqr \lrcorner \ulcorner \phi \Big(\phi \big(p \phi (qr) \big) \phi \big(\phi (pq) r \big) \Big) \urcorner t \Big) \phi \Bigg(k \Bigg(\llcorner gp \lrcorner \ulcorner \phi \Bigg(\llcorner f \lrcorner \ulcorner \phi$$

$$\Bigg(g(pp) \phi \Bigg(\llcorner r \lrcorner \ulcorner \phi \Big(f(rr) g(pp) \Big) \urcorner \llcorner r \lrcorner \ulcorner \phi \Big(f(rr) g \Big(\phi (p \llcorner q \lrcorner \ulcorner q \urcorner$$

$$) p \Big) \Big) \urcorner \Bigg) \Bigg) \urcorner \llcorner q \lrcorner \ulcorner g(qp) \urcorner \Bigg) \urcorner t \Bigg) \llcorner pqr \lrcorner \ulcorner \phi \Big(\phi \big(p \phi (qr) \big) \phi \big(\phi (pq)$$

$$r))^{\urcorner}\Bigg)\Bigg)^{\urcorner} s\Bigg) \llcorner pqr \lrcorner\ {}^{\ulcorner}\phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big)^{\urcorner}\Bigg)\Bigg)^{\urcorner} p\Bigg)$$

$$\llcorner pq \lrcorner\ {}^{\ulcorner}\phi\big(\phi(pq)\phi(qp)\big)^{\urcorner}\Bigg)\Bigg)^{\urcorner}.$$

The observations referred to in (2) enabled me to replace this axiom by a shorter one, consisting of 232 signs, which is the logical product of the proposition A1* and the axioms A2 and A3:

$$(\mathrm{A}_b)\quad \llcorner hs \lrcorner\ {}^{\ulcorner}\phi\Bigg(h\bigg(\llcorner pqr \lrcorner\ {}^{\ulcorner}\phi\Big(\phi(pq)\phi\big(\phi(rq)\phi(pr)\big)\Big)^{\urcorner} s\bigg)\phi\Bigg(h\Bigg(\Bigg($$

$$\llcorner kt \lrcorner\ {}^{\ulcorner}\phi\Bigg(k\bigg(\llcorner pqr \lrcorner\ {}^{\ulcorner}\phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big)^{\urcorner} t\bigg)\phi\Bigg(k\Bigg(\llcorner gp \lrcorner$$

$${}^{\ulcorner}\phi\Bigg(\llcorner f \lrcorner\ {}^{\ulcorner}\phi\bigg(g(pp)\phi\bigg(\llcorner r \lrcorner\ {}^{\ulcorner}\phi\big(f(rr)g(pp)\big)^{\urcorner} \llcorner r \lrcorner\ {}^{\ulcorner}\phi\big(f(rr)g\big(\phi$$

$$(p \llcorner q \lrcorner \ulcorner q \urcorner)p\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner} \llcorner q \lrcorner \ulcorner g(qp)^{\urcorner}\Big)^{\urcorner} t\Big) \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(p\phi(qr)\big)$$

$$\phi\big(\phi(pq)r\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\Big) s\Big) \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\big(\phi(rq)\phi(pr)\big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}$$

From the axiom (A_b), which dates from the year 1923, one can deduce the proposition A1* in the way that '*P*' was deduced from the formula (a), as sketched in §10 of my article. The Law of Commutativity and the axiom A1 can be obtained from A1*, as I established in (2). Once the Law of Commutativity is available, the 'logical factors' A2 and A3, contained in (A_b), can be easily obtained by the methods given in §10 of my article.

(4) In 1923, drawing on all the results summarized under (1), as well as other theoretical constructions conceived *ad hoc* which revealed definite possibilities for simplifying the axiom of protothetic, I discovered an axiom (A_c) composed of 156 signs and thus shorter than axiom (A_b). As the subsequent simplifications of the axiom of protothetic pertained to the axiom (A_b) and, in this way, to a large extent deprived the axiom (A_c) of theoretical importance, I cite this axiom without further comment:

$$(A_c) \quad \llcorner fpqr \lrcorner \ulcorner \phi\Bigg(f\big(\phi(pp)q\big)\phi\Bigg(f\Bigg(\llcorner g \lrcorner \ulcorner \phi\Bigg(g(pp)\phi\Bigg(g\Big(\phi\Big(r\phi$$

$$(pr)\Big)p\Big) \llcorner h \lrcorner \ulcorner \phi\Big(\llcorner k \lrcorner \ulcorner \phi\Big(\llcorner s \lrcorner \ulcorner \phi\big(k(ss)h(pp)\big)^{\urcorner}\phi\Big(h(pp) \llcorner s \lrcorner$$

$$\ulcorner\phi\Big(k(ss)h\Big(\phi(p\llcorner t\lrcorner\ulcorner t\urcorner)p\Big)\Big)^{\urcorner}\Big)\Big)^{\urcorner}\llcorner t\lrcorner\ulcorner h(tp)^{\urcorner}\Big)^{\urcorner}\Big)\Big)^{\urcorner} q\Big)\phi$$

$$\Big(\phi(pq)\phi\Big(\phi(rq)\phi(pr)\Big)\Big)\Big)\Big)^{\urcorner}.$$

(5) Once Tarski learned of the results referred to in (1), he proved in 1924, and on the same basis on which he had earlier constructed the proof of his theorem mentioned in the résumé of §1 of my article, which states:

$[p,q] \because p . q . \equiv .\therefore [f] .\therefore p \equiv : [r] . p \equiv f(r) . \equiv . [r] .$
$q \equiv f(r),$

the proposition analogous to this theorem, namely:

$[p,q] \because p . q . \equiv .\therefore [f] .\therefore p \equiv : [r] . f(r) . \equiv . [r] . q \equiv f(r).$

(6) In the year 1926 Wajsberg informed me of his proof that the axiom system of the system $\mathfrak{S}$, given by me in the résumé of §3 of my article, could be replaced by either one of the two following propositions:

(W) $p \equiv q . \equiv r : \equiv s .\therefore \equiv .\therefore s \equiv : p \equiv . q \equiv r,$

(W*) $p \equiv . q \equiv r : \equiv : r \equiv s . \equiv s .\therefore \equiv . p \equiv q.$[3]

(7) Wajsberg's result, mentioned in (6), enabled me in 1926 to replace (A_c) by a shorter axiom of 124 signs:

[3] See M. Wajsberg, *Metalogische Beiträge*, Warsaw 1936, pp. 34–36.

$$(\mathrm{A}_d)\quad \llcorner fhpqrx \lrcorner\ \phi \ulcorner\Bigg(f \Bigg(\llcorner k \lrcorner\ \phi \ulcorner\Bigg(\llcorner s \lrcorner\ \ulcorner \phi\big(k(ss)h(pp)\big)\urcorner\ \phi\Bigg(h(pp)$$

$$\llcorner s \lrcorner\ \ulcorner \phi\Big(k(ss)h\big(\phi(p\,\llcorner t \lrcorner\ \ulcorner t \urcorner)p\big)\Big)\urcorner \Bigg)\Bigg)\urcorner\ q \Bigg)\phi\Bigg(f\Big(\llcorner t \lrcorner\ \ulcorner h(tp)\urcorner$$

$$q\Big)\phi\Bigg(\phi\bigg(\phi\Big(p\phi(qr)\Big)\phi\Big(\phi(rx)x\Big)\bigg)\phi(pq)\Bigg)\Bigg)\Bigg)\urcorner .$$

The axiom (A_d) is a 'logical product' of my axiom A3 and of Wajsberg's axiom (W*). This logical product is related to those formulae which, according to one of the propositions introduced in (1), are equivalent to the corresponding formulae of the type '$p \,.\, q \equiv r$'. If we wish to derive the axioms A1–A3 from (A_d), we can do so in the following stages:

(a) We define, as in §10 of my article, a propositional function of the type '$\Phi(pq)$' which is satisfied for all values of its arguments.

(b) We substitute the sign of the function mentioned in (a) for the variable 'f' in the axiom (A_d). Then, applying several times the directives for distribution of the quantifier, substitution and detachment, we easily obtain Wajsberg's axiom (W*), which forms one of the 'logical factors' of (A_d). This axiom is obtained according to a pattern which approximates to the one already familiar to us.

(c) We derive the axioms A1 and A2 from the axiom (W*), using Wajsberg's method.

(d) Since, as a result of (c), the whole system $\mathfrak{S}$ stands at our disposal, we transform the axiom (A_d) in such a way that by detachment we obtain the proposition:

$$\llcorner fhpq \lrcorner \ulcorner \phi\Bigg(f\Bigg(\llcorner k \lrcorner \ulcorner \phi\Bigg(\llcorner s \lrcorner \ulcorner \phi\big(k(ss)h(pp)\big)\urcorner \phi\Bigg(h(pp) \llcorner s \lrcorner$$

$$\ulcorner \phi\Big(k(ss)h\big(\phi(p \llcorner t \lrcorner \ulcorner t \urcorner)p\big)\Big)\urcorner\Bigg)\Bigg)\urcorner q\Bigg) f\big(\llcorner t \lrcorner \ulcorner h(tp)\urcorner q\big)\Bigg)\urcorner$$

.

(e) We substitute the sign 'ϕ' for the variable 'f' in the proposition just obtained. Similarly, using the theorems of the system and the directives (α)–(γ), we quite easily obtain a proposition which differs from axiom A3 in variables but in no important way, and which can be proved equivalent to A3 with the help of A1 and A2.

(8) In one of my 1926 university lectures, attended by Wajsberg, I remarked that if the correlate of Wajsberg's proposition (W*) in the axiom (A_d) were replaced by some other formula, from which followed not only all theorems of the system $\mathfrak{S}$, but also some correspondingly formulated 'extensionality proposition' analogous to the proposition

$$\llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pq) \llcorner f \lrcorner \ulcorner \phi\big(f(p)f(q)\big)\urcorner\Big)\urcorner ,$$

one could likewise make a change in (A_d), without altering the deductive power of this axiom. This change would replace the universal proposition which begins with the quantifier $\llcorner k \lrcorner$, and agrees more or less with the first of Tarski's formulae for the 'logical product' (see the résumé of §1 of my article), by a corresponding logical product constructed like the second, and shorter, of these formulae. At the time I mentioned that I saw no reason why (A_d) could not be thus shortened.

(9) Considering my remarks reviewed in (8), Wajsberg established in 1926 that all theorems of the system $\mathfrak{S}$ could be derived from the proposition

$$\llcorner pqrst \lrcorner \ulcorner \phi\left(\phi(pq)\, \llcorner g \lrcorner \ulcorner \phi\left(g\left(\phi\left(\phi(rs)t\right)q\right)g\left(\phi\left(\phi(st)r\right)p\right)\right)\urcorner\right)\urcorner .$$

This proposition, although no 'extensionality proposition' in the strict sense, nevertheless makes it possible — as Wajsberg proved — to replace the longer by the shorter form of the logical product in the axiom (A_d). As a result of Wajsberg's changes in (A_d), he obtained the following shorter axiom consisting of 120 signs:

$$(A_e)\quad \llcorner fhpqrst \lrcorner \ulcorner \phi\left(f\left(\llcorner g \lrcorner \ulcorner \phi\left(h(pp)\phi\left(g\left(qh(pp)\right)g\left(qh\left(\phi(p\, \llcorner t \lrcorner \ulcorner t \urcorner)p\right)\right)\right)\right)\urcorner r\right)\phi\left(f\left(\llcorner t \lrcorner \ulcorner h(tp) \urcorner r\right)\phi\left(\phi(pq)\, \llcorner g \lrcorner \ulcorner \phi\left(g\left(\phi\left(\phi(rs)t\right)q\right)g\left(\phi\left(\phi(st)r\right)p\right)\right)\urcorner\right)\right)\right)\urcorner .$$

(10) Starting again directly with the axiom (A_d) and using Tarski's proposition introduced in (5), I constructed, in 1926, an axiom of 116 signs, shorter than Wajsberg's axiom (A_d). Having introduced in the axiom (A_d), as we saw, a correlate of the proposition (W*) given in (6), I was able to use in a similar way the proposition (W) in the new axiom. However, I could proceed vice versa with equal success. The new axiom runs thus:

$$(\mathrm{A}_f)\quad \llcorner fhpqrs \lrcorner\ \phi \ulcorner\Bigg(f\Big(\llcorner t \lrcorner\ \ulcorner h(tp)\urcorner\ q\Big)\phi\Bigg(f\Bigg(\llcorner k \lrcorner\ \ulcorner\phi\Bigg(h(pp)\phi\Bigg(\llcorner s \lrcorner$$

$$\ulcorner k(ss)\urcorner\ \llcorner s \lrcorner\ \ulcorner\phi\Big(k(ss)h\Big(\phi(p\llcorner t \lrcorner\ \ulcorner t \urcorner)p\Big)\Big)\urcorner\Bigg)\Bigg)\urcorner\ q\Bigg)\phi\Big(\phi\Big(\phi\Big(\phi$$

$$(pq)r\Big)s\Big)\phi\Big(s\phi\Big(p\phi(qr)\Big)\Big)\Big)\Bigg)\Bigg)\urcorner .$$

(11) During the same year, 1926, Wajsberg reduced the 116 signs of the axiom (A_f) to 106. The shortened axiom is:

$$(\mathrm{A}_g)\quad \llcorner fp \lrcorner\ \phi \ulcorner\Bigg(\llcorner s \lrcorner\ \ulcorner f(sp)\urcorner\ \llcorner g \lrcorner\ \ulcorner\phi\Bigg(f(pp)\phi\Bigg(\llcorner t \lrcorner\ \ulcorner g(tt)\urcorner\ \llcorner qrt \lrcorner$$

$$\ulcorner\phi\Bigg(g\Big(\phi\Big(\phi(\phi(tt)t\Big)t\Big)\phi\Bigg(f\Big(\phi(p\llcorner s \lrcorner\ \ulcorner s \urcorner)p\Big)\phi\Big(\phi\Big(p\phi(qr)\Big)\phi\Big(r\phi$$

$$(qp)\Big)\Big)\Big)\Bigg)\urcorner\Bigg)\Bigg)\urcorner\Bigg)\urcorner .$$

(12) Making essential use of Wajsberg's axiom (A_e) and at the same time appealing to Tarski's theorem, according to which every truth function f is such that

$[p] . f(p) . \equiv . f(f(Fl)),$[4]

I constructed the 82-sign axiom introduced in the résumé of §11 of my article. To preserve consistency with the names of the other axioms considered here, I call this axiom (A_h). It happened that in one and the same conversation Wajsberg and I informed each other of the two axioms (A_g) and (A_h), which had been discovered independently of one another, yet at approximately the same time.

As I have already mentioned, the axiom (A_h) possesses the peculiarity of being the shortest anyone has yet thought of. However, it has in practice a disadvantage, quite similar to that of Wajsberg's axiom (A_e), its genetic predecessor: namely that the method of deriving any familiar basis of the system $\mathfrak{S}$ from this axiom is rather too complicated. The result is that it is not as quickly understood how this axiom provides an adequate basis for protothetic. In this respect the axiom (A_i) has the advantage, as will be discussed later. The remarks concerning this axiom should throw some light on the deductions which can be made from the axioms (A_e), (A_g), and (A_h).

(13) Łukasiewicz proved in 1933 that Wajsberg's propositions (W) and (W*) (see (6)), each of which can be the sole axiom of the system $\mathfrak{S}$, may be further shortened, and that it is possible to derive this system from any one of the following three axioms:

Ł. $p \equiv q . \equiv :r \equiv q . \equiv . p \equiv r.$

(This axiom corresponds to my proposition A1*, given in (2).)

Ł*. $p \equiv q . \equiv :p \equiv r . \equiv . r \equiv q,$

Ł**. $p \equiv q . \equiv :r \equiv p . \equiv . q \equiv r.$

From this result of Łukasiewicz's it became clear to me that:

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[4] See (i) Alfred Tajtelbaum-Tarski, 'O wyrazie pierwotnym logistyki' (On the primitive term of logistic), Doctoral thesis. Offprint from *Przegląd Filozoficzny* (1923), pp. 19 and 20. (ii) Alfred Tajtelbaum-Tarski, 'Sur les truth-functions au sens de MM. Russell et Whitehead', *Fundamenta Mathematicae* 5 (1924), p. 69.

namely A1* with different variables. This gives us the entire theory $\mathfrak{S}$, by Łukasiewicz's result of (13). In the derivation of the above proposition we exploit the idea used also by Wajsberg in proving, from the following proposition referred to in (9);

$$\llcorner pqrst \lrcorner \ulcorner \phi\left(\phi(pq) \llcorner g \lrcorner \ulcorner \phi\left(g\left(\phi\left(\phi(rs)t\right)q\right)g\left(\phi\left(\phi(st)r\right)p\right)\right)\urcorner\right)\urcorner$$

the proposition

$$\llcorner rst \lrcorner \ulcorner \phi\left(\phi\left(\phi(st)r\right)\phi\left(\phi(rs)t\right)\right)\urcorner .$$

(d) Substituting 'p' for 'q' and 'Φ' for 'f' in the axiom (A_i) we obtain, just as easily, the proposition

$$\llcorner gprst \lrcorner \ulcorner \phi\left(g\left(\phi\left(\phi(rt)\phi(sr)\right)p\right)g\left(\phi(st)p\right)\right)\urcorner .$$

(e) Substituting 'p' for 'q' in (A_i) we manipulate according to the system $\mathfrak{S}$ the proposition obtained by detachment, in such a way that we can further detach from it the proposition

$$\llcorner fp \lrcorner \phi\left(f\left(pf(p \llcorner u \lrcorner \ulcorner u \urcorner)\right) \llcorner u \lrcorner \ulcorner f(pu)\urcorner\right)\urcorner .$$

(f) We introduce the following three definitions:

$$(D_a) \quad \llcorner pq \lrcorner \ulcorner \phi\left(\llcorner gr \lrcorner \ulcorner \phi\left(p\phi\left(g\left(\phi\left(\phi(rr)\phi(rr)\right)q\right)g\left(\phi(rr)p\right)\right)\right)\urcorner \varphi(pq)\right)\urcorner ,$$

$$(D_b) \quad \llcorner pq \lrcorner \ulcorner \phi\left(\phi\left(p\phi(qq)\right) - \varphi(pq)\right)\urcorner ,$$

$$(\mathrm{D}_c)\quad \llcorner pq \lrcorner \ulcorner \phi\Big(\phi\big(\Phi(pq)\llcorner u\lrcorner\ulcorner u\urcorner\big)\circ(pq)\Big)\urcorner .$$

The first of these three is one of the possible definitions of a 'logical product'. The second is the definition of a function equivalent to its own first argument. And the third, as is seen in connexion with (b), is the definition of a function which yields a false proposition for all values of its arguments.

(g) Substituting in the proposition

$$\llcorner pq \lrcorner \ulcorner \phi\Big(q\phi\big(p\phi(pq)\big)\Big)\urcorner ,$$

the expression

$$\phi\bigg(g\Big(\phi\big(\phi(rr)\phi(rr)\big)p\Big)g\big(\phi(rr)p\big)\bigg)$$

for the variable 'q', and making use of the proposition introduced in (d), we obtain by detachment, distribution of the quantifier, and application of the definition (D_a), the proposition

$$\llcorner p \lrcorner \ulcorner \phi\big(p\text{ϙ}(pp)\big)\urcorner .$$

(h) Substituting in (A_i) the expression '$\llcorner u\lrcorner\ulcorner u\urcorner$' for '$p$', the expression '$\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)$' for '$q$', the sign '-ϙ' for '$f$', and the sign '$r$' for '$s$' and '$t$', we obtain, with the help of theorems of the system $\mathfrak{S}$ and the definitions (D_b) and (D_a), the theorem

$$\phi\Big(\llcorner u\lrcorner\ulcorner u\urcorner\text{ϙ}\big(\llcorner u\lrcorner\ulcorner u\urcorner\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\big)\Big).$$

(i) Similarly, substituting in $(\mathrm{A}_i)$the expression

'$\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)$'

for 'p', the expression '$\llcorner u\lrcorner\ulcorner u\urcorner$' for '$q$', the sign 'o' for '$f$', and the sign '$r$' for '$s$' and '$t$' we obtain, with the help of the definitions (D_c) and (D_a), the analogous theorem

$$\phi\Big(\llcorner u\lrcorner\ulcorner u\urcorner\text{ϙ}\big(\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\llcorner u\lrcorner\ulcorner u\urcorner\big)\Big).$$

If an arbitrary propositional function '$F(u)$' of one propositional argument possesses the characteristic that its values '$F(\llcorner u \lrcorner\ \ulcorner u \urcorner)$' and '$F\Big(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\Big)$' have already been proved in the system, then it is possible to obtain in the system the universal proposition '$\llcorner u \lrcorner\ \ulcorner F(u) \urcorner$'. The procedure to be adopted may be sketched as follows. In accordance with our hypothesis we have that

$\alpha.\ F(\llcorner u \lrcorner\ \ulcorner u \urcorner)$

and

$\beta.\ F\Big(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\Big).$

We define a new functional sign, let us say 'G', by means of an auxiliary definition of the type

$$\gamma.\ \llcorner pq \lrcorner\ \ulcorner \phi\Big(\phi\big(F(q)\phi(pp)\big)G(pq)\phi\Big) \urcorner.$$

From γ we deduce, in accordance with the system $\mathfrak{S}$,

$$\delta.\ \llcorner pq \lrcorner\ \ulcorner \phi\big(F(q)G(pq)\big) \urcorner.$$

From δ and α we infer that

$$\varepsilon.\ G\Big(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\ \llcorner u \lrcorner\ \ulcorner u \urcorner\Big);$$

from ε, that

$$\zeta.\ \phi\Bigg(G\Big(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\ \llcorner u \lrcorner\ \ulcorner u \urcorner\Big)\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\Bigg);$$

and from δ and β, that

$$\eta.\ G\Bigg(\phi\Big(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\Big)\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\Bigg).$$

We substitute in (A_i) the expression '$G\Big(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\ \llcorner u \lrcorner\ \ulcorner u \urcorner\Big)$' for '$p$', the expression '$\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)$' for '$q$', the sign '$\Phi$' for '$f$', and the expression '$\llcorner u \lrcorner\ \ulcorner u \urcorner$' for '$r$', '$s$', and '$t$'. Then in the proposition resulting from a single detachment using ζ, we substitute the sign 'G' for 'g'. We can then assert, in virtue of the definition mentioned in (b) and the proposition η, that

$$\theta.\ G\left(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)G\left(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)\ \llcorner u \lrcorner\ \ulcorner u \urcorner\right)\right).$$

Finally, from the proposition introduced in (e) and the proposition θ, we conclude that

$$\llcorner u \lrcorner\ \ulcorner G\left(\phi(\llcorner u \lrcorner\ \ulcorner u \urcorner\ \llcorner u \lrcorner\ \ulcorner u \urcorner)u\right)\urcorner,$$

from which, on the basis of the proposition δ, it follows that

$$\llcorner u \lrcorner\ \ulcorner F(u)\urcorner.$$

The results of the deductions sketched here being at our disposal, we can see without difficulty that the axiom (A_i) suffices for the construction of the given system of protothetic.

SUPPLEMENTARY REMARK V

In the course of my seminar on the foundations of mathematics in the academic year 1924–1925 at the University of Warsaw, I and other participants analysed Łukasiewicz's *Two-valued logic*, mentioned above in the résumé of §5 of my article. In this connexion I constructed in 1924 a system of protothetic based on principles quite different from those according to which the system $\mathfrak{S}_2$–$\mathfrak{S}_5$ has been constructed. Using Łukasiewicz's construction to some considerable extent as a model, I was concerned to employ an 'algorithmic' or 'computative' style in my new system, as opposed to the much more common 'substitution-detachment' style. At this point I would like to mention one or two of the peculiarities of my system which differentiate it from Łukasiewicz's 'Two-valued logic' — apart, of course, from the obvious inequivalence of the two systems, the second being not a system of protothetic at all.

(a) In my system there are no signs '*U*' ('I accept') or '*N*' ('I reject', 'I deny') which are prefixed to the theses of Łukasiewicz's system.[5]

[5] See Jan Łukasiewicz, 'Logika dwuwartościowa' (*Two-valued logic*), offprint from the volume in memory of Twardowski (*Przegląd Filozoficzny* 23), Lwów 1921, pp. 4 and 5.

(b) My system is based on two primitive terms, 'ϕ-' and '$\wedge$' (the logical 'zero'), whereas in Łukasiewicz's system, along with the correlates of these two terms, the logical 'one' appears as a primitive term.[6] (Inasmuch as the four propositions appearing in Łukasiewicz's system which he calls definitions, as well as the author's axioms at the beginning of the system, were introduced with no reference to any directives,[7] I feel compelled to regard these 'definitions' simply as axioms of a special kind, and also to include among the primitive terms of 'Two-valued logic', besides the three terms mentioned above, the four 'defined' terms, as well as the equality sign used by Łukasiewicz in definitions. Cf. the résumé of §1 of my article.)

(c) While Łukasiewicz introduces three theses as axioms of his system, of which two are formulated with the help of variables, my system is based on one axiom alone, and in this axiom no variables appear.

(d) One of the directives of my system makes possible the addition to the system of any number of definitions.

(e) My system has no substitution directive.[8]

The system of protothetic in question, which I have formalized with the same degree of precision with which I formulated the directives of the system $\mathfrak{S}_5$ in §11 of my article, could be summarized in the following sketchy and inexact fashion.

Axiom

(A) $\phi(\wedge\wedge)$.

Directives

Directive a: if any expressions 'p' and 'q' are already theses of the system, the corresponding expression '$\phi(pq)$' may be added.

[6] *Op. cit.*, pp. 4 and 16.

[7] *Op. cit.*, pp. 15 and 16.

[8] Cf. *op. cit.*, the 'said rule' a on p. 11.

Directive b: if any expressions 'p' and '$\phi\!\!-\!(q\wedge)$' are already theses of the system, the corresponding expression '$\phi\!\!-\!\big(\phi\!\!-\!(pq)\wedge\big)$' may be added.

Directive c: if any expressions '$\phi\!\!-\!(p\wedge)$' and 'q' are already theses of the system, the corresponding expression '$\phi\!\!-\!(pq)$' may be added.

Directive d: if any expressions '$\phi\!\!-\!(p\wedge)$' and '$\phi\!\!-\!(q\wedge)$' are already theses of the system, the corresponding expression '$\phi\!\!-\!(pq)$' may be added.

Directive e: this directive permits the addition to the system of definitions having the form of expressions of the type

$$\phi\!\!-\!\Bigg(\phi\!\!-\!\bigg(\phi\!\!-\!(pq)\phi\!\!-\!\Big(\phi\!\!-\!(qp)\wedge\Big)\bigg)\wedge\Bigg),$$

or of the same type preceded by universal quantifiers. The well-formed propositions represented here by the signs 'p' and 'q' constitute, respectively, the *definiens* and the *definiendum* of the definition.

Directive f: if the *definiens* of any definition already belonging to the system (or of some substitution instance of such a definition) is a thesis of the system, then the *definiendum* of the definition in question (or of its substitution instance) may be added to the system.

Directive g: if in some expression '$\phi\!\!-\!(p\wedge)$', which is already a thesis of the system, the expression 'p' is the *definiens* of some definition already belonging to the system (or of some substitution instance of such a definition), then the expression '$\phi\!\!-\!(q\wedge)$' may be added to the system, where 'q' is the *definiendum* of the definition in question (or of its substitution instance).

Directive h: this directive, which is a correlate of the directive η of the system $\mathfrak{S}_2$, generalized for the semantic category of propositions, presupposes (as does the last directive i) certain constant terms that may be called basic constants, the latter including in particular the primitive terms '$\wedge$' and '$\phi\!\!-$'. A finite number of these basic constants should be defined for each semantic category occurring in the system, according to a scheme inductively characterized in advance. For the semantic category of propositions there are two such basic constants. For the semantic category of signs of propositional functions of one propositional argument, there are four basic constants. For the semantic category of signs of propositional functions of one argument, the latter being itself of the semantic category of such function signs, there are sixteen basic constants, etc. This accords with the observations on p. 37 of my article. The directive in question states that an expression beginning with a universal quantifier can be added to the system as a new thesis, if all those expressions that can be obtained from the given expression by substituting for its variables the basic constants of the same semantic categories are already theses of the system.

Directive i: if 'q' is an expression beginning with a universal quantifier, if an expression 'p' can be obtained from the expression 'q' by the substitution for its variables of basic constants of the same semantic categories, and if the expression '$\multimap\!\!\vdash(p\wedge)$' is already a thesis of the system, the corresponding expression '$\multimap\!\!\vdash(q\wedge)$' may be added to the system.

In order to illustrate procedures valid in the system of protothetic described here, I deduce from the axiom A, as an example the thesis

$$\llcorner f \lrcorner \ulcorner \multimap\!\!\vdash \Big(f\big(f(\llcorner p \lrcorner \ulcorner p \urcorner) \big) \llcorner p \lrcorner \ulcorner f(p) \urcorner \Big) \urcorner .$$

Because the scheme for defining basic constants, mentioned above in the discussion of directive h, has not been effectively formulated by me here, I shall assume (truly) in my derivation that the terms defined by means of the definitions D1–D5 satisfy the scheme. I shall also assume that these terms, together with the primitive term '$\wedge$', exhaust the basic constants belonging to the semantic categories of (i) propositions, and (ii) signs of propositional functions with one propositional argument. The derivation in question proceeds as follows:

A. $\phi\text{-}(\wedge\wedge)$.

D1. $$\phi\text{-}\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi\text{-}(\wedge\wedge)\vee\big)\phi\text{-}\big(\phi\text{-}\big(\vee\phi\text{-}(\wedge\wedge)\big)\wedge\big)\Big)\wedge\Big)$$

(on the basis of directive e).

D2. $$\llcorner p\lrcorner\ \ulcorner\phi\text{-}\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi\text{-}(pp)\vdash\!\dashv(p)\big)\phi\text{-}\big(\phi\text{-}\big(\vdash\!\dashv(p)\phi\text{-}(pp)\big)\wedge\big)\Big)\wedge\Big)\urcorner$$

(Dir. e).

D3. $$\llcorner p\lrcorner\ \ulcorner\phi\text{-}\Big(\phi\text{-}\Big(\phi\text{-}\big(p\dashv(p)\big)\phi\text{-}\big(\phi\text{-}\big(\dashv(p)p\big)\wedge\big)\Big)\wedge\Big)\urcorner$$

(Dir. e).

D4. $$\llcorner p\lrcorner\ \ulcorner\phi\text{-}\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi\text{-}(p\wedge)\vdash(p)\big)\phi\text{-}\big(\phi\text{-}\big(\vdash(p)\phi\text{-}(p\wedge)\big)\wedge\big)\Big)\wedge\Big)\urcorner$$

(Dir. e).

D5. $$\llcorner p\lrcorner\ \ulcorner\phi\text{-}\Big(\phi\text{-}\Big(\phi\text{-}\big(\phi\text{-}\big(\phi\text{-}(pp)\wedge\big)-(p)\big)\phi\text{-}\Big(\phi\text{-}\Big(-(p)\phi\text{-}\big(\phi\text{-}(pp)$$

$$\wedge\big)\Big)\wedge\Big)\Big)\wedge\Big)\urcorner$$

(Dir. e).

T1. $\phi\text{-}(\llcorner p\lrcorner\ulcorner p\urcorner\wedge)$ (A, Dir. i)

T2. $\phi\text{-}(\llcorner p\lrcorner\ulcorner p\urcorner\llcorner p\lrcorner\ulcorner p\urcorner)$ (T1, Dir. d)

T3. $\vdash\!\dashv(\llcorner p\lrcorner\ulcorner p\urcorner)$ (D2, T2, Dir. f)

T4. $\phi\text{-}\big(\vdash\!\dashv(\llcorner p\lrcorner\ulcorner p\urcorner)\vdash\!\dashv(\llcorner p\lrcorner\ulcorner p\urcorner)\big)$ (T3, a)

T5. $\vdash\!\dashv\big(\vdash\!\dashv(\llcorner p\lrcorner\ulcorner p\urcorner)\big)$ (D2, T4, f)

T6. $\vdash\!\dashv(\wedge)$ (D2, A, f)

T7. $\vee$ (D1, A, f)

T8. $\phi\text{-}(\vee\vee)$ (T7, a)

T9. $\vdash\!\dashv(\vee)$ (D2, T8, f)

T10. $\llcorner p \lrcorner \ulcorner \vdash\!\dashv(p) \urcorner$ (T6, T9, h)

T11. $\phi\!-\!\left(\vdash\!\dashv\left(\vdash\!\dashv(\llcorner p \lrcorner \ulcorner p \urcorner)\right) \llcorner p \lrcorner \ulcorner \vdash\!\dashv(p) \urcorner\right)$ (T5, T10, a)

T12. $\phi\!-\!\left(\dashv(\llcorner p \lrcorner \ulcorner p \urcorner)\wedge\right)$ (D3, T1, g)

T13. $\phi\!-\!\left(\dashv\left(\dashv(\llcorner p \lrcorner \ulcorner p \urcorner)\right)\wedge\right)$ (D3, T12, g)

T14. $\phi\!-\!\left(\dashv(\wedge)\wedge\right)$ (D3, A, g)

T15. $\phi\!-\!\left(\llcorner p \lrcorner \ulcorner \dashv(p)\wedge\right)$ (T14, i)

T16. $\phi\!-\!\left(\dashv\left(\dashv(\llcorner p \lrcorner \ulcorner p \urcorner)\right) \llcorner p \lrcorner \ulcorner \dashv(p) \urcorner\right)$ (T13, T15, d)

T17. $\vdash(\llcorner p \lrcorner \ulcorner p \urcorner)$ (D4, T1, f)

T18. $\phi\!-\!\left(\phi\!-\!\left(\vdash(\llcorner p \lrcorner \ulcorner p \urcorner)\wedge\right)\wedge\right)$ (T17, A, b)

T19. $\phi\!-\!\left(\vdash\left(\vdash(\llcorner p \lrcorner \ulcorner p \urcorner)\right)\wedge\right)$ (D4, T18, g)

T20. $\phi\!-\!\left(\phi\!-\!(\vee\wedge)\wedge\right)$ (T7, A, b)

T21. $\phi\!-\!\left(\vdash(\vee)\wedge\right)$ (D4, T20, g)

T22. $\phi\!-\!\left(\llcorner p \lrcorner \ulcorner \vdash(p) \urcorner \wedge\right)$ (T21, i)

T23. $\phi\!-\!\left(\vdash\left(\vdash(\llcorner p \lrcorner \ulcorner p \urcorner)\right) \llcorner p \lrcorner \ulcorner \vdash(p) \urcorner\right)$ (T19, T22, d)

T24. $\phi\!-\!\left(\phi\!-\!\left(\phi\!-\!(\llcorner p \lrcorner \ulcorner p \urcorner \llcorner p \lrcorner \ulcorner p \urcorner)\wedge\right)\wedge\right)$ (T2, A, b)

T25. $\phi\!-\!\left(-(\llcorner p \lrcorner \ulcorner p \urcorner)\wedge\right)$ (D5, T24, g)

T26. $\phi\!-\!\left(-(\llcorner p \llcorner \ulcorner p \urcorner) - (\llcorner p \lrcorner \ulcorner p \urcorner)\right)$ (T25, d)

T27. $\phi\!-\!\left(\phi\!-\!\left(\phi\!-\!\left(-(\llcorner p \lrcorner \ulcorner p \urcorner) - (\llcorner p \lrcorner \ulcorner p \urcorner)\right)\wedge\right)\wedge\right)$ (T26, A, b)

T28. $\phi\!-\!\left(-\left(-(\llcorner p \lrcorner \ulcorner p \urcorner)\right)\wedge\right)$ (D5, T27, g)

T29. $\phi\!-\!\Big(\phi\!-\!\big(\phi\!-\!(\wedge\wedge)\wedge\big)\wedge\Big)$ (A, b)

T30. $\phi\!-\!\big(-(\wedge)\wedge\big)$ (D5, T29, g)

T31. $\phi\!-\!\big(\llcorner p \lrcorner\ \ulcorner -(p) \urcorner\ \wedge\big)$ (T30, i),

T32. $\phi\!-\!\Big(-\big(-(\llcorner p \lrcorner\ \ulcorner p \urcorner)\big)\ \llcorner p \lrcorner\ \ulcorner -(p) \urcorner\Big)$ (T28, T31, d),

T33. $\llcorner f \lrcorner\ \ulcorner \phi\!-\!\Big(f\big(f(\llcorner p \lrcorner\ \ulcorner p \urcorner)\big)\ \llcorner p \lrcorner\ \ulcorner f(p) \urcorner\Big) \urcorner$

(T11, T16, T23, T32, h).

On the model of the system of protothetic sketched here, which is based upon the primitive terms '$\wedge$' and '$\phi\!-$', I constructed in 1924 nine further systems of this theory, based upon the following combinations of primitive terms: (1) '$-\phi$' and '$\wedge$'; (2) 'ϕ' and '$\wedge$'; (3) '-ϕ-' and '$\wedge$'; (4) '-ϕ-' and '$\vee$'; (5) '-o-' and '$\vee$'; (6) '-o' and '$\vee$'; (7) 'o-' and '$\vee$'; (8) 'ó' and '$\wedge$'; (9) 'o-' and '$\vee$'. The single axiom, forming the correlate of the axiom A, was constituted in each of these systems by the shortest true proposition that could be formulated with the help of the primitive terms of the given system, together perhaps with parentheses of the type '(' and ')'. (From this it can be seen that the axiom of a system in which the sign '$\vee$' appeared as one of the primitive terms was itself the sign '$\vee$'.) Corresponding to the directives a–d there were four analogous directives in each of the new systems. In the four possible cases in which the arbitrary expressions 'p' and 'q', or their negations expressed in the form '$\phi\!-\!(p\wedge)$' and '$\phi\!-\!(q\wedge)$', appeared as theses in the system, the directives a–d permitted adding to the system, in accordance with the meaning of the function sign '$\phi\!-$', the corresponding expressions '$\phi\!-\!(pq)$' or '$\phi\!-\!\big(\phi\!-\!(pq)\wedge\big)$', the latter being the negation of the former. In the four possible cases in which the arbitrary expressions 'p' and 'q', or their negations expressed with the help of the primitive terms of the given system, appeared as theses, analogous directives in each of the new systems permitted adding to the system, in accordance with the meaning of the primitive function sign '*f*', the corresponding expression '*f*(*pq*)' or its negation expressed with the help of the primitive terms. The correlate of the directive e permitted adding to the system definitions formulated according to an analogous general scheme, adapted to the primitive terms of the system in question. To the directives f–i corresponded four wholly analogous directives. Whereas two of the directives belonging to the first group (directives g and i) concerned certain negations taking the form of expressions of the type 'ϕ-(*p*∧)', the correlates of these directives appealed to corresponding negations, formed with the help of the primitive terms of the system to be constructed. The role of the negation of any given expression '*p*', expressed with the help of the primitive terms of the system in question, was played, in the systems constructed with the help of the above combinations 1–9 of primitive terms, by expressions of the following types: (1) '-ϕ(∧*p*)'; (2) 'ϕ(*p*∧)' or, in a second parallel system, 'ϕ(∧*p*)'; (3) '-ȯ-(*pp*)'; (4) '-ȯ-(*pp*)'; (5) '-o-(*p*∨)'; or, in a second parallel system, '-o-(∨*p*)'; (6) '-o(∨*p*)'; (7) 'o-(*p*∨)'; (8) 'ȯ(*pp*)'; (9) 'ȯ(*pp*)'.

During my university course in 1933–1934 entitled 'Introduction to Mathematical Logic' (see supplementary remark I) I showed that two further systems of protothetic based on the primitive terms '-ȯ-' and '∨' could be constructed, using negations formulated with the help of expressions of the type '-ȯ-(*p*∨)' and '-ȯ-(∨*p*)'respectively. I also showed that two other systems based on the primitive terms 'ȯ' and '∧' could be constructed, using negations formulated by means of the expressions 'ȯ(*p*∧)' or 'ȯ(∧*p*)'.

During my lectures entitled 'Foundations of the Propositional Calculus' given in the University of Warsaw during the academic year 1934–1935, I remarked that those systems of protothetic mentioned above whose axiom is the sign '∨' possess what I would call, with an eye to the harmony of these systems, a rather annoying characteristic, namely that the axiom in these systems is the only thesis which cannot be repeated. In order to eliminate this disharmony I specified two methods:

Method 1. The axiom '∨' is replaced in the systems in question by a new directive which states that an expression may be added to the system if it is equiform with the eliminated axiom. (In systems modified in this way, each thesis can be repeated any number of times.)

Method 2. The directives which are correlates of a–d and f–i are restricted in such a way that only theses not equiform with any of the theses already belonging to the system may be added. (In systems modified in this way, no thesis can be repeated.)

SUPPLEMENTARY REMARK VI.

While editing the 'terminological explanations' of my article, which were discussed above in the résumé of §11, and which concerned those terms important for the formulation of the directives of protothetic, I often observed how, the directives remaining unaltered, the careless overlooking of this or that condition or restriction contained in these explanations could lead to an unavoidable contradiction in my system. As a result I began to investigate various other familiar deductive systems from this point of view, and to look for contradictions in these systems which could arise from similar carelessness in their formalization. The central point in these considerations lay in the analysis of the differences in directives between my system and the systems of other authors, in particular in the method of definition and, in general, the method of introducing into the system non-primitive constants. Among other systems, I analysed more closely in this respect the system of arithmetic published by von Neumann (see above the résumé of §11).

The directives of my system of protothetic permit only single signs and functional expressions of the type '$f[kl\ldots]$',

'$f\{xy\ldots\}[kl\ldots]$', etc., to be used in the system as function signs.[9] As function signs in von Neumann's system (these are 'operations',[10] 'transformations' of operations,[11] transformations of these transformations,[12] etc.) there can appear, according to the directives of this system, any symbols, with the exception of the parentheses '(' and ')', commas, and expressions of the type 'x_m', 'C_m' and 'A_m', where an arbitrary numeral replaces the letter 'm'.[13] Taking into account the fact that those symbols which play the role of function signs in the system under consideration are not limited by this restriction in the number of signs they contain, new function signs could be introduced into the system by the process of transformation. These would be composed of different combinations of arbitrarily many signs, the latter already appearing in other function signs of the system. (Examples of sign combinations of this kind would be the expressions '+', '1', '++', '1+', '11', '+ + +', '+ + 1', '+1+', etc., which consist of signs appearing in the function sign '+1' introduced explicitly in the system by von Neumann[14] as a transformation of the operation '$O_3^{(1)}$'.) After comparing the two systems I suspected that von Neumann's rules regulating the form of function signs might be so liberal as to lead, in the context of his entire system, to

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[9] Cf. note 1, p. 66, of my article. See also (i) M. Schönfinkel, 'Über die Bausteine der mathematischen Logik', *Mathematische Annalen* 92, no. 3/4, Berlin, 1924, pp. 307–315; (ii) B. Sobociński, 'O kolejnych uproszczeniach aksjomatyki "ontologji" Prof. St. Leśniewskiego' ('Successive simplifications of the axiom-system of Leśniewski's Ontology'), offprint from the volume in commemoration of fifteen years' teaching in the University of Warsaw by Prof. T. Kotarbiński, Warsaw 1934, p. 159. I did not cite Schönfinkel's work in the above-mentioned footnote to my article because at the time I was not acquainted with it.

[10] Cf. J. von Neumann, 'Zur Hilbertschen Beweistheorie', *Mathematische Zeitschrift* 26, no. 1, Berlin 1927, pp. 4 and 5.

[11] *Op. cit.*, pp. 8 and 9.

[12] *Op. cit.*, p. 8

[13] *Op. cit.*, pp. 4, 8, and 9.

[14] *Op. cit.*, p. 15.

a logical catastrophe. Further analysis substantiated my suspicion, inasmuch as von Neumann's system is not developed with sufficient care to preclude the derivation of two mutually contradictory propositions. In fact I derived them explicitly in §11 of my article — see the résumé of §11 above.

These remarks concerning von Neumann's system of arithmetic incited him to publish his 'Bemerkungen zu den Ausführungen von Herrn St. Leśniewski über meine Arbeit 'Zur Hilbertschen Beweistheorie''.[15] This article in turn evoked Lindenbaum's 'Bemerkung zu den vorhergehenden 'Bemerkungen...' des Herrn J. v. Neumann'.[16]

In his comments on my article,[17] von Neumann introduced certain changes, backed up by sentences such as 'I am taking the liberty of changing his argument somewhat, in ways which I believe are insignificant, but more practical for the following discussion',[18] and 'I write □, ● (instead of 1, + as Leśniewski does) in order to avoid giving rise to any arithmetical association'.[19] To prevent any misunderstanding on the part of the reader, may I remind him, in connexion with the second of the sentences quoted here, that the symbol '+1', consisting of the signs '+' and '1', was introduced into von Neumann's system by the author himself.

In answer to my thesis that the system under consideration is inconsistent, von Neumann has the following to say.[20]

"May I note here that, in my opinion, his objection rests on a misunderstanding of the concept 'sign': if mathematics is to be symbolically formulated, it is imperative that the different signs be discriminable. {Compare D. Hilbert, *Hamb. Abh.* I, pages 162–163.} This discriminability requires not only that two

[15] *Fundamenta Mathematicae* 17 (1931).

[16] *Fundamenta Mathematicae* 17 (1931).

[17] Cf. von Neumann, *op. cit.*, pp. 331 and 332.

[18] *Op. cit.*, p. 331.

[19] *Op. cit.*, p. 332, note 1.

[20] The footnotes to the paragraphs quoted here are given in the accompanying brackets.

differently written signs should be separately distinguishable from each other, but, more important, that each combination (linear succession) of signs should be unambiguously analysable into its constituent (printed) parts. Leśniewski selected the symbols □●, □, ●□ (which I designate for the moment as α, β, γ) in accordance with my general transformation rules, but at the same time violated an elementary law of every symbolic 'language': the signs α, β, γ, are not sufficiently distinguishable from one another, since $\alpha\beta$ is identical with $\beta\gamma$. That is, it is impossible to determine whether the combination □●□ is $\alpha\beta$ or $\beta\gamma$. {This naturally does not alter the fact that the mathematical formalism is to be regarded as in principle meaningless.}

"Once again it is important to stress: every symbolic system, my own included, must be constructed out of signs such that two sign combinations $\alpha\beta\gamma \ldots \rho$ and $\lambda\mu\nu \ldots \xi$ can have the same appearance only if they consist of the same number of signs, and if α coincides with λ, β with μ, γ with ν,..., ρ with ξ.

"As this has nothing to do with my object, namely proving the consistency of mathematics, but belongs rather to an earlier and in my opinion unmathematical stage of formalism, I didn't feel particularly compelled to refer to it in my work. However, because a misunderstanding arose, I have nevertheless discussed the matter."[21]

These paragraphs compel me to make a few comments of an interpretative and terminological nature.

In discussing the construction of his system of arithmetic in *Zur Hilbertschen Beweistheorie*, von Neumann has established quite precisely what he means by 'simple signs'.[22] This he has not done for 'signs' in general. Nor has he given the reader any indication of the sense in which he uses the word 'symbol'. The question of what 'sign' and 'symbol' mean, in the terminology of the author, is left for the reader to decide on the basis of various

[21] *Op. cit.*, p. 332.

[22] See von Neumann, *Zur Hilbertschen Beweistheorie*, pp. 4–6.

characteristic contexts. Having seen nothing in these contexts to require making a qualified interpretation, I interpreted, in fact, the words 'sign' and 'symbol' simply as exact correlates of my term 'expression' (compare pages 60 and 61 of my article). Using this terminology, I wrote the comments on von Neumann's system which I published in my article. I saw at the time, and I see today, no traditional meaning of the terms 'sign' and 'symbol' better adapted to the totality of von Neumann's explanations containing these terms in *Zur Hilbertschen Beweistheorie*.

Using this interpretation of the terms, the long quotation above from von Neumann's later publication contains a series of seemingly paradoxical incongruities. If 'sign' means the same as 'expression' each 'linear succession of signs' which contains at least three words[23] can be decomposed in at least three different ways into disjoint components which are signs. (For the moment I ignore combinations of only two words, in order to avoid borderline questions concerning von Neumann's use of the expressions 'component' and 'succession'.) For example, the linear succession '0+1' of the two signs '0' and '+1', which contains the three words '0', '+' and '1', is decomposable into (a) the disjoint components '0', '+' and '1', which are signs, (b) the disjoint components '0' and '+1', which are signs, and (c) the disjoint components '0+' and '1', which are signs. The postulate formulated by von Neumann can be satisfied by no two 'sign combinations' of the same form each containing at least three words. According to this postulate 'two sign combinations $\alpha\beta\gamma\ldots\rho$ and $\lambda\mu\nu\ldots\xi$ can have the same appearance only if they consist of the same number of signs, and if α coincides with λ, β with μ, γ with ν, ..., ρ with ξ'. (Thus, the sign combination made up of the sign 'Z' and the subsequent sign '0+1', and the sign combination made up of

[23] [*Ed. note*: Up to this point *Wort* has been translated as 'sign'. But Leśniewski now begins to use it in his technical sense of 'simple expression', and it will henceforth in this article be translated as 'word'.]

the sign '$Z0$' and the subsequent sign '+1' look exactly alike, although neither does the sign 'Z' from the first sign combination coincide with the sign '$Z0$' from the second, nor does the sign '0+1' from the first coincide with the sign '+1' from the second.) These signs, which should be 'discriminable' from other signs in von Neumann's sense, cannot possibly exist. The fact that the signs '1+', '1', and '+1', which I made use of in my article to derive a contradiction in von Neumann's system, and which as we saw were replaced in von Neumann's reply by the signs '□ ●', '□', and '● □', are not 'sufficiently distinguishable' from one another, cannot be taken as implying any 'misunderstanding of the concept "sign" '. (I shall not here take up what is for me a very obscure question, what von Neumann means by the 'unmathematical stage of formalism'.)

To one unfamiliar with the editorial details of the directives of von Neumann's system, and the author's commentary on his system, but who has only read von Neumann's answer to my critique cited here, and on this basis interprets the word 'sign' according to the postulated 'elementary law of every symbolic 'language', it might seem that in von Neumann's terminology 'sign' is exactly the same as 'word' in my terminology (compare pages 60 and 61 of my article). Such an opinion would be incorrect, as the following facts show:

(a) In von Neumann's terminology the expression 'x_1' is a 'sign',[24] and in this expression the numerical index is also a 'sign',[25] so that, according to this interpretation of the word 'sign', both the expression 'x_1' and its index would be words. But not according to my terminology, in which no letter or index that is merely part of a word is itself a word.

(b) As we saw above, von Neumann introduced into his system the expression '+1' as a transformation of the operation '$O_3^{(1)}$'. According to the directives concerning transformations

[24] *Op. cit.*, p. 4.

[25] *Op. cit.*, p. 6.

given by the author, this expression must be a sign.[26] At the same time, in keeping with our interpretation of the word 'sign', it must be a word. This would not be compatible with my terminology, where no expression consisting of two words is a word.

(c) The expression '$Z0$', consisting also of two words, is thus no word according to my terminology, although in the terminology of von Neumann it is a sign.[27]

I am inclined to believe that the interpretation of 'sign' as 'expression' can be carried out in a thoroughly consistent way within the context of *Zur Hilbertschen Beweistheorie*, which work provided the point of departure for my critical remarks about von Neumann's system. (Note that even a thoroughly consistent terminology in no way excludes the possibility of obtaining contradictory theses in a system whose directives, based on this terminology, are not formulated with sufficient care.) This interpretation in turn forces us to reject those postulates of the author which were first published *ex post facto* by him in his reply to my critical remarks. On the other hand, the attempt to correlate the term 'sign' with my term 'word' leads to obvious incompatibilities in interpretation even when we consider the specific contexts of von Neumann's first, fundamental publications. All this says nothing in favour of interpreting 'signs' as 'words'.

The complications and obscurity described here over the meaning of the word 'sign', which arose through conflating different explanations drawn from the author's two different works, require some radical hypotheses to throw light on the true source of the confusion, and to prepare the ground for its more or less reasonable settlement. As the one who to some extent brought on this tangle through my critique, I feel the need of imparting to the reader one or two confidences concerning the heuristic conception

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26 *Op. cit.*, p. 9.

27 *Op. cit.*, pp. 41 and 15.

which was helpful to me in overcoming some of the wealth of difficulties described above. This heuristic conception could be summarized more or less as follows. In making use of the terms 'sign' and 'symbol' in *Zur Hilbertschen Beweistheorie*, von Neumann made no preliminary attempt exactly to circumscribe the use of these terms. And, using these terms in a completely unrestricted and intuitive way, he employed them in practice as correlates of my term 'expression', so that they concurred with my original interpretation of them given above. The author first gave a precise statement of the universal postulate, which the 'signs' of every 'symbolic language' or every 'symbolic system' should satisfy, in the publication of the reply to my critique of his system. The reply was carelessly carried out, and the way in which the author used the word 'sign' is conspicuously inconsistent. The explanation given by the author holds good for 'signs' as 'words', but not for 'signs' as 'expressions'. A successful method of eliminating all these inconsistencies would be to consider the paragraph analysed above as non-existent, and obstinately to treat 'signs' and 'symbols' as 'expressions', at least until the author gives some clearly formulated and convincing reasons for doing otherwise. On the basis of a terminology determined in this way, one can anticipate weakening the directives of von Neumann's system to eliminate the contradiction.

Here I would like to draw to the reader's attention that the impossibility of harmonizing von Neumann's standpoint in *Zur Hilbertschen Beweistheorie* with the 'elementary law of every symbolic language', which was formulated in his reply, can also be proved, if I may so express it, in a more 'immanent' way, without recourse to any hypotheses as to interpretation. The argument proceeds as follows. The words '$\sim$', 'Z', and '0' constitute, in von Neumann's system, transformations of the operations '$O_1^{(1)}$' and

'$O_2^{(1)}$' and of the constant 'C_1'.[28] According to the author's directives concerning transformations, these words must be signs.[29] In von Neumann's terminology, the expression '$Z0$' is likewise a sign, as we saw in the discussion of 'signs' as 'words'. The last two steps indicate that the sign combination '$\sim Z0$' can be analysed on the one hand into the successive signs '$\sim$', 'Z', and '0', and on the other into '$\sim$' and '$Z0$'. The 'elementary law of every symbolic language', postulated by the author, is obviously incompatible with this fact.

The article containing von Neumann's polemic against my critique also contains one or two theoretical points worth considering. What is essential in the author's arguments can be brought together in the following quotations:

'As the subject has already been raised, I should like to say one more thing on the question of signs, and at the same time to correct a real oversight in my work.'[30]

'That the system may function at all, it is essential that the construction of any formula should be unique, and that, in particular, this formula should not result in two different ways.'[31]

'I have already mentioned (*loc. cit.*) that the formal system which I gave, without the "transformations" (pages 8–9), satisfies the postulate.'[32] (The page numbers cited in these paragraphs refer to the earlier of von Neumann's two works discussed here.)

'But in order to include the "transformations" and the simplifying convention concerning parentheses (*loc. cit.*, pages 8–9) one must again be convinced of the validity of the principle of

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[28] *Op. cit.*, pp. 10 and 15.

[29] *Op. cit.*, pp. 8 and 9.

[30] Von Neumann, 'Bemerkungen zu den Ausführungen von Herrn St. Leśniewski über meine Arbeit 'Zur Hilbertschen Beweistheorie'', p. 333.

[31] *Loc. cit.*

[32] *Loc. cit.*

univocity. Here indeed I permitted too much. Thus, for example, the three operations

'$O_r^{(2)}(\cdot,\cdot)$, $O_s^{(1)}(\cdot)$, $O_t^{(1)}(\cdot)$,

can be transformed into

$(\cdot \bullet \cdot)$, $\bullet(\cdot)$, $(\cdot)\bullet$,

and then (with some constant C_p) the formulae

$O_r^{(2)}(C_p, O_s^{(1)}(C_p))$ and $O_r^{(2)}(O_t^{(1)}(C_p), C_p)$

can be constructed. Both assume the form

$(C_p \bullet \bullet C_p)$.

To avoid such improprieties it suffices, for example, to make the following addition to my 'transformation rule': no Γ may be applied concurrently in a transformation $\Gamma(:, \ldots, :)$ and in a transformation $(:, \ldots, :)\Gamma$. (The second case could even be limited to the occurrence of a single blank place.)

'It is not difficult to prove the univocity of the systems modified by these precautions. However, with all due respect to the reasons given here, I feel that to carry out the proof is superfluous, particularly as these *pro-domo* discussions already take up too much space.'[33]

My observations on his article, of which a significant part has been included here for the convenience of the reader, already occupy far more space than von Neumann's '*pro-domo*' discussions. Unfortunately, however, I cannot yet conclude these remarks, for I am as sceptical of von Neumann's system, reformed by these precautions, as I was earlier of the author's original system.

I am inclined to believe that, in making no attempt to produce an explicit proof of univocity for his new system because of its alleged simplicity, the author once again was too careless. Contrary to his opinion on this question, I shall attempt to prove that the restriction on the directives, as he introduced it, renders his system in no way univocal; that is to say, the reformed system is as inconsistent as the first one.

[33] *Op. cit.*, p. 334.

This would be a banal task had von Neumann not questioned the correctness of the proof, presented in my article, that his original system was inconsistent. The whole proof, as well as the simple considerations designed to show that the formula '1+1' appearing in the proof can be 'constructed' in von Neumann's sense in at least two different ways, would hold true also for the second of his systems. In reply to my critique, von Neumann complicated the situation by charging me, as we know, with proceeding incorrectly (never mind in what respect) in treating the signs '1+' and '1' as transformations, along with the sign '+1' which was introduced by himself. As I don't wish to make the validity of my argument depend upon whether or not this debatable reproach is justified, I am replacing my original proof by a new construction, in which the transformations in question no longer appear. To fend off all similar reproofs I introduce, in accordance with the directives of both von Neumann's systems, only those transformations of signs which have already been explicitly introduced by the author himself.

In von Neumann's systems, two mutually contradictory propositions can be obtained in the following way:

From 'schema' 1 of Group III[34]

(α) $Z0$.

Using the transformation rules[35] we transform the symbol '$\Gamma_{1,1}$'[36] into the symbol 'Z'. Schema 3 of Group III,[37] in conjunction with the rule b concerning the omission of parentheses,[38] shows us that

(β) $\sim (Z + 1 = 0)$.

[34] Von Neumann, *Zur Hilbertschen Beweistheorie*, p. 15.

[35] See *op. cit.*, p. 8.

[36] See *op. cit.*, p. 20.

[37] *Op. cit.*, p. 15.

[38] *Op. cit.*, p. 9.

Considering the sequences of the formulae $b_{u,v}^{(n)}$[39] discussed by von Neumann we can assert[40] the existence of a natural number l such that the formula '$x_1+1 = 0$' is identical with the formula $b_{l,1}^{(1)}$, the latter being one of the elements of the sequence $b_{u,1}^{(1)}$. This, together with schema 2 of Group VI[41] and rule b concerning the omission of parentheses, allows us to establish that

(γ) $\Omega_{l,1}^{(1)} Z = (Z + 1 = 0)$.

According to the rules,[42] we transform the symbol '$\Omega_{l,1}^{(1)}$' in such a way that expressions of the type '$\Omega_{l,1}^{(1)} a_1$' are changed into corresponding expressions of the type '$(a_1)0$'. We can then assert, in conformity with (γ) and with the rule b, that

(δ) $Z0 = (Z + 1 = 0)$.

Schema 1 of Group II[43] allows us to assert that

(ε) $Z0 = Z0$,

and schema 2′ of group II,[44] with the substitution of '$(x_1 = Z0)$' for 'c', that

(ζ) $(Z0 = (Z + 1 = 0)) \rightarrow ((Z0 = Z0) \rightarrow ((Z + 1 = 0) = Z0))$.

From (δ) and (ζ) we conclude that

(η) $(Z0 = Z0) \rightarrow ((Z + 1 = 0) = Z0)$,[45]

and from (ε) and (η), that

(θ) $(Z + 1 = 0) = Z0$.

From schema 2′ of Group II, when the formula '$\sim x_1$' is substituted for 'c', it follows that

(ι) $((Z + 1 = 0) = Z0) \rightarrow (\sim (Z + 1 = 0) \rightarrow \sim Z0)$.

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39 *Op. cit.*, p. 20.

40 See *op. cit.*, p. 21.

41 *Op. cit.*, p. 20.

42 See *op. cit.*, pp. 8 and 9, and the above-mentioned precautions.

43 *Op. cit.*, p. 15.

44 *Loc. cit.*

45 *Op. cit.*, p. 11.

From (θ) and (ι) we conclude that

(κ) $\sim (Z+1=0) \rightarrow \sim Z0$,

and from (β) and (κ) that

$\sim Z0$,

which contradicts (α).

We may assure ourselves that von Neumann's system, even after the above restrictions of the author, by no means becomes an univocal system, by establishing that the formula '$Z0$' in the proof just given can be derived, using appropriate transformations, on the one hand from the formula '$\Omega^{(1)}_{l,1}\Gamma^{(1)}_{1,1}$', and on the other hand from the formula '$O^{(1)}_2 C_1$'.[46]

Lindenbaum, in his above-mentioned paper, compares my proof of the inconsistency of von Neumann's first system with von Neumann's proof of the 'equivocity' of this system given in one of the paragraphs quoted above, and states the following conclusion:

'Now the entire trouble derives wholly from a different source — again in both constructions in the same way — so that, for example, the rules concerning the elimination of the parentheses[1] {[1]'*Zur Hilbertschen Beweistheorie*, §4, rule III.} often permit too much.[2] {[2] Nevertheless — in an entirely different treatment — parentheses are completely dispensable.} Should parentheses always be retained, such constructions would be quite impossible.'[47]

It seems to me that Lindenbaum's diagnosis doesn't hit the nail on the head. If I somewhat modified the deduction with the help of which I proved the inconsistency of von Neumann's system in my article, I could obtain the same result in the following way without using the rules (a)–(c)[48] concerning the omission of parentheses.

[46] See *op. cit.*, p. 15.

[47] Lindenbaum, *op. cit.*, p. 336.

[48] See von Neumann, *op. cit.*, p. 9.

On the basis of considerations analogous to those in the commentary to thesis (α) on page 81 of my [original] article, we can add to the system the thesis which states that

(a) $(\Omega^{(1)}_{k,1}0 = 0)$.

The transformation of the symbol '$\Omega^{(1)}_{k,1}$', according to which expressions of the type '$\Omega^{(1)}_{k,1}(a_1)$' transform into corresponding expressions of the type '$(a_1) + 0$', allows us to infer from *a* that

(b) $((0) + 0 = 0)$.

According to schema 3 of group III,

(c) $\sim ((0) + 1 = 0)$.

The transformation of the symbol '0' into the symbol '1' transforms the formulae *b* and *c* into formulae which state respectively that

$$((1) + 1 = 1)$$

and

$$\sim ((1) + 1 = 1).$$

SUPPLEMENTARY REMARK VII

While the already published part of my article contains the axioms and the directives of the system $\mathfrak{S}_5$, formulated as precisely as possible, and hence contains *in potentia* the entire formalized system of protothetic, my somewhat later above-mentioned publication entitled *Über die Grundlagen der Ontologie* gives the axioms and directives (and thus implicitly the whole formalized system) of ontology. Sobociński's basic monograph of 1934, also mentioned above, entitled 'Successive simplifications of the axiom-system of Leśniewski's Ontology' reports on the results of axiomatic research in this field by myself, by Tarski, and by the author. As for mereology, the third of the theories discussed above in the résumé of the introduction to my article, I have devoted to it the major part of the already published sections of my work 'On the Foundations of Mathematics'.[49] This work has been appearing since 1927 in *Przegląd Filozoficzny*, and to date comprises the following parts (altogether 171 pages):

1. *Introduction. Section I.* On certain questions concerning the meaning of 'logistic' theses.

Section II. On Russell's 'antinomy' concerning the 'class of classes which are not elements of themselves'.

Section III. On different ways of understanding the words 'class' and 'set'.[50]

2. *Section IV.* On 'Foundations of general set theory. I.'.[51,52]

3. *Section V.* Further theorems and definitions of 'general set theory' from the period up to 1920 inclusive.[53]

4. *Section VI.* Axiomatization of 'general set theory' from the year 1918.

Section VII. Axiomatization of 'general set theory' from the year 1920.

Section VIII. On certain conditions, established by Kuratowski and Tarski, necessary and sufficient for *P* to be the class of *a*.

Section IX. Further theorems of 'general set theory' from the years 1921–3.[54]

5. *Section X.* Axiomatization of 'general set theory' from the year 1921.

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49 S. Leśniewski, '*O podstawach matematyki*' ('On the Foundations of Mathematics'), *Przegląd Filozoficzny* 30–34.

50 Vol. 30, 1927.

51 Vol. 31, 1928.

52 [*Ed. note*: This is the title of Leśniewski's 1916 paper referred to in the résumé of the introduction to his article.]

53 Vol. 32, 1929.

54 Vol. 33, 1930.

Section XI. On 'singular propositions' of the type '$A \in b$'.[55]

In connexion with the expression 'general set theory', which appears here in the titles of different individual sections, I should like to mention that the theory which I now call 'mereology' I formerly called 'general set theory', or 'general class theory'. I ceased using these two names long ago because, in order not to arouse needless misunderstanding, I wanted to distinguish my theory clearly from the various 'set theories' and 'class theories' which, if I may so express myself, possess an 'official' character. As I began to use the word 'class' in mereology in a way incompatible with the tradition of these theories, I made an effort to rely on the most precise analysis possible of the meaning which in practice I employed, employ, and will continue to employ in the future. As it happened, the meaning I used would seem to harmonize to a great extent with common intuition. I used this word in particular in discussions concerning the 'evidence' or 'non-evidence' of various theses which play a part in the different 'antinomies' constructed by class theorists. I was never able to conceive of a sense of the word 'class' in which I should be at all inclined to ascribe to classes the totality of the properties postulated in these theses. Expressions of the type 'class of objects a' are, on the basis of my mereology, names denoting definite and quite ordinary objects. These expressions naturally have nothing in common either with any mythology of 'classes', considered as objects of some 'higher type' or 'higher order', or with a use of the word 'class' in which the latter is not the name of any object(s), but rather a surrogate *façon de parler* of some entirely different syntactical type, as for example in the system of Whitehead and Russell.[56] The totality of theorems of my system of the foundations of mathematics, which in practice can be handled as theoretical correlates of this

55 Vol. 34, 1931.

56 See Alferd North Whitehead and Bertrand Russell, *Principia Mathematica*, vol. 1, 2nd edition (Cambridge, 1925), pp. 71 and 72.

or that thesis of these authors' 'theory of classes', forms a proper part of my ontology.[57]

[57] [*Ed. note*: The two pages of bibliographical abbreviations and typographical corrections to his original article of 1929 with which Leśniewski concluded his paper are here omitted.]

AN ANNOTATED LEŚNIEWSKI BIBLIOGRAPHY

V. Frederick Rickey

This bibliography is intended to be a complete list of papers relevant to Leśniewski's work in logic. Until the publication of the present volume, the papers of Stanisław Leśniewski (1886–1939) have been very difficult to obtain. As much of his work was unpublished at the time of his death and subsequently destroyed in the war, it has been even more difficult to obtain a proper understanding of his work and views on the foundations of mathematics. That this has been possible at all is due mainly to the publications and lectures of three of his disciples who, fortunately, have carried on his work: Czesław Lejewski in England, Jerzy Słupecki in Poland, and Bolesław Sobociński in the United States. Today there is a great wealth of literature dealing with the logical systems of Leśniewski. But even this is difficult to master, for it appears in several languages in nearly a hundred periodicals from the fields of mathematics, philosophy, and linguistics. It is hoped that the present work will make these papers more accessible.

This bibliography is intended to be complete and comprehensive. If a paper concerns Protothetic, Ontology, or Mereology, then it should clearly be included. But what about papers on the equivalential calculus, type theories, free logics, definitions, semantical categories, etc.? Some papers from each of these areas have been included, but the criteria is whether the paper contains information relevant to Leśniewski's work and its development. In general technical papers have been included more readily than philosophical ones; older papers have been included more readily than newer ones. No attempt has been made to record every time that Leśniewski's name is mentioned in the literature — though, at first, it may seem that I have done so. Neither have we listed every paper that Leśniewski cited.

This bibliography has been annotated to enhance its usefulness. But the annotations also have their difficulties. They are not intended to be brief reviews or even synopses. The length of the annotation should not be taken as a measure of the value of the paper. The annotations only concern those parts of the papers which are relevant to Leśniewski's logic, so many important results outside of this scope are not noted.

Reviews have been included if they contain useful information. But no attempt has been made to search for reviews or author's abstracts, much less to list every review of a work. None have been included merely for the sake of cross reference.

Considerable effort has been expended to eliminate errors of fact from the titles and of interpretation from the annotations. My experience with the preliminary versions of this bibliography indicates that I have not been entirely successful. Although I have tried, I have not been able to examine copies of all of the papers listed here. Though I have examined the great majority of the papers, I have not read all of these carefully. When I was unable to examine a paper I tried to verify the reference from several sources, but even this has not always been possible. Therefore, I would greatly appreciate it if authors would notify me at the following address of errors of fact or interpretation that they find here:

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I would also appreciate it if authors would send reprints of their papers as someday I hope to publish a supplement to this bibliography.

The following abbreviations have been used throughout this bibliography:

MR	*Mathematical Reviews,*
JSL	*The Journal of Symbolic Logic,*
NDJFL	*Notre Dame Journal of Formal Logic,*
Z	*Zentralblatt für Mathematik und ihre Grenzgebiete.*

Papers are cited by giving title, journal, volume number, and pages, in that order. More complete information is given for items which are difficult to identify.

This bibliography could never have been published without the assistance of the many people who informed me of additional items and of errors in the two preliminary versions which were circulated in July 1972 and June 1976. I have forgotten completely who the 'several friends and colleagues' were who urged me to circulate the first preliminary version, but they are to be thanked again. But I cannot forget, nor can I adequately thank, the many who have helped by sending reprints, titles, suggestions, and errata: T. Batóg, C. Davis, D. Henry, K. Iseki, A. Ishimoto, G. Kalinowski, J. Kearns, T. Kubiński, G. Küng, C. Lejewski, E. Luschei, S. McCall, R. Myers, M. O'Neil, T. Scharle, V. Sinisi, B. Sobociński, J. Srzednicki,

S. Surma, R. Wolf, and R. Zuber. I have intentionally not cited the names of these individuals in the annotations, for I alone must remain responsible for the errors that surely remain. Finally, I must say that this work could not have been completed without the excellent work of my two typists: Linda Shellenbarger, at Bowling Green State University, who typed the two preliminary versions and Libby DeMyer, at Indiana University at South Bend, who typed the final version.

V. F. R.
South Bend
January, 1978

AJDUKIEWICZ, KAZIMIERZ (1890–1963)

[1923] 'O intencji pytania 'co to jest P', (Referat z odczytu)' *Ruch Filozoficzny* 7, 152b–153a.

[1926] 'Założenia logiki tradycyjnej' ('Foundations of Traditional Logic'), *Przegląd Filozoficzny* 29, 200–229.

The directive for distributing quantifiers that is discussed on page 210 is credited to Leśniewski.

[1928] *Główne zasady metodologii nauk i logiki formalney* (*Essential Principles of the Methodology of Science and Formal Logic*), authorized typescript, Warsaw, 304 pp.

[1934a] 'W sprawie 'uniwersaljów', *Przegląd Filozoficzny* 37, 219–234. Reprinted in [1960], 169–210.

Using Leśniewski's theory of semantic categories, Ajdukiewicz refutes Kotarbiński's 'proof' of the non-existence of individuals. He introduces here, for the first time, a convenient notation for the description of particular semantical categories.

[1934b] 'Logistyczny antyirracjonalizm w Polsce', *Przegląd Filozoficzny* 37, 399–408. [1935a] is a German translation.

[1935a] 'Der logistische Antiirrationalismus in Polen', *Erkenntnis* 5, 151–161. Translation of [1934b]. See the bibliography, pp. 199–203.

Contains interesting historical remarks about the Polish school of logic.

[1935b] 'Die syntaktische Konnexität', *Studia Philosophica* 1, 1–27, English translation in McCall [1967]. Partial English translation in the *Review of Metaphysics* 20, 635–647. Reviewed by Weinberg, *JSL* 3, 58.

The seminal paper on the application of the notation for the semantical categories to ordinary language.

[[1949] 'On the Notion of Existence. Some Remarks Connected with the Problem of Idealism', *Studia Philosophica* 4 (for 1949–50) published 1951), 7–22. Reviewed *JSL* 17, 141–142 by Quine.

Ajdukiewicz considers several kinds of existence and applies them to the question of fictitious objects and also the metaphysical controversy over what is real. He uses Ontology. In the review Quine interprets Ontology in set theory and thus concludes that quantification commits Leśniewski to abstract entities.

[1960] *Język i poznanie*, Warsaw. Volume 1 published 1960, volume 2, 1965.

Selected papers from 1920–1939 and 1945–1963. Contains a reprint of [1934b] and Polish translations of [1935b] and [1949].

[1967] 'Syntactic Connexion', in McCall [1967], 207–231. English translation of [1935b].

[1973] *Problems and Theories of Philosophy*, Cambridge University Press. Reviewed by Giedymin, *British Journal for the Philosophy of Science* 25, 189–206.

A basic philosophy text. The problem of ideal objects is treated.

Note: For a complete list of the publications of Ajdukiewicz, see *Studia Logica* 16, 39–43.

ANDREWS, PETER

[1963] 'A Reduction of the Axioms for the Theory of Propositional Types', *Fundamenta Mathematicae* 52, 345–350. Reviewed *JSL* 30, 385 by J. R. Guard.

A simplification of Henkin [1963].

ANGELELLI, IGNACIO

[1967] *Studies on Gottlob Frege and Traditional Philosophy*, D. Reidel Publishing Company.

Leśniewski had "the only philosophically acceptable manner of planning a 'way out' of the antinomies" (p. 218).

APOSTEL, LEO

[1960] 'Logic and Ontology', *Logique et Analyse* 3, #11–12, 202–225.

He claims that for Leśniewski and Heinrich Scholz, logic was ontology (in the philosophical sense). Claims that Leśniewski [1930a] states that "the science of logic has quite explicitly as its object the study of certain very general laws of being" (quoting Apostel who doesn't quote Leśniewski). Cites Lejewski [1954b] and [1958b] as clear explanations of Leśniewski's position. The discussion ends by saying that Leśniewski defines existence to exclude the null class and then goes on to make some incomprehensible remarks about methodology.

[1976] 'Mereology, Time, Action and Meaning', *Festschrift Gerhard Frey Zum 60. Geburtstag*, Innsbruck, 189–233.

ARAI, YOSHINARI and TANAKA, SHOTARO

[1966] 'A Remark on Propositional Calculi with Variable Functors', *Proc. Japan Acad.* 42, 1056–1057. Reviewed *MR* 35 #4089, by B. Lercher.

The title refers to the system of Leśniewski's Protothetic in Meredith [1951]. The authors derive several equivalences in that system.

ASENJO, F. G.

[1962] *El Todo y Las Partes: Estudios de Ontologia Formal* (Whole and Parts: Studies in Formal Ontology), Editorial Martínez de Murguía, Madrid, 276 pp.

[1965] 'Theory of Multiplicities', *Logique et Analyse* 8, #30, 105–110.

[1969] 'Mathematical Organisms', *Logique et Analyse* 12, #48, 301–310.

There are connections between Mereology and the systems developed in these three papers.

[1976] 'Leśniewski's Work on Non-Classical Set Theories', *XXIInd Conference on the History of Logic*, 5–9 July 1976, Kraków (Abstract of a lecture).

BACON, JOHN

[1976] Syllogistic without existence, *NDJFL* 8, 195–219.

[1974] The untenability of genera, *Logique et Analyse* 17, #65–66, 197–208.

BAR-HILLEL, JEHOSHUA (1915–1975)

[1950] 'On Syntactical Categories', *JSL* 15, 1–16. Reprinted in [1964]. Reviewed by Lorenzenen, *MR* 11, 635.

[1953] 'A Quasi-Arithmetical Notation for Syntactic Description', *Language* 29, 47–58. Reprinted in [1964].

[1954] 'Indexical Expressions', *Mind* 63, 359–379. Reprinted in [1970]. Reviewed by J. F. Thomson, *JSL* 22, 320–321.

Stresses the importance of *inscriptional* semantics for Philosophy.

[1960] 'On Categorical and Phase Structure Grammars', *The Bulletin of the Research Council of Israel* 9F, 1–16. Reprinted in [1964]. With Gaifman and Shamir.

[1964] *Language and Information: Selected Essays on their Theory and Application*. Addison–Wesley Publ. Co. Contains reprints of [1950], [1953], and [1960].

[1970] *Aspects of Language: Essays in Philosophy of Language, Linguistic Philosophy, and Methodology of Linguistics*. The Magnes Press, Jerusalem. Contains a reprint of [1954].

BARNETT, DENE

[1967] 'An Outline of Nominalistic Arithmetic', *JSL* 32, 575 (abstract).

Uses Mereology and Tarski's concatenation theory to define natural number, addition, multiplication, rational and real numbers, and derivations are indicated of interpretations for standard axioms for first order arithmetic.

[1976] 'Leśniewski's Mereology, Applications and Problems', *XXIInd Conference on the History of Logic*, 5–9 July 1976, Kraków. (Abstract of a lecture).

BATÓG, TADEUSZ

[1961a] 'Logiczna rekonstrukcja pojęcia fonemu' (A Logical Reconstruction of the Concept of Phoneme), *Studia Logica* 11, 139–183. With Russian and English summaries.

Uses Mereology in the axiomatization of phonology.

[1961b] 'Critical Remarks on Greenberg's Axiomatic Phonology', *Studia Logica* 12, 195–205.

This is a criticism of J. H. Greenberg's 'An Axiomatization of the Phonologic Aspect of Language', which appears in *Symposium on Sociological Theory*, ed. L. Gross, Evanston–New York, 1959. Batóg closes his paper by remarking that "Greenberg's system would gain much in simplicity and naturality if it were based on Leśniewski's mereology."

[1962] 'A Contribution to Axiomatic Phonology', *Studia Logica* 13, 67–80.

With Polish and Russian summaries. Reviewed by S.-Y. Kuroda, *JSL* 31, 251.

The system is based on Leśniewski's Mereology and is a modification of the system of axiomatic phonology in the author's [1961a].

[1967] *The Axiomatic Method in Phonology*, Routledge and Kegan Paul, London.

The axiomatic system of phonology presented in this monograph is based on Mereology as extended by Tarski in Woodger [1937].

[1969] 'A Reduction in the Number of Primitive Concepts of Phonology', *Studia Logica* 25, 55–60. With Polish and Russian summaries.

BERGMANN, GUSTAW

[1967] *Realism, a Critique of Brentano and Meinong*, University of Wisconsin Press.

Deals with similar problems as Leśniewski.

BETH, E. W.

[1966] 'Remarks on the Paradoxes of Logic and Set Theory', *Essays on the Foundations of Mathematics*, dedicated to A. A. Fraenkel on his 70th birthday, Jerusalem, 307–311.

Mentions pseudodefinitions and credits them to Leśniewski (cf. Tarski [1956], 223, 283).

BIRD, OTTO ALLEN

[1975] Leśniewski, Stanisław, *Encyclopedia Brittanica*, Micropedia VI, 166 and Macropedia X, 832–834.

BLACK, ROBERT

[1973] 'In Defense of *Principia Mathematica*', *Mind* 82, 611–612.

Comments on Nemesszeghy [1971].

BOCHEŃSKI, INOCENTY M.

[1939] 'La Logique de Théopraste', *Collectanea Logica* 1, 195–304.

As the publishing house was bombed this was only known through a review of H. Scholz, *Z* 22, 290–291, until republished as [1949a].

[1947] *Pologne 1919–1939*, Vol. III, *Vie Intellectuelle et Artistique*, Éditions de la Baconnière, Neuchâtel.

Chapter 3.1, written by Bocheński, contains a good discussion of Polish Philosophy and Logic.

[1948] 'On the Categorical Syllogism', *Dominican Studies* 1, 35–57.

[1949a] 'La Logique de Théophraste', *Collectanea Friburgensia*, Nouvele série, 32.

[1949b] 'On the Syntactical Categories', *The New Scholasticism* 23, 257–280. Reviewed by J. Bendiek, *JSL* 16, 221–222. Reprinted in Menne [1962].

Contains a nice introduction to the theory of syntactical categories. He argues that a theory of syntactical categories can resolve the logical, but not the semantical, antinomies.

[1956] *The Problem of Universals,* University of Notre Dame Press. Reprinted in Menne [1962].

BORKOWSKI, LUDWIK

[1968] Kilka uwag o pojęciu definicji (Some Remarks About the Notion of Definition), *Studia Logica* 23, 59–70. With Russian and English summaries. Reviewed by P. Materna, *JSL* 35, 468.

Extends the translatability condition so that the non-creativity of definitions is guaranteed.

[1970] *Logika formalna* (Formal Logic), PWN, Warsaw.

Contains a chapter on Ontology and many remarks on Leśniewski's method in the propositional calculus.

BORNSTEIN, BENEDYKT

[1914] Podstawy filozoficzne teorji mnogości (Philosophical Foundation of the Theory of Sets), *Przegląd Filozoficzny* 17, 183–193.

This prompted Leśniewski [1914b].

[1915] 'Polemika. W sprawie recenzji p. St. Leśniewskiego rozprawy mojej p. t. "Podstawy filozoficzne teorji mnogości" ', *Przegląd Filozoficzny* 18, 121–140.

A reply to Leśniewski [1914b].

BOUDREAU, JACK C.

[1976] 'Set Theoretical Models for Leśniewski's Logical Systems', *XXIInd Conference on the History of Logic*, 5–9 July 1976, Kraków, p. 2–5. (abstract of a lecture).

Provides a set theoretical Model Theory for Ontology and proves a soundness theorem (consistency).

[19..] 'A Model-Theoretic Analysis of Leśniewski's Logical Systems', *NDJFL*, *Z* 332.02013.

Provides a "set-theoretic model for Leśniewski's logical systems, which I believe, is in keeping with their constructive, or 'nominalistic' spirit". A soundness theorem is proved.

BURGE, TYLER

[1972] 'Truth and Mass Terms', *The Journal of Philosophy* 69, 263–282.

The Calculus of Individuals is used here.

[1975] 'Truth and Singular Terms', *Noûs* 8, 309–325.

[1977] 'A Theory of Aggregates', *Noûs* 11, 97–117.

Canty, John Thomas

[1967] *Leśniewski's Ontology and Gödel's Incompleteness Theorem*, Ph. D. Dissertation, University of Notre Dame, under the direction of Sobociński. Published as [1969a], [1969b].

[1968] 'On Symbolizing Singularity S5 Functions', *NDJFL* 9, 340–342.

Leśniewski's wheel and spoke notation is used to symbolize the 16 unary functors of S5. This is done in such a way that the symbolism indicates the intended interpretation and also the syntactical connections between the functors. This work is based on a normal form representation of G. J. Massey.

[1969a] 'The Numerical Epsilon', *NDJFL* 10, 47–63. Abstract, *JSL* 32, 432. *MR* 39 #2608.

Ontology, extended by an axiom of infinity, is used to derive Peano's arithmetic. Section one gives the main theses of this derivation, which parallels the work of *Principia Mathematicae*. In section two a numerical epsilon is defined and it is shown that an internal ontological model for this epsilon exists. Using the numerical epsilon, the paper concludes by providing a characteristically ontological model for Peano's arithmetic.

[1969b] 'Leśniewski's Terminological Explanations as Recursive Concepts', *NDJFL* 10, 337–369.

[1969c] 'Ontology: Leśniewski's Logical Language', *Foundations of Language* 5, 455–469.

Autorreferrat, *Z* 198, 15.

[1971] 'Elementary Logic Without Referential Quantification', *NDJFL* 12, 441–446. Abstract, *JSL* 38, 352 and Autorreferrat *Z* 205, 304.

[1976] 'The Proper Interpretation of Ontology', *XXIInd Conference on the History of Logic*, 5–9 July 1976, Kraków, pp. 6–8, (abstract of a lecture).

[1977] 'The Proper Interpretation of Ontology', *Studia Logica* 36.

Chikawa, Kazuo

[1967] 'On Equivalences of Laws in Elementary Protothetics I, II', *Proceedings of the Japan Academy* 43, 743–747; 44, 56–59. Reviews: *MR* 36 #4960, *MR* 37 #2576, *Z* 197, 3 (J. Bacon).

Gives generalizations of Słupecki's six laws that describe the properties of functions of one variable in elementary Protothetics. Shows each law of functions of one argument is equivalent to its corresponding law of functions in two arguments.

CELIŠČEV, V. V.

[1974] 'Logičeskaja istina i empirizm' (The Logical Truth and Empirism), 'Nauka' publ., Novosibirsk.

In §3 (pp. 24–34) several systems with substitutional and referential quantification are briefly discussed.

[1976] 'Logika suščestvovanija' (Logic of Existence), 'Nauka' publ., Novosibirsk.

Ch. III, §7: Leśniewski's theory of existence (pp. 82–93). Based on Prior [1955a], [1962], and Lejewski [1954b].

Ch. IV, §4: Leśniewski's theory of descriptions (pp. 111–118). The use of the copula in Ontology and, referring to Lejewski [1954b], Luschei [1962], Prior [1955a] and [1962], substitution of descriptions for variables are explained.

Ch. V, §3: Interrelations of two methods of quantification and two conceptions of logic (pp. 132–138). A comparison of substitutional and referential quantification.

CHISHOLM, RODERICK

[1973] 'Parts as essential to their wholes', *The Review of Metaphysics* 26, 581–603. See Plantinga [1975].

[1975] 'Mereological Essentialism: Some Further Considerations', *The Review of Metaphysics* 28, 477–484.

CHURCH, ALONZO

[1951] 'The need for abstract entities in semantic analysis', *Proceedings of the American Academy of Arts and Sciences* 80, 100–112.

[1956] *Introduction to Mathematical Logic*, Volume 1, Princeton University Press.

There are numerous comments about Leśniewski and his views on definitions and Protothetic.

[1972] 'Axioms for Functional Calculi of Higher Order', *Logic and Art: Essays in Honor of Nelson Goodman*, ed. by Richard Rudner and Israel Scheffler, Bobbs-Merrill, 97–213.

CHWISTEK, LEON (1884–1944)

[1922] 'Zasady czystej teorji typów' (Principles of the Simple Theory of Types), *Przegląd Filozoficzny* 25, 359-391.

Leśniewski's work is discussed on p. 372.

[1924] 'The theory of constructive types. Principles of Logic and Mathematics', *Rocznik Polskiego Towarzystwa Matematycznego (Annales de la Société Polonaise de Mathématiques)* 2, 9–48 and 3, 92–141.

There are several comments about Leśniewski.

[1935] *Granice nauki. Zarys logiki i metodologii nauk ścisłych* (The Limit of Science. Outlines of Logic and the Methodology of the Exact Sciences), Lwów–Warszawa, Książnica — Atlas. [1948] is an English translation.

[1948] *Limits of Science*, Routledge and Kegan Paul. Revised and enlarged translation of [1935]. Reviewed by Myhill, *JSL* 14, 119–125.

Criticizes Leśniewski's Ontology (p. 103) and his views on the empty class (pp. 113–114).

CIRULIS, JANIS P.

[1975] *Logika s Vključeniem* (Logic with Inclusion). In Russian. *Zeitschrift für math. Logik und Grundlagen der Math.* 21, 247–266.

Can be regarded as a realization of some ideas of Canty [1971].

CLAY, ROBERT E.

[1961] *Contributions to Mereology*, Ph.D. Dissertation, University of Notre Dame, under the direction of Sobociński.

[1965] 'The Relation of Weakly Discrete to Set and Equinumerosity in Mereology', *NDJFL* 6, 325–340. Part of [1961].

It is shown that under the condition of weakly discrete the collective and distributive classes become alike with respect to equinumerosity. Hence we can prove analogues of set-theoretical formulae. Also, for a certain type of statement, discrete and weakly discrete are equivalent.

[1966] 'On the definition of Mereological class', *NDJFL* 7, 359–360. Reviewed *MR* 38 #2003.

If the usual mereological definition of class is replaced by the short definition of class:

$[Aa] \therefore A \in Kl(a) . \equiv \because A \in A \because [B] \because a \subset el(B) . \equiv . A \in el(B)$,

then the resulting system is *not* equivalent to Mereology. Models are given to show this system is weaker than Mereology.

[1968] 'The Consistency of Leśniewski's Mereology Relative to the Real Number System', *JSL* 33, 251–257. Reviewed by Canty, *Z* 182, 318.

As the base for the model take the set of all real numbers whose decimal expansion contain only zeros and ones with the exception of 0. This guarantees that representations are unique. Then define 'A is an element of B' to mean that every place where A has a one in its decimal expansion, B does also. All axioms and rules are verified under this interpretation. The real number system is introduced axiomatically into Ontology; thus the rules of Ontology go over.

[1969] 'Sole Axioms for Partially Ordered Sets', *Logique et Analyse* 12, #48, 361–375.

[1970] 'The Dependence of a Mereological Axiom', *NDJFL* 11, 471–472.

In the standard axiom system based on element the axiom which says that every individual is an element of itself is dependent. This is not true in the standard axiom system based on part, even though Tarski [1929] claims otherwise.

[1971] 'A Model for Leśniewski's Mereology in Functions', *NDJFL* 12, 467–478. Also see the corrections, *NDJFL* 16, 269–270. Reviews: *Z* 188, 15; *MR* 50 #12708; *Z* 301, 02028 (Autorreferrat).

[1972] 'On Inductive Finiteness in Mereology', *NDJFL* 13, 88–90. Reviewed *MR* 45 #6582.

In Mereology he proves that if a is finite then the set of a is finite. Sobociński had previously proved this under the hypothesis that a is discrete.

[1973] 'Two Results in Leśniewski's Mereology', *NDJFL* 14, 559–564. *Z* 267, 02008 (Autorreferrat).

The short definition of class can be proved without the use of auxiliary definitions. (Leśniewski [1927] used the mereological notion of 'set' in his proof). Also, the results of [1965] still hold in the weakened system using the short definition of class.

[1974a] 'Relation of Leśniewski's Mereology to Boolean Algebra', *JSL* 39, 638–648.

Disproves the claim of Tarski [1935] and Grzegorczyk [1955] that the model of Mereology and the models of complete Boolean algebra with zero deleted are identical.

[1974b] 'Some Mereological Models', *NDJFL* 15, 141–146.

The non-empty regular sets of any topological space form a Boolean algebra with zero deleted. Thus, by [1974a], we have a variety of models of Mereology. For example, Euclidean space provides a model of atomless Mereology.

[1975] 'Single Axioms for Atomistic and Atomless Mereology', *NDJFL* 16, 345–351.

Provides single axioms shorter than those of Lejewski [1973a].

COHEN, LAURENCE JONATHON

[1966] 'Does Logic Deny the Possibility of an Empty Universe?', in his *The Diversity of Meaning*, 255–264. Methuen & Co., London, 2nd ed.

asically a criti ue of Lejewski [1954b]

Sympathetic but mistaken.

CORCORAN, JOHN; FRANK, WILLIAM and MALONEY, MICHAEL

[1974] 'String Theory', *JSL* 39, 625–637.

A valuable paper dealing with the same subject as Rickey [1972].

CRESSWELL, M. J.

[1966] 'Functions of Propositions', *JSL* 31, 545–560.

This has some connections with Protothetic.

CROSSLEY, JOHN, compiler

[1975] 'Reminisces', *Algebra and Logic*, Lecture Notes on Mathematics, #450, Springer, 1–...

An edited transcript of a group of logicians, including S. C. Kleene, M. Morley, and A. Mostowski, reminiscing about the early history of Mathematical Logic, especially recursion theory. ncluded is information about the logical climate in Poland and personal information about E. Post. Leśniewski is mentioned.

CURRY, HASKELL B.

[1961] 'Some Logical Aspects of Grammatical Structure', *Struccture of Language and its Mathematical Aspects*, ed. by R. Jacobson, Proc. 12th Sympos. in Applied Mathematics, Providence, R. ., American Mathematical Society, 56–68.

CZEŻOWSKI, TADEUSZ

[1949] *Logika. Podręcznik dla studiujących nauki filozoficzne* (Logic. A Textbook for Philosophers), Warszawa, Państwowe Zakłady Wydawnictw Szkolnych. Reviewed *JSL* 15, 206.

Contains a brief exposition of Protothetic and the theory of semantic categories.

[1974] 'Polish Philosophy in the nterwar Period 1919–1939', *Dialectics and Humanism* 1–27–35, Summer '74.

DĄMBSKA, IZYDORA

[1948] 'W sprawie tzw. nazw pustych' (About the So-called Objectless Names), *Przegląd Filozoficzny* 44, 77–81. English summary, p. 289.

DAVIS, CHARLES C., JR.

[1973] *An Investigation Concerning the Hilbert–Sierpiński Logical Form of the Axiom of Choice*, Ph.D. Dissertation, University of Notre Dame, under the direction of Sobociński.

[1974] 'Some Semantically Closed Languages', *Journal of Philosophical Logic* 3, 229–240.

[1975] 'An Investigation Concerning the Hilbert–Sierpiński Logical Form of the Axiom of Choice, *NDJFL* 16, 145–184.

[1976] 'A Note on the Axiom of Choice in Leśniewski's Ontology', *NDJFL* 17, 35–43.

Concerns the Axiom of Choice for many-link functors.

DEMBOWSKI, J.

[1952] *Science in New Poland*, London, Lawrence and Wishart, 1952, 59pp.

DE PATER, W. A.

[1974] 'Semiotiek in Polen', *Tijdschrift voor Philosophie* 36, 762–777.

DITCHEN, RYSZARD; GLIBOWSKI, EDMUND, and KOŚCIK, STANISŁAW

[1963] 'O pewnym układzie pojęć pierwotnych geometrii elementarnej' (On a System of Foundations of Elementary Geometry), *Acta Universitatis Wratislaviensis. Matematyka, fizyka, astronomia* 4, no. 17, 5–11.

DJANKOV B.

[1974] 'Rol' teoriji semantičeskich kategorij v obosnovanii sovremennich logičeskich teorij' (The Role of Semantical Categories in Foundations of Modern Logical Theories), *Philosophy in the Contemporary World. Philosophy and Logic* (in Russian), 'Nauka', Moscow, 439–457.

"The problem stated there is considered more as historical-logical than purely theoretical" (p. 439). Sections 3, 4 (pp. 442–452) contain a brief review of the development of the theory of semantic categories in the works of Husserl, Leśniewski, Ajdukiewicz, and Tarski.

DREWNOWSKI, JAN FRANCISZEK

[1934] 'Zarys programu filozoficznego' (Outline of a Philosophic Program), *Przegląd Filozoficzny* 37, 3–38, 150–181, 262–292.

DUDMAN, V. H.

[1973] 'Frege on Definition', *Mind* 82, 609–610.
Comments on Nemesszeghy [1971].

DUMMET, MICHAEL

[1973] 'Frege's Way Out: A Footnote to a Footnote', *Analysis* 33, 139–140.

DUPRAZ, M and ROUAULT, J.

[1968] *Lexis–Affirmation–Négation: Edute fondée sur les classes*, Centre d'étude pour la traduction automatique Grenoble, document G. 2400–A.
Calls attention to the value of Leśniewski's logic for linguistics.

EBERLE, ROLF A.

[1965] *Nominalistic Systems — the Logic and Semantics of Some Nominalistic Positions*, Ph.D. dissertation, University of California at Los Angeles, under the direction of Donald Kalish.

[1967] 'Some Complete Calculi of Individuals', *NDJFL* 8, 267–278.

[1968] 'Yoes on Non-Atomic Systems of Individuals', *Noûs* 2, 399–403.
Tries to formulate a principle of individuation suitable for non-atomic systems. The problem was raised by Yoes [1967]. The solution was criticized by Schuldenfrei [1969].

[1969a] 'Non-Atomic Systems of Individuals Revisited', *Noûs* 3, 431–434.
An improvement of [1968].

[1969b] 'Denotationless terms and predicates expressive of positive qualities', *Theoria* 35, 104–124. Reviewed by J. Corcoran, *MR* 43 #31.
Presents a first order logic which permits empty universes in interpretations. The interesting and plausible semantic theory can account for partially defined operators.

[1970] *Nominalistic Systems*, Synthese Library, Dordrecht, Holland.

[1974] 'Ontologically Neutral Arithmetic', *Philosophia* 4, 67–94.

EDWARDS, PAUL, ED.

[1967] *The Encyclopedia of Philosophy*, The Macmillan Company and the Free Press, New York, and Collier–Macmillan Limited, London, 8 vols.

The following articles are of interest (they are not listed separately in this bibliography):

'Ajdukiewicz, Kazimierz', I, 62–63, by Z. A. Jordan.

'Brentano, Franz', I, 363–368, by Roderick M. Chisholm.

'Chwistek, Leon', II, 112–113, by H. Hiż.

'Definition', II, 314–324, by Raziel Abelson.

'Existence', IV, 509–513, by A. N. Prior.

'Goodman, Nelson', II, 225–237, by Richard S. Rudner.

'Kotarbiński, Tadeusz', IV, 361–363, by Z. A. Jordan.

'Leśniewski, Stanisław', IV, 441–443, by C. Lejewski.

'Polish Logicians', IV, 566–568, by A. N. Prior.

'Łukasiewicz, Jan', V, 104–107, by C. Lejewski.

'Polish Philosophy', VI, 363–370, by George Krzywicki-Herburt.

'Semantics, history of', VII, 358–406, by Norman Kretzmann.

'Syntactical and semantical categories', VIII, 57–61, by Y. Bar-Hillel.

'Tarski, Alfred', VIII, 77–81, by A. Mostowski.

'Twardowski, K.', VIII, 166–167, by George Krzywicki-Herburt.

'Types, theory of', VIII, 168–172, by Y. Bar-Hillel.

Reviews of the above articles can be found in *JSL* 35, 295–310, by W. Craig and B. Mates.

EVENDEN, J.

[1962] 'A Lattice Diagram for the Propositional Calculus', *Mathematical Gazette* 46, 119–122.

EVENDEN, J. and HUBBELING, H. G.

[1969] 'A Synthesis of Truth-Function Diagrams', *Logique et Analyse* 12, #46, 123–128.

FARBER, MARVIN

[1943] *The Foundations of Phenomenology*, Third Edition, 1967, State University of New York Press, Albany.

Chapter X (pp. 283–312) contains an analysis of wholes and parts as presented by Husserl.

FLOYD, W. F. and HARRIS, F. T. C., ED.

[1964] Joseph Henry Woodger, Curriculum Vitae, *Form and Strategy in Science, Studies Dedicated to Joseph Henry Woodger on the Occasion of his Seventieth Birthday*, D. Reidel, 1–6.

"Amongst the philosophers with whom Woodger now came into contact was Professor K. R. Popper who introduced him to A. Tarski. In his analysis of the relation 'part of', a pre-requisite for the study of theories involving statements about structure, Woodger had independently developed a system that was similar to Leśniewski's *Mereology*. Tarski's excitement at the first development of an application of such a system was a considerable stimulus to Woodger. In 1935 he went to Poland in order to meet with the Polish school of Logicians and to discuss mutual ideas, especially with Łukasiewicz and Tarski with whom he had been in correspondence." (p. 4).

"In 1949 he was invited to give the Tarner Lectures by Trinity College Cambridge. In these he expanded more fully his view that a nominalistic attitude was the correct basis for the language of science." (p. 5).

FRAENKEL, ABRAHAM A. and BAR-HILLEL, YEHOSHUA

[1958] *Foundations of Set Theory*, North Holland.

Pp. 185–188 discuss Ontology in barest outline. They comment: "We seem to stand at the verge of a real interest in the work of these two logicians [Leśniewski and Chwistek] that has already fertilized the thought of many a worker in the foundations of Mathematics" (p. 186). Pp. 168–171 contain a discussion of Leśniewski's semantical categories.

FRAENKEL, ABRAHAM A.; BAR-HILLEL, YEHOSHUA; and LEVY, AZRIEL

[1973] *Foundations of Set Theory*, North Holland, second revised edition of Fraenkel and Bar-Hillel [1958].

FRANZKE, NORBERT and RAUTENBERG, WOLFGANG

[1972] 'Zur Geschichte der Logik in Polen', *Quantoren — Modalitäten — Paradoxien*, Beiträge Zur Logik, 39–94, *Z* 305, 02002.

FREGE GOTTLOB (1848–1925)

[1893] *Grundgesetze der Arithmetik*, Jena, 2 volumes, reprinted 1962 by Georg Olms Verlagsbuchhandlung, Hildesheim. Parts are available in English translation.

Leśniewski, who was aware of Frege's work from the beginning, read this carefully. See especially §33 on definitions.

[1895] 'Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra der Logik', *Archiv für systematische Philosophie* 1, 433-456. Also in *Translations from the Philosophical Writing's of Gottlob Frege*, ed. P. Geach and M. Black, 2nd ed. 1960, Oxford, 86–106.

Compare Frege's notions of Class with Leśniewski's.

GALLIE, R. D.

[1973] 'A. N. Prior and Substitutional Quantification', *Analysis* 34, 65–69.

[1975] 'Substitutionalism and Substitutional Quantification', *Analysis* 35, 97–101.

GEACH, PETER T.

[1956] 'On Frege's Way Out', *Mind* 63, 408–409.

A generalized form of Leśniewski's proof that Frege's way out of Russell's antinomy only generates new contradictions.

[1960?] 'A Program for Syntax', *Syntheses* 22, 3–17.

Makes use of Ajdukiewicz' notation.

[1976] 'On So-Called Ontological Definitions' *XXIInd Conference on the History of Logic*, 5–9 July 1976, Kraków, p. 1. (Abstract of a lecture).

Points out a close similarity between the ways that Ockham and Leśniewski wrote definitions. Claims that Leśniewski failed to observe the Fregean canons of definition. He cites an example in Prior [1956] which obeys Leśniewski's terminological explanations for definitions, yet which leads to contradictions. He suggests definitions be treated in the style of Quine's abbreviative definitions. The consistency proof of Kruszewski [1925] as well as remarks in Rickey [1976] show that Geach is not interpreting Prior correctly.

GILES-PETERS, ANDREW ROBERT

[1972] *Nominalistic Philosophy of Logic, with Particular Reference to the Systems of Stanisław Leśniewski*, Master of Arts thesis, Philosophy Department, La Trobe University, Bundoora, Victoria.

GLIBOWSKI, EDMUND

[1969] 'The Application of Mereology to Grounding of Elementary Geometry', *Studia Logica* 24, 109–129. With Polish and Russian summaries.

GLIBOWSKI, E. and SŁUPECKI, J.

[1956] 'Geometria sześcianów' (Cube Geometry), *Zeszyty Naukowe — Matematyka, Wyższa Szkoła Pedagogiczna*, Opole, 38–47.

Based on Mereology.

GOODELL, JOHN D.

[1952] 'The Foundations of Computing Machinery', *The Journal of Computing Systems* 1, 1–13. Reviewed, *JSL* 18, 283.

Leśniewski's wheel and spoke notation is adopted here.

[1953a] 'The Foundations of Computing Machinery', Part II, *The Journal of Computing Machinery* 1, 86-110. Reviewed, *JSL* 18, 348.

This paper, which deals with the calculus of propositions with quantifiers, uses an adoption of Leśniewski's wheel and spoke notation.

[1953b] 'Notes on Decision Element Systems Using Various Practical Techniques', *The Journal of Computing Systems* 1, 196–199. Reviewed, *JSL* 19, 143.

The wheel and spoke notation is used.

GOODMAN, NELSON

[1951] *The Structure of Appearance*, Harvard University Press. Second Edition, 1966, Bobbs–Merrill.

There is much of value here on the Calculus of Individuals and its applications, as well as the simplicity of primitive terms. There are important changes in the second edition.

GOODMAN, NELSON and QUINE, W. V.

[1947] 'Steps Toward a Constructive Nominalism', *JSL* 12, 105–122. Reviewed by Fitch, *JSL* 13, 49–50, and by Beth *MR* 9, 262.

GÖTLIND, ERIK

[1951] 'A Leśniewski–Mihailescu-Theorem for m-Valued Propositional Calculi', *Portugaliae Mathematica* 10, 97–102. Reviewed by Alan Rose *Z* 43, 249, Gene Rose, *JSL* 22, 329, and by A. Robinson, *MR* 13, 615.

GRELLING, KURT and NELSON, LEONARD

[1908] 'Bemerkungen zu den Paradoxien von Russell and Burali-Forti. Bemerkungen zur vorstehenden Abhandlung vor Gerhard Hessenberg, *Abhandlungen der Fries'schen Schule*, n. s., 2, 300–334.

Leśniewski has adopted Nelson's definition of an antinomy. See p. 314.

GRENIEWSKI, HENRYK

[1925] 'Próba dedukcyjnej teorji przyczynowości', (Attempt at a Deductive Theory of Causality), *Przegląd Filozoficzny* 28, 82–105.

[1949] 'Certain Notions of the Theory of Numbers as Applied to the Propositional Calculus', *Časopis Pěst. Mat. Fys.* 74, 132–136. Reviewed by Curry, *MR* 13, 198.

[1950] 'Functors of the Propositional Calculus', *Ann. Soc. Polon. Math.* 22, supplement, 78–86. Reviewed by Curry, *MR* 13, 198.

This has some connections with Protothetic.

[1953] 'Logika formalna w Polsce w dobie Odrodzenia', (The Renaissance of Formal Logic in Poland), *Problemy* 10, 658–664.

GRIZE, JEAN-BLAISE

[1972] *Notes sur l'ontologie et la méréologie de Leśniewski*, Travaux du Centre de Récherches Semiologiques, No. 12, 35 pp.

A clear brief introduction to Ontology and Mereology written especially for linguists.

GROMSKA, DANIELA

[1948] 'Philosophes polonais morts entre 1938 et 1945', *Studia Philosophica* 3, 31–91. Reviewed, *JSL* 18, 93–94.

Contains obituaries of L'Abbé Stanisław Kobyłecki, Edward Stamm, St. Leśniewski, Leon Chwistek, Władysław Hetper, Jan Salamucha, Mme. Janina Lindenbaum, Adolph Lindenbaum, Z. Schmierer, J. Metallmann, St. Schayer.

GROSSMANN, REINHARDT SIEGBERT

[1963] 'Common Names', *Essays in Ontology*, E. B. Allaire, et. al., ed., Iowa Publications in Philosophy 1, 64–75.

[1965] *The Structure of Mind*, University of Wisconsin Press.

Deals with similar problems as Leśniewski. Sections on Twardowski, Meinong, etc., have more philosophical and historical relevance than might be apparent. (But they can't be taken uncritically as an accurate account.)

[1969] *Reflections on Frege's Philosophy*, Northwestern University Press.

Contains a section on definitions.

GRZEGORCZYK, ANDRZEJ

[1950] 'The Pragmatic Foundations of Semantics', *Synthese* 8, 300–324. Reviewed by Chisholm, *JSL* 16, 292.

Ontology is mentioned several times. He says that we can frequently regard the sign of inclusion "$\subset$" as equivalent to the sign of "$\in$" of membership (pp. 316–317). Formally this is correct because of the thesis

$[mb] : m \in b \,.\equiv .\, m \subset b \,.\, m \in V$

of Ontology. Without the word 'frequently' he would be wrong. Even with it, the wrong impression is given.

[1955] 'The Systems of Leśniewski in Relation to Contemporary Logical Research', *Studia Logica* 3, 77–97. Reviewed by Hiż, *MR* 17, 1171–1172, and by Prior, *JSL* 27, 117–118.

This paper has been criticized by Clay [1974a], Luschei [1962], pp. 154–166, and Rickey [1977].

[1959] 'O pewnych formalnych konsekwencjach reizmu' (On Certain Formal Consequences of Reism), *Fragmenty Filozoficzne*, seria druga, Księga pamiątkowa ku uczczeniu czterdziestolecia pracy nauczycielskiej w Uniwersytecie Warszawskim Profesora Tadeusza Kotarbińskiego, PWN, Warsaw, 7–14. Reviewed by Lejewski, *JSL* 38, 536.

Reism calls for a geometry of solids.

[1961a] 'Axiomatizability of Geometry Without Points', *The Model in Mathematics*, D. Reidel, 104–111, and *Synthese* 12, 228–235. Reviewed by J. Diller, *Z* 201, 231, and by W. Schwabhäuser, *JSL* 37, 201.

Lemma II of part I is false, and this is used in the imported theorem 2. Mereology is mentioned p. 231.

[1961b] 'Aksjomatyczne badanie pojęcia przedłużenia czasowego', (Le traitment axiomatique de la notion de prolongment temporal), *Studia Logica* 11, 23–35. Polish and French with a Russian summary.

This theory is based on Mereology. (There are some differences between the Polish and French versions).

[1964] 'A Note on the Theory of Propositional Types', *Fundamenta Mathematicae* 54, 27–29. Reviewed by Peter Andrews, *JSL* 51, 502.

Shows how to reduce the number of primitive types in Henkin [1963].

GRZEGORCZYK, ANDRZEJ; MOSTOWSKI A. and RYLL-NARDZEWSKI C.

[1958] 'The Classical and the ω-Complete Arithmetic', *JSL* 23, 188–206. Reviewed, *JSL* 27, 80.

In this second order arithmetic they have a 'Leśniewski Schemata': $(\exists\alpha^k)(x_1,\ldots,x_n)[\alpha^k(x_1,\ldots,x_n) = \pi]$, which "is a form of definability corresponding to Leśniewski's rule of ontological definability."

GUMAŃSKI, L.

[1960] 'Logika klasyczna a założenia egzystencjalne' (Classical Logic and Existential Presuppositions), *Zeszyty Naukowe Uniwersytetu Mikołaja Kopernika w Toruniu, Filozofia I*, Z. 4.

This interesting and exhaustive study shows that the traditional logic cannot be *treated as a part of* Leśniewski's elementary ontology, of the algebra of classes, of the theory of relations or of the first-order functional calculus.

[1965] 'Jedynkowe systemy aksjomatyczne', *Prace Wydziału filologiczno-filozoficznego* 15, No. 1, Towarzystwo Naukowe w Toruniu, Toruń, 75pp. Reviewed *JSL* 31, 115–117 by Pavel Materna.

Among other things, propositional calculi with quantifiers and variable functions are studied.

HAACK, SUSAN

[1974] 'Mentioning Expressions', *Logique et Analyse* 18, #67–68, 277–294.

Suggests that if propositional quantifiers are interpreted substitutionally and if quotations are treated as functions, then this "might provide some relief to the ontological difficulties which Quine [1934] finds in the interpretation of protothetic" (p. 293)

HALLDÉN, SÖREN

[1949] 'An Analogy in Modal Logic to the Leśniewski–Mihăilescu Theorem', *Norsk. Mat. Tidsskr.* 31, 4–9. Reviewed by Hasenjaeger, *Z* 40, 147, McKinsey *JSL* 15, 70 and by Curry, *MR* 10, 585.

HALPERN, IGNACY

[1911] 'Metafizyka, dzieje jej nazwy, pojęć, prądów' ( ), *Ruch Filozoficzny* 1, 13–14.

In this report of a lecture there are several comments by Leśniewski.

HAMBLIN, C. L.

[1973] 'A Felicitous Fragment of the Predicate Calculus', *NDJFL* 14, 433–447.

There are some similarities with many-link functors here.

HAUSMAN, ALAN and ECHELBARGER, CHARLES

[1968] 'Goodman's Nominalism', *Studies in Logical Theory*, ed. Nicholas Rescher (American Philosophical Quarterly Monograph Series #2), 113–124.

They argue that no extension of Goodman's nominalistic ontology is adequate.

HELLMAN, GEOFFREY

[1969] 'Finitude, Infinitude, and Isomorphism of Interpretation in Some Nominalistic Calculi', *Noûs* 3, 413–425.

HELMER, OLAF

[1935] 'On the Theory of Axiom-Systems', *Analysis* 3, 1–11.

[1936] 'A Few Remarks on the Syntax of Axiom-Systems', *Actes du Congrés International de Philosophie Scientifique, VII, Logique*, 12–17. Reviewed by C. H. Langford, *JSL* 2, 84.

Those two papers treat the same topic as Sobociński [1955].

HEMPLE, CARL G.

[1953] 'Reflections on Nelson Goodman's *The Structure of Appearance*', *The Philosophical Review* 62, 108–116.

HENKIN, LEON

[1953] 'Banishing the Rule of Substitution for Functional Variables', *JSL* 18, 201–208. Reviewed by Church, *JSL* 20, 179–180, and by Heyting, *MR* 15, 277.

[1953] 'Some Notes on Nominalism', *Journal of Symbolic Logic* 18, 19–29.

[1955] 'The Nominalistic Interpretation of Mathematical Language', *Bull. Soc. Math. Belg.* 7, 137–142. Reviewed *MR* 19, 111, by A. Robinson.

[1962] 'Nominalistic Analysis of Mathematical Language', *Logic, Methodology and Philosophy of Science*, Stanford University Press, 187–193.

After a historical sketch of nominalism (which mentions Leśniewski, p. 187), he considers the following points (which were also considered by Goodman and Quine): 1. Provide a description of the conditions under which mathematical sentences may be affirmed, without reference to abstract entities. 2. Eschew any assumption on the finitude or infinitude of physical objects.

[1963] 'A Theory of Propositional Types', *Fundamenta Mathematicae* 52, 323–334. See Errata, 53, 119.

A system very closely related to Protothetic.

HENRY, DESMOND PAUL

[1962] 'An Anselmian Regress', *NDJFL* 3, 193–198. Reviewed by Luschei, *JSL* 36, 509–513.

Ontology, and in particular many-link functors, is used in the discussion.

[1963] 'Saint Anselm's Nonsense', *Mind* 72, 51–61. Reviewed by Luschei, *JSL* 36, 509–513.

[1964a] 'Ockham, Suppositio, and Modern Logic', *NDJFL* 5, 290–292.

Uses Ontology to refute the constantly occurring complaint that modern logic cannot analyze certain theses or forms of expressions which occur in medieval logic. The crucial difficulty is usually presented as "Ockham quantifies over terms whereas modern logicians quantify over variables (individuals)." Ontology can handle this.

[1964b] 'Being, Essence, and Existence', *Logique et Analyse* 7, #27, 104–110.

[1964c] *The De Grammatico of St. Anselm. The theory of Paronymy*, Publications in Mediaeval studies no. 18. University of Notre Dame Press, Notre Dame, Ind., XV + 169 pp. Reviewed in *Foundations of Language* 4, 78–79, and by Luschei, *JSL* 36, 509–513.

Ontology, in particular, higher semantical categories, is used to clarify Anselm's arguments and views.

[1965] 'Ockham and the Formal Distinction', *Franciscan Studies* 25, 285–292.

[1967] *The Logic of St. Anselm*, Oxford University Press, 1967, VI + 258 pp. Reviewed by Norman Kretzmann, *JSL* 34, 312–313.

[1969] 'Leśniewski's Ontology and some Medieval Logicians', *NDJFL* 10, 324–326.

[1972] *Medieval Logic and Metaphysics: A Modern Introduction*, Hutchinson University Library. Reviewed by F. C. Copleston, *Bibl. Phil.* 20, #242 (79–80), Ervin Nemesszeghy, *The Heythrop Journal* 15, 196–198, Ivo Thomas, *Phil. Quart.* 24, 71–72, and by M. J. Loux, *Mind* 83, 607–608.

Contains an excellent introduction to Ontology with examples of its applications for the elucidation of problems in medieval logic and metaphysics. Modifications of Henry [1964a] and [1969] have become chapters of this book.

[1974] *Commentary on 'De Grammatico': The Historical-Logical Dimensions of a Dialogue of St. Anselm's*, D. Reidel Publishing Co., 351 pp.

Ontology is used a great deal to make arguments precise.

[1975] 'The Singular Syllogisms of Garlandus Compotista', *Revue Internationale de Philosophie* 29, 243–270.
Ontology is used here.

HIŻ, HENRY

[1948] *An Economic Foundation for Arithmetic*, Ph.D. Dissertation, Harvard University.

[1952] 'On Primitive Terms of Logic', *JSL* 17, 156–157. (Abstract)
Extends Tarski's doctoral thesis.

[1957] 'Types and Environments', *Philosophy of Science* 24, 215–220.

[1959] 'O rzeczach' (On Things), *Fragmenty Filozoficzne* 20, 15–24.

[1960] 'The Intuitions of Grammatical Categories', *Methodos* 12, 311–319. Reviewed by G. H. Matthews, *JSL* 32, 115–116.

[1961a] 'Steps Toward Grammatical Recognition', *Advances in Documentation and Library Science*, vol. 3, part 2, *Information Retrieval and Machine Translation*, Interscience Publishers, New York and London, 811–822.

[1961b] 'Congrammaticality, Batteries of Transformations and Grammatical Categories', *Structure of Language and its Mathematical Aspects*, ed. Roman Jakobson, Providence, R. I., American Mathematical Society, 43–50.
Gives a definition of semantical categories based on substitutability in *many* (not 'all') sentences without loss of sentencehood.

[1961c] 'Syntactic Completion Analysis', *Transformations and Discourse Analysis Papers* 21, University of Pensilvania.

[1964] 'A Linearization of Chemical Graphs', *Journal of Chemical Documentation* 4, 173–180.

[1965] 'Ontological Definitions in Augmented Protothetics', *JSL* 31, 149–150. (Abstract).

[1967] 'Grammar Logicism', *The Monist* 41, 110–127. Reviewed *JSL* 39, 180 by Alec Fisher.
Recommends the use of Protothetic.

[1968] 'Computable and Uncomputable of Elements of Syntax', *Logic, Methodology and Philosophy of Science III*, ed. by Rootselaar and Staal, Amsterdam (North-Holland), 239–254.

[1971] 'On the Abstractness of Individuals', *Identity and Individuation*, Milton K. Munitz, ed. New York University Press, 251–261.

[1973] 'On Assertions of Existence', *Logic and Ontology*, Milton K. Munitz, New York University Press, 175–191.

[1976] 'Descriptions In Russell and Leśniewski', *XXIInd Conference on the History of Logic*, 5-9 July, 1976, Kraków, 62–67.

HODGES, WILFRED and LEWIS, DAVID

[1968] 'Finitude and Infinitude in the Atomic Calculus of Individuals', *Noûs* 2, 405–410.

There is no sentence in Goodman's calculus of individuals which says whether there are finitely or infinitely many individuals.

HORWICH, PAUL

[1975] 'A Formalization of 'Nothing'', *NDJFL* 15, 363–368.

This is a discussion of Henry [1967]. He objects that some of Henry's statements in Ontology "do not capture exactly Anselm's statements." After presenting reasons for this view alternate formulations are suggested.

HUGLY, PHILIP

[1975] 'Quine's Way Out', *Analysis* 36, 28–37.

HUSSERL, EDMUND

[1913] *Logische Untersuchungen*, English translation of the Second German Edition (the first was 1900) by J. N. Findlay is entitles *Logical Investigations*, Humanties Press, 1970, 2 vols.

Investigation III is on the theory of wholes and parts. Leśniewski's is not mentioned, but someone should take time to bring out any connections that may exist.

ISÉKI, KIYOSHI

[1966a] 'On Axiom Systems of the Propositional Calculus, XV', *Proceedings of the Japan Academy* 42, 217–220.

Shows that the equivalential calculus can be based on *Epp*, *EEpqEqp*, *EEpqEEqrEpr*. It is amazing that this very intuitive axiom system was not discovered previously.

[1966b] 'Algebraic Formulations of Propositional Calculi with Variable Forming Functors', *Proceedings of the Japan Academy* 42, 1058–1059.

[1968] *Kigô ronrigaku —meidai ronri* (Symbolic Logic — Propositional Logic) (Japanese), Vol. I. Maki Publishion Co., Tokyo, 303pp. Reviewed by Nakamura, *JSL* 35, 580–581.

Chapter 4 contains a discussion of Protothetic (244–274), Ontology (275–290) and Mereology (290–297).

[1968] 'General Theory of Mappings', *Proceedings of the Japan Academy* 44, 663–666.

"Some of his (Büchi — 'Die Boole'sche Partialordnung und die Paarung von Gefügen', *Portugaliae Mathematica* 7, 119–180) results are true for both the set theories in the senses of G. Cantor and S. Leśniewski." This paper deals with the Cantor type.

[1974] 'Remarks on Axioms of Magnitudes', *Math. Sem. Notes Kobe Univ.* 2, no. 3, paper no. 33, 7 pp.

ISHIMOTO, ARATA

[1976] 'A Propositional Fragment of Leśniewski's Ontology and Related Systems I'. Abstract of this 12 page manuscript appears in the proceedings of the *XXIInd Conference on the History of Logic*, Kraków, 5–9 July 1976, 12–15.

Shows that a certain fragment of Ontology is complete with respect to the interpretation proposed by Prior [1965].

IWANUŚ, BOGUSŁAW

[1969a] 'Remarks About Syllogistic With Negative Terms', *Studia Logica* 24, 131–141. With Polish and Russian summaries.

[1969b] 'An Extension of the Traditional Logic Containing the Elementary Ontology and the Algebra of Classes', *Studia Logica* 25, 97–139. With Polish and Russian summaries. Reviewed by Canty *Z* 261, 02009.

[1973a] 'On Leśniewski's Elementary Ontology', *Studia Logica* 31, 73–125. With Polish and Russian summaries. Reviewed by Canty *Z* 275, 02019.

[1973b] 'Proof of Decidability of the Traditional Calculus of Names, *Studia Logica* 32, 131–147. With Polish and Russian summaries.

JARDINE, CHARLES J. and JARDINE, NICHOLAS

[1971] 'The Matching of Parts of Things', *Studia Logica* 27, 123–132. With Polish and Russian summaries. Reviewed by E. Koppelman *Z* 264, 02013.

JAŚKOWSKI, STANISŁAW (1906–1965)

[1934] 'On the Rules of Supposition in Formal Logic', *Studia Logica* 1, 5–32. Reprinted, with considerable changes in notation, in McCall [1967], 232–258.

This famous paper which initiates work on natural deduction techniques, is obviously inspired by Leśniewski's proof technique.

[1948a] 'Une modification des définitions fondamentales de la géométrie des corps de M. A. Tarski', *Annales de la Société Polonaise de Mathématique* 21, 298–301. Reviewed by Blumenthal *MR* 11, 123, and by K. Schröter and G. Asser, *Z* 40, 368.

[1948b] 'Sur certains axioms de la géométrie élémentaire', *Annales de la Société Polonaise de Mathématique* 21, 349–350. (Abstract).

[1948c] 'Geometria Brył' (Geometry of Solids), *Matematyka: Czasopismo dla nauczycieli* 1, 1–7.

Sphere is the primitive term.

[1949] 'Quelques problémes actuels concernant les fondements des Mathématique', *Časopis pro Pěstováni Matematiky a fysiky* 74, 74–78. With a Polish summary.

[1950] 'Sur les axiomes de la géométrie des corps', *Dodatek do Rocznika Polskiego Towarzystwa Matematycznego* 22, 86–87. VI Zjazd Matematyków Polskich, Warszawa 20–23, IX, 1948. (Abstract).

JORDAN, ZBIGNIEW A. (1906–1965)

[1945] *The Development of Mathematical Logic and of Logical Positivism in Poland between the Two Wars*, Polish Science and Learning, no. 6, Oxford University Press, 47 pp. The first six (of ten) sections and the relevant parts of the bibliography are reprinted in McCall [1967], 346–406.

[1963a] 'Logical Determinism', *NDJFL* 4, 1–38.

[1963b] 'O logicznym determinizmie', *Studia Logica* 14, 59–96. With a Russian summary. This is slightly different from [1963a].

[1963c] *Philosophy and Ideology: The Development of Philosophy and Marxism-Leninism in Poland since the Second World War*, D. Reidel, Dordrecht–Holland.

A comprehensive history of the Warsaw school. The influence of Twardowski is clearly seen here. Contains a good bibliography.

KALINOWSKI, GEORGES

[1973] 'La Logique de Leśniewski et la Theologie de Saint Anselm', *Archives de Philosophie* 36, 407–416.

[1977] 'La Grammaire Pure et les Catégories Sémantiques', *Archives de Philosophie* 40, 467–475.

[19...] *Protothétique, ontologie, méréologie. Textes*, A. Colin, Paris.

Contains French translations of Leśniewski [1927], [1929b], and [1930a], as well as Lejewski [1958b].

Kamiński, Stanisław

[1977] 'The Development of Logic and the Philosophy of Science in Poland after the Second World War', *Zeitschrift für allgemeine Wissenschaftentheorie* 8, 163–171.

Kaplan, David

[1970] 'Nominalistic Set Theory', *Noûs* 4, 225–240.

Kearns, John Thomas

[1962] *Leśniewski, Language and Logic*, Ph.D. dissertation, Yale, 163 pp.

[1966] 'Quantifiers and Universal Validity', *Logique et Analyse* 9, #35–36, 298–309.

[1967] 'The Contributions of Leśniewski', *NDJFL* 8, 61–93. Contains a portion of [1962].

"a brief, sympathetic, and relatively complete account of Leśniewski's work." (p. 61). The comments at the end about 'structure' are misdirected. He, and many others, tries to make cardinality a Mereological notion, while it is really an Ontological one.

[1968a] 'A Universally Valid System of Predicate Calculus with no Existential Presuppositions', *Logique et Analyse* 11, #43, 367–389. Reviewed by da Costa, *MR* 39 #2609.

A System of predicate logic in the spirit of Leśniewski.

[1968b] 'The Logical Concept of Existence', *NDJFL* 9, 313–324. *MR* 39 #2596.

[1969] 'Two Views of Variables', *NDJFL* 10, 163–180.

A reply to Lejewski [1954b].

[1970] 'Substance and Time', *The Journal of Philosophy* 67, 277–289.

Many-link functors are used in this criticism of an argument of G. Bergmann which purports to show that a substance ontology is untenable.

[19..] 'A Little More Like English', 1975, manuscript.

Presents a formal system wherein quantified general terms can significantly be used in the same places as proper names. He claims this system is more like English that Ontology is. There are reasons to believe that deep structure is closer to the surface than had previously been thought.

Kelley, John L.

[1955] *General Topology*, D. Van Nostrand, 1955.

On p. 251 of the appendix on (Morse) set theory there is the following footnote: "Actually, an axiom scheme for definition is also assumed without explicit statement. That is, statements of a certain form, which in particular involve one new constant and are either an equivalence or an identity, are accepted as definitions and are treated in precisely the same fashion as theorems. The axiom scheme of definition is in the fortunate position of being justifiable in the sense that, if the definitions conform with the prescribed rules, then no new contradictions and no real enrichment of the theory results. These results are due to S. Leśniewski (sic)."

Kelley is, of course, wrong about the non-creativity of definitions. Unfortunately this belief persists in the literature.

KIELKOPF, CHARLES F.

[1976] 'Interpretations of the Quantifiers in Versions of Leśniewski's Ontology', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, p. 16 (abstract of a lecture).

A reaction to Küng and Canty [1976].

[1977] 'Quantifiers in Ontology', *Studia Logica* 36,

KLIBANSKY, RAYMOND, ed.

[1968] *Contemporary Philosophy, A Survey*, Vol. I — Logic and Foundations of Mathematics, 1968, Firenze, Italy.

KOKOSZYŃSKA, MARIA

[1968] 'Kazimierz Ajdukiewicz', in Klibansky [1968], 202–208.

Contains a portrait and a list of eleven articles *about* Ajdukiewicz, all but two of, which appeared after his death in 1963.

KORCIK, ANTONI

[1954] 'Zdania egzystencjalne u Arystotelesa' (Existential Propositions in Aristotle), *Polonia Sacra* (Kraków) 6, 46–50. Reviewed, *JSL* 20, 172 by Lejewski.

This paper mentions Leśniewski's views on existential propositions.

KORTLANDT, FREDERIK HERMAN HENRI

[1972] *Modelling the Phoneme: New Trends in East European Phonemic Theory*, The Hague–Paris, Mouton.

The fifth chapter is devoted to the exposition and critical analysis of Batóg's axiomatic system of phonology. It gives an account of some mereological concepts.

KOTARBIŃSKI, TADEUSZ

[1921] 'Sprawa istnienia przedmiotów idealnych' (Question of Existence of Ideal Objects), *Przegląd Filozoficzny* 24. Reprinted in [1957a] 2, 7–39.

Contains a summary of Leśniewski's arguments and views.

[1923] 'Prawdziwość i fałszywość definicyj' (Truth and Falsehood of Definitions), *Przegląd Filozoficzny* 27, 263–264.

This summary of a discussion contains some remarks by Leśniewski. Photostat in Leśniewski [1967].

[1929] *Elementy teorji poznania, logiki formalnej i metodologji nauk* (Elements of the Theory of Knowledge, Formal Logic, and Methodology of Science), Lwów. Reprinted 1947 and revised 1961. [1966b] is an English translation.

[1935] 'Zasadnicze myśli pansomatyzmu' (The Fundamental Ideas of Pansomatism), *Przegląd Filozoficzny* 38, 283–294. [1955] is an English translation.

[1948] 'Sur l'attitude réiste (ou concrétiste)', *Synthèse* 7, 262–273.

[1949] 'O postawie reistycznej' (On the Foundations of Reism),

[1955] 'The Fundamental Ideas of Pansomatism', *Mind* 64, 488–500. English translation of [1935].

[1956a] 'Sprawność i błąd. Z myślą o dobrej robocie nauczyciela' (Cleverness and Errors. Łukasiewicz), Warszawa, Państwowe Zakłady Wydawnictw Szkolnych, 102 pp.

In the chapter 'Nauczyciele sztuki nauczania' (Teachers of the art of teaching) Kotarbiński characterizes Leśniewski as a teacher.

[1956b] 'Garstka wspomnień o Stanisławie Leśniewskim' (Handful of Memories of Stanisław Leśniewski), *Ruch Filozoficzny* 24, 155–163.

[1957a] *Wybór pism* (Selected Works), Warsaw, vol. I, 733 pp., vol. II (1958), 936 pages. Contains, among others, [1921].

[1957b] 'La philosophie dans la Pologne contemporarine', *Synthèse* 12, 29–38.

Discusses the work of Twardowski, Łukasiewicz and Leśniewski.

[1957c] *Wykłady z dziejów logiki* (Outlines of the History of Logic), Societas Scientiarum Lodziensis, Łódź, 28, 244 pp. Reviewed by Rose Rand, *JSL* 25, 62–63. [1964] is a French translation.

"Contains personal recollections of Leśniewski's unpublished treatments of certain topics in semantics, together with brief informal accounts of his theories", Luschei [1962], p. 320.

[1958a] 'La Logique en Pologne (1945–1955). Les Études Philosophiques, *Philosophy in the Mid-Century*, Florence 1958, II, 234–241.

[1958b] 'Fazy rozwojowe konkretyzmu' (..), *Studia Filozoficzne* 4, 3–13.

Reprinted in the second edition of Kotarbiński [1929].

[1959] *La logique en Pologne. Son originalité et les influences étrangères*, Academia Polacca di Scienze e Lettere, Biblioteca di Roma, Conferenze, Fascicole 7, Angelo Signorelli Editore, Rome, 24 pp. Reviewed by Tadeusz Czeżowski, *JSL* 25, 259.

Mentions Łukasiewicz and Leśniewski as the outstanding members of the Polish school and describes their work.

[1964] *Leçons sur l'historie de la logique*, Paris. French translation of [1957c].

[1966a] 'Sur l'Attitude réiste ou concrétiste le Language', *Actes du 13e Congres des Sociétés de Philosophie de Langue Française*, Neuchatel, I, 100–102.

[1966b] *Gnosiology, The Scientific Approach to the Theory of Knowledge*, Pergamon Press, xiii + 548. English translation of [1929].

Pp. 190–211 provide an introduction to Ontology.

[1967] Notes on the Development of Formal Logic in Poland in the years 1900–39, in McCall [1967], 1–14.

KOWALSKI, JAMES G.

[1975] *Leśniewski's Ontology Extended With the Axiom of Choice*, Ph.D. dissertation under Sobociński at Notre Dame. Published as [1977].

[1977] 'Leśniewski's Ontology Extended With the Axiom of Choice', *NDJFL* 18, 1–78. *Z* 321, 02015 (Autorreferrat).

Shows, in Ontology, that the Axiom of Choice, Zorn's Lemma, and the Well Ordering Principle are equivalent. In a type theory like Ontology, the Axiom of Choice cannot be added as a single sentence, but it must be added for each type. A rule of procedure for doing this is provided.

KRASZEWSKI, ZDZISŁAW and SUSZKO, ROMAN

[1966] 'O klasach normalnych i nienormalnych na terenie języka potocznego (Z badań nad pojęciem klasy I)' (Normal and Non-Normal Classes in Natural Language — Investigations Into the Concept of a Class I), *Studia Logica* 19, 127–146. With English and Russian summaries. *MR* 38 #2004.

[1968] 'Klasy normalne i nienormalne a teoriomnogościowe i mereologiczne pojęcie klasy (Z badań nad pojęciem klasy II)' (Normal and Non-Normal Classes and the Set-Theoretical and the Mereological Concept of Class — Studies on the Concept of Class II), *Studia Logica* 22, 85–97. With English and Russian summaries. *MR* 38 #2005.

These papers discuss the relationships between different notions of class.

KROKIEWICZ, A.

[1948] 'O logice stoików' (On Stoic Logic), *Kwartalnik Filozoficzny* 17, 173–197.

Leśniewski is mentioned p. 186.

KRUSZEWSKI, Z.

[1925] 'Ontologja bez aksjomatów' (Ontology Without Axioms), *Przegląd Filozoficzny* 28, 136 (abstract).

He showed that Ontology is consistent with respect to Protothetic by interpreting ε as ϙ. Scharle proved (1967, unpublished) that if ε is interpreted as a binary functor and the lowest type variables range over propositions then ȯ is the only other interpretation. If the primitive category of Ontology is interpreted as protothetical function then there are other interpretations. Kruszewski did this work for his master's thesis. Then he took a trip around the world and died in Ceylon of some tropical disease.

KRZYŻANOWSKI, JULIUSZ

[1939] 'Symbolika Ontologiczna czy Algebra logiki' (Ontological Symbolism or Algebra of Logic), *Przegląd Klasyczny* 5, 85–89.

KUBIŃSKI, TADEUSZ

[1958] 'Nazwy nieostre' (Vague Terms), *Studia Logica* 7, 115–179. With English and Russian summaries. Reviewed by Lejewski, *JSL* 24, 270–271.

This system contains a fragment of Ontology.

[1959] 'Systemy pozornie sprzeczne' (Quasi-Inconsistent Systems), *Zeszyty Naukowe Uniwersytetu Wrocławskiego, Seria B, Matematyki, Fizyki i Astronomii*, 1959, 53–61.

A family of systems is presented each of which is an extension of elementary ontology without definitions. Each system contains theorems of the form '*a* is *b* and *a* is non-*b*'. The constant 'is' is defined by means

of Leśniewski's epsilon and a special form of weak negation. Consistency proofs are given.

[1960] 'An Attempt to Bring Logic Nearer to Colloquial Language', *Studia Logica* 10, 61–75. With Polish and Russian summaries.

An extension of elementary ontology without definitions is considered. It contains formulas which could be read 'x is rather y than z' or 'x is more similar to y than to z', or, in special cases, 'x is more similar to y than to non-y'. A proof of consistency is given.

[19xx] 'An Extension of the Theory of Syntactic Categories', *Acta Universitatis Wratislaviensis* 12, 19–36.

[1964] 'Cudzysłów i prawda' (Quotation Marks and Truth), *Ruch Filozoficzny* 23, 70–72.

An application of ontology to the definition of supposition.

[1965] 'Two Kinds of Quotation Mark Expressions in Formalized Languages', *Studia Logica* 17, 31–51. With Polish and Russian summaries. Reviewed by Canty *Z* 299, 02033.

Three systems are considered, each of which contains elementary Ontology without definitions. All three systems contain formulas some of whose parts are enclosed in quotation marks. Quotation marks of the first kind blind variables, while those of the second kind do not.

[1966] 'Przegląd niektórych zagadnień logiki pytań' (A Review of Some Problems on the Logic of Questions), *Studia Logica* 18, 105–137. With Russian and English summaries. Reviewed, *JSL* 32, 548–549, by Pavel Materna.

A connection between this problem and Protothetic is stated.

[1968] 'Uwagi o modelach systemu mereologii Leśniewskiego' (Remarks on Models of the Leśniewskian System of Mereology), *Ruch Filozoficzny* 26, 336–338.

This is an abbreviated version of Kubiński [1971c].

[1969] 'Pewna teoriomnogościowa własność ontologii' (Some Model Theoretic Properties of Ontology), *Ruch Filozoficzny* 27, No. 4.

Ontology is absolutely non-categorical given some definition of noncategoricity.

[1970] 'Pewne klasy relacji między pytaniami' (A Certain Class of Relations Between Questions), *Ruch Filozoficzny* 28, no. 3–4.

[1971a] 'Teoria identyczności i ontologia elementarna' (Theory of Identity and Elementary Ontology), *Acta Universitatis Wratislaviensis* 139, *Prace Filozoficzne* VIII, 3–8.

Contains a proof that the theory of identity and the theory based on the single axiom of elementary Ontology and the axiom 'for every x, x is x', where the 'is' is that of Ontology, are equivalent.

[1971b] 'Trzy elementarne rachunki nazw' (Three Elementary Calculi of Names), *Acta Universitatis Wratislaviensis* 139, *Prace Filozoficzne* VIII, 9–24.

The first calculus of names considered, calculus R, is based on a single axiom which is a special shortening of the single axiom of elementary Ontology. The theory based on the single axiom of elementary Ontology is a fragment of a slight definitional extension of the calculus R. The second calculus S, is a proper extension of R. Its decidability is proved. The third, calculus T, is an extension of S. It is axiomless and contains a description operator. Its decidability is proved.

[1971c] 'A Report on Investigations Concerning Mereology', *Acta Universitatis Wratislaviensis* 139, *Prace Filozoficzne* VIII, 48–69.

Mereology is compared with a system of the least upper bound. The main theorem of the first part of the paper states that the notion of mereological whole is stronger than the notion of the least upper bound.

[1971d] 'O pseudodefinicjach aksjomatycznych stałej 'jest' w teoriach elementarnych' (On Pseudo-Axiomatic Definitions of the Constant 'is' in Elementary Theories), *Ruch Filozoficzny* 29, 263–269.

A family of elementary systems is considered. The simplest ones are based on a single axiom which is a special shortening of the single axiom of elementary Ontology. Other systems contain signs for operations (e.g., join, meet, and various forms of complementations). Interconnections between these systems are characterized. Various model-theoretic theorems are given. Arithmetical classes of the shortenings of the single axiom of elementary Ontology are defined.

KUBIŃSKI, T. and ZABSKI, EUGENIUSZ

[1971] 'Próby aksjomatycznego ujęcia pojęcia nieodróżnialności empirycznej' (Tentative Axiomatic Treatment of the Concept of Empirical Indiscernability), *Ruch Filozoficzny* 29, 270–274.

KÜNG, GUIDO

[1963] *Ontologie und Logistische Analyse der Sprache. Eine Untersuchung zur Zeitgenössischen Universaliendiscussion*, Springer. [1967] is an English translation.

[1967] *Ontology and the Logistic Analysis of Language*, The Humanities Press. Revised English translation of [1963]

Symbolic logic as applied to classical philosophical problems. Contains sections on Russell, Wittgenstein, Carnap, Leśniewski, Quine, and Goodman.

Chapter 8 (pp. 84–104 of the German; pp. 102–126 of the English) is on Leśniewski. Topics considered are "the paradox of 'general individuals'", Mereology, Ontology, quantifiers without existential import, and nominalism. This is a well documented and careful study. He even mentions Mill (1843, Bd I, Kap. 3, §5) on denotation (Bezeichnen, oznaczać) and connotation (Mitbezeichnen, współoznaczać).

[1972] *Noema und Gegenstand, Jenseits von Sein und Nichtsein, Beiträge zur Meinongforschung*, ed. R. Maller, Graz.

[1974] 'Prologue-Functors', *Journal of Philosophical Logic* 3, 241–254.

[1976] 'The Meaning of the Quantifiers in the Logic of Leśniewski', *XXII-nd Conference on the History of Logic*, 5–9 July 1976, Kraków, p. 15. (abstract of a lecture).

[1977a] 'Nominalistische Logik heute', *Allgemeine Zeitschrift für Philosophie* 1, 29–52.

Discusses the nominalism of Goodman and Quine, Eberle, and especially Leśniewski.

[1977b] 'The Meaning of the Quantifiers in the Logic of Leśniewski', *Studia Logica* 36, –.

[19..] 'Funktory prologowe i kwantifikatory u Stanisława Leśniewskiego, *Studia Semiotyczne*

KÜNG, GUIDO and CANTY, JOHN THOMAS

[1970] 'Substitutional Quantification and Leśniewskian Quantifiers', *Theoria* 36, 165–182. Autorreferrat, *Z* 213, 14.

KURATOWSKI, K.

[1970] 'The Polish Mathematical Society Between the Two World Wars', *Rev. Polish Acad. Sci.* 15, 73–77.

KURATOWSKI, K. and MOSTOWSKI, A.

[1968] *Set Theory*, North Holland.

They credit Leśniewski (p. 297) with showing the following formula equivalent to the axiom of the choice: $\bigwedge_{m,n \notin N}(m \cdot n = m \vee m \cdot n = n)$.

KUZAWA, MARY GRACE

[1968] *Modern Mathematics. The Genesis of a School in Poland*, College and University Press, 1968, 143 pp.

LAFORGE, JEAN-MARC

[1974] 'Fondements pour une méréologie ensembliste', *Logique et Analyse* 17, #65–66, 165–174.

LAMBEK, JOACHIM

[1958] 'The Mathematics of Sentence Structure', *American Mathematical Monthly* 65, 154–170.

[1959] 'Contributions to a Mechanical Analysis of the English Verb-Phrase', *Journal of the Canadian Linguistic Association* 5, 83–89.

[1961] 'On the Calculus of Syntactical Types', *Structure of Language and its Mathematical Aspects*, R. Jakobsen, ed., 166–178.

LAMBERT, KAREL

[1963] 'Existential Import Revisited', *NDJFL* 4, 288–292.

Lambert reviewes and rejects several methods of handling nondesignating terms in quantification theory, especially quantification theory with identity. Then he suggests another solution. His solution is akin to views held by Leśniewski.

[1965] 'On Logic and Existence', *NDJFL* 6, 135–141.

[1967] 'Free Logic and the Concept of Existence', *NDJFL* 8, 133–144. *MR* 38 #2006.

[1969] *The Logical Way of Doing Things*, Yale University Press.

This is more relevant than it seems.

[19..] 'Explaining Away Singular Non-Existence Statements', *Dialogue* 1, No. 4.

LAMBERT, KAREL and SCHARLE, THOMAS

[1967] 'A Translation Theorem for Two Systems of Free Logic', *Logique et Analyse* 10, #39–40, 328–341. Reviewed by S. McCall, *MR* 37 #1229. [1976] is a translation.

They prove that every formula of the system L4′ of Lejewski [1965] has a translation in the system FL of Lambert [...] which is equivalent.

[1976] 'Un teorema di traduzione ter due sistemi di logica libera', *La Logiche Libere*, ed. Ermanno Bencivenga, editore Boringhieri, Turin, 337–350. Italian translation of [1967].

LARGEAULT, JEAN

[1972] *Enquête sur le nominalisme*, Louvain, Préface de R. Poirier, Paris, Beatric Nauwelaerts, Louvain.

Mentions Leśniewski 359–360, 364.

LEBIEDIEWA, SWIETŁANA

[1969a] 'The Systems of Modal Calculus of Names I', *Studia Logica* 24, 83–107. Reviewed by E. Melis, *Z* 252, 02010.

[1969b] 'The Systems of Modal Calculus of Names, II: Modal Calculi of Names Based on the Classical Calculus of Propositions', *Studia Logica* 25, 79–96. Reviewed by M. J. Cresswell, *MR* 44 #37 and E. Melis, *Z* 252, 02011.

These systems are based on Ontology.

[19xx] 'Syllogistic of Systems of Modal Ontology',

A treatment of Aristotle's modal syllogistic and of modal syllogistic based on Leśniewski's Ontology.

LEDNIKOV, E. E.

[1973] *Kritičeskij analiz nominalističeskich tendencij v sovremennoj logike* (Critical Analysis of Nominalistic Tendencies in Contemporary Logic), 'Naukova dumka', Kiev.

Ch. LV, §2: The Mereological conception of abstractions. Calculus of individuals (pp. 166–175). Retelling of Sinisi [1969], Słupecki [1958] and Rvačov [19..]. Discussion of Goodman, Quine [1947], Henkin [1962].

LEHRBERGER, JOHN

[1974] *Functor Analysis of Natural Language*, Mouton and Co. (=*Janua Linguarum*, Series Minor, #197).

Discussion of the work of Husserl, Leśniewski, Ajdukiewicz, Bar-Hillel, and Hiż on grammatical categories.

LEJEWSKI, CZESŁAW

[1953] 'O pojęciu istnienia w logice' (On the Concept of Existence in Logic), *Polskie Towarzystwo Naukowe Na Obczyźnie*, Rocznik 4, 15–17.

A discussion of quantification and existence, comparing Quine and Leśniewski.

[1954a] 'A Contribution to Leśniewski's Mereology', *Polskie Towarzystwo Naukowe Na Obczyźnie*, Rocznik 5, 48–50. Reviewed by Prior, *JSL* 21, 325–326.

Discusses the equivalence of several single axioms for Mereology: two based on element, and one each based on Kl, extra, and ov.

[1954b] 'Logic and Existence', *British Journal for the Philosophy of Science* 5, 104–119.

An excellent and often cited paper.

[1955] 'A New Axiom of Mereology', *Polskie Towarzystwo Naukowe Na Obczyźnie* 6, 65–70.

Two single axioms based on extr are given and proved equivalent to the pairs given by Leśniewski for that term.

[1957] 'Proper Names', *A symposium, Proceedings of The Aristotelian Society*, Supp. vol. 31, 229–256.

[1958a] 'On Implicational Definitions', *Studia Logica* 8, 189–211. With Polish and Russian summaries. Reviewed, *JSL* 24, 246. This is part of his doctoral dissertation at the University of London, 1955, under the direction of Popper and Łukasiewicz (external examiner).

Contains a good synopsis of Leśniewski's views on definitions.

[1958b] 'On Leśniewski's Ontology', *Ratio* (Oxford) 1, 150–176. Reviewed, *JSL* 34, 647–648 by Iwanuś. [1958c] and [1972] are translations. French translation in Kalinowski [19..].

Contains a valuable introduction to Ontology. Introduces his useful 'Ontological Table'.

[1958c] 'Zu Leśniewskis Ontologie', *Ratio* (Frankfurt a.M.) 2, 50–78. Translation of [1958b].

[1958d] Reviews of W. T. Parry's 'A New Symbolism for the Propositional Calculus' and G. B. Standley's 'Ideographic Computation in the Propositional Calculus', *JSL* 23, 63.

Parry claims that his trapezoid notation has all the ideographic advantages of Leśniewski's wheel and spoke symbolism. Lejewski points out that Leśniewski's rules do not dictate the notation and that "The shape of constant terms was a purely contingent matter of typographical elegance".

[1960a] 'A Re-Examination of the Russellian Theory of Descriptions', *Philosophy* 35, 14–29. Russell's reply, p. 146.

[1960b] 'Studies in the Axiomatic Foundations of Boolean Algebra I, II, III', *NDJFL* 1, 23–47 and 91–106, and 2, 79–93.

[1963a] 'A Note on a Problem Concerning the Axiomatic Foundations of Mereology', *NDJFL* 4, 135–139.

In 1961 Sobociński set the problem of finding an axiom for Mereology based on el which used quantification only over nominal variables. Sobociński's 1948 axiom involved quantification over variables of type N/N (as do all other single axioms). Lejewski introduces the term elKl by the definition:

$[ABa] \because A \in el(B) . B \in Kl(a) . \equiv . A \in elKl(Ba)$

and then gives a single axiom for Mereology using this term. He also gives an axiom for el without variables of type N/N.

[1963b] 'Aristotle's Syllogistic and Its Extensions', *Synthèse* 15, 125–154. Also appears in *Form and Strategy In Science; Studies dedicated to Joseph Henry Woodger on the Occassion of his Seventieth Birthday*, ed. by John R. Gregg and F. T. C. Harris, D. Reidel, 1964, 203–232.

[1965] 'Parts of Speech', *Proceedings of the Aristotelian Society*, Supp. vol. 39, 189–204.

Interesting paper dealing with Leśniewski's semantic categories. Contrasts with Russell's theory.

[1967a] 'A Theory of Non-Reflexive Identity and its Ontological Ramifications', *Grundfragen der Wissenschaften und ihre Wurzeln in der Methaphysik*, ed. Paul Wiengartner. Salzburg–München, 65–102.

[1967b] 'The Problems of Ontological Commitment', *Fragmenty Filozoficzne* (Third Series), 147–164.

Contains a criticism of Quine's doctrine of ontological commitment.

[1967c] 'A Single Axiom for the Mereological Notion of Proper Part', *NDJFL* 7, 279–285.

A new single axiom based on pt is given for Mereology. Leśniewski's 1918 system based on pt, but without any defined term, is also given.

[1969] 'Consistency of Leśniewski's Mereology', *JSL* 34, 321–328.

[1970] 'Quantification and Ontological Commitment', *Physics, Logic and History*, ed. by Wolfgang Yourgrau and Allen D. Breck, Plenum Press, New York and London, 173–181. Discussion 181–190. There is also a comment by Lejewski on pp. 293–294 of this volume.

[1972] 'Leśniewski no Sonzairon ni tsuite', *Ronrishisô no Kakumê Risei no Bunseki* (The Revolution of Logical Thought), ed. Arata Ishimoto, Tokai University Press, 201–236. Japanese translation of [1958b].

[1973a] 'A Contribution to the Study of Extended Mereologies', *NDJFL* 14, 53–61.

[1973b] 'Leśniewski, Stanisław', *Dictionary of Scientific Biography* 8, 262–263. C. C. Gillispie, ed. Charles Scribner's Sons, New York.

[1974a] 'Popper's Theory of Formal or Deductive Inference', *The Philosophy of Karl Popper*, ed. P. A. Schilpp, Open Court Publishing Co., 632–670 of Vol. I. There is a reply by Popper, Vol. II, 1095–1096.

[1974b] 'A System of Logic For Bicategorical Ontology', *Journal of Philosophical Logic* 3, 265–283.

Although Leśniewski's system are not mentioned, some connections can be seen.

[1975a] 'Logic, History of', *Encyclopedia Brittanica*, Macropedia XI, 70–71.

[1975b] 'Syntax and Semantics of Ordinary Language', *The Aristotelian Society*, Supp. vol. 49, 127–146. A commentary by William Haas follows, 147–169.

Uses Leśniewski's theory of semantical categories to analyse ordinary language.

[1976a] 'On Prosleptic Premises', *NDJFL* 17, 1–18. Autorreferrat, *Z* 313, 02003.

Not directly related to Leśniewski's system, but they are mentioned and used in a proof at the end of this paper.

[1976b] 'Systems of Leśniewski's Ontology With the Functor of Weak Inclusion as the Only Primitive Term', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, 38 pages.

[1976c] 'Ontology and Logic' (with comments by Michael Dummett and Dag Prawitz and a reply by the author), *Philosophy of Logic* (Proceedings of the Third Bristol Conf. in Critical Phil.), 1–63. U. Calif. Press, Berkeley.

LEONARD, HENRY S.

[1967] *Principles of Reasoning: An Introduction to Logic, Methodology and Theory of Signs*, Dover, revised edition.

LEONARD, HENRY S. and GOODMAN, NELSON

[1936] 'A Calculus of Individuals', *JSL* 2, 63. (Abstract).

[1940] 'The Calculus of Individuals and Its Uses', *JSL* 5, 45–55. Reviewed by Lafleur, *JSL* 5, 113–114.

It is often said that the calculus of individuals is 'akin' to Mereology, but the precise nature of that affinity has never been investigated.

LEŚNIEWSKI, STANISŁAW (1886–1939)

[1911] 'Przyczynek do analizy zdań egzystencjalnych' (A Contribution to Analysis of Existential Propositions), *Przegląd Filozoficzny* 14, 329–345.

[1912] 'Próba dowodu ontologicznej zasady sprzeczności' (An Attempt at a Proof on the Ontological Principle of Contradiction), *Przegląd Filozoficzny* 15, 202–226.

[1913a] *Logičeskia razsuždenia* (Russian), St. Petersburg, 87 pp.

Contains Russian versions of [1911] and [1912].

[1913b] 'Czy prawda jest tylko wieczna czy też wieczna i odwieczna?' (Is All Truth Only True Eternally or it is also True Without a Beginning?), *Nowe Tory* 18, [1963] is an English translation.

[1913c] 'Krytyka logicznej zasady wyłączonego środku' (The Critique of the Logical Principle of the Excluded Middle), *Przegląd Filozoficzny* 16, 315–352.

[1914] 'Czy klasa klas, nie podporządkowanych sobie, jest podporządkowana sobie?', (Is a Class of Classes not Subordinate to Themselves, Subordinated to Itself?), *Przegląd Filozoficzny* 17, 63–75.

[1914b] 'Teoria mnogości na 'podstawach filozoficznych' Benedykta Bornsteina', *Przegląd Filozoficzny* 17, 488–507.

[1916] *Podstawy ogólnej teoryi mnogości. I* (Foundations of a General Theory of Sets), Prace Polskiego Koła Naukowego w Moskwie. Sekcya matematyczno-przyrodnicza, no. 2, 42 pp., Moscow.

[1927] 'O podstawach matematyki' (On the Foundations of Mathematics), *Przegląd Filozoficzny* 30, 164–206; 31, 261–291; 32, 60–101; 33, 77–105, and 142–170.

[1929a] 'Über Funktionen, deren Felder Gruppen mit Rücksicht auf diese Funktionen sind', *Fundamenta Mathematicae* 13, 319–332.

[1929b] 'Grunzüge eines neuen Systems der Grundlagen der Mathematik', *Fundamenta Mathematicae* 14, 1–81.

[1929c] 'Über Funktionen, deren Felder Abelsche Gruppen in bezug auf diese Funktionen sind', *Fundamenta Mathematicae* 14, 242–251.

[1930a] 'Über die Grundlagen der Ontologie', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Classe III, 23, 111–132.

[1930b] 'Über Definitionen in der sogenannten Theorie der Deduktion', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Classe III, 23, 289–309.

[1938a] 'Einleitende Bemerkungen zur Fortsetzung meiner Mitteilung u. d. T. 'Grunzüge eines neuen Systems der Grundlagen der Mathematik'', *Collectanea Logica* 1 (Offprint 1938), 1–60. Reviewed by Quine, *JSL* 5, 83–84, and by H. Scholz *Z* 23, 98–99. [1967b] is an English translation.

[1938b] 'Grundzüge eines neuen Systems der Grundlagen der Mathematik, §12', *Collectanea Logica* 1 (Offprint 1938), 61–144. Reviewed by Quine, *JSL* 5, 84.

[1967a] *Stanisław Leśniewski: Collected Papers.*

Canty has collected Leśniewski's papers, with the exception of [1913a], [1913b], [1916], which could not be located, and the bound photostats have been deposited in the University of Notre Dame Library. BC/135/L637. vi + 297 pages.

[1963] 'Is Truth only Eternal or Both Eternal and Sempiternal', English translation of [1913b], by Rose Rand, *Polish Review* 8, 23–43.

[1967a] 'Introductory Remarks to the Continuation of my Article: Grundzüge eines neuen Systems der Grundlagen der Mathematik', in McCall [1967], 116–169. English translation of [1938a].

[1967b] 'On Definitions in the So-Called Theory of Deduction', in McCall [1967], 170–187. English translation of [1930b].

LEWIS, DAVID

[1970] 'Nominalistic Set Theory', *Noûs* 4, 225–240.

[1972] 'General Semantics', *Semantics of Natural Language*, ed. by Davidson and Harman, D. Reidel.

One of the most interesting extensions of Ajdukiewicz's ideas.

LINDENBAUM, ADOLF (1904–1941)

[1931] 'Bemerkungen zu den vorhergehenden "Bemerkungen..." des Herrn J. v. Neumann', *Fundamenta Mathematicae* 17, 335–336.

[1936] 'Sur la simplicité formelle des notions', *Actes du Congrès International de Philosophie Scientique, VII Logique*, Paris, 29–38.

Cf. Sobociński's paper on well constructed axioms [1954].

LINDENBAUM A. and TARSKI, A.

[1926] 'Communication sur les récherches de la théorie des ensembles', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Classee III, 19, 299–330.

LINSKY, L. and SCHUMM, G.

[1971] 'Frege's Way Out: A Footnote', *Analysis* 32, 5–7.

LIPPERT, BERNHARD MATTHÄUS

[1976] *Rekonstruktionen zur Lesniewski'schen Logik*, Zulassungsarbeit zum Staatsexamen, Konstanz.

An interesting synopsis of work pertaining to Leśniewski's logic.

LODE, TENNY

[1952] 'The Realization of a Universal Decision Element', *The Journal of Computing Systems* 1, 14–22. Reviewed, *JSL* 18, 284, by N. M. Blachman and W. W. Boone.

Leśniewski's wheel and spoke notation is adopted here.

LORENZ, KUNO

[1976] 'Some Remarks on the Relation Between the Dichotomy of Part and Whole With the Dichotomy of Property and Object', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, p. 17. (Abstract of a lecture).

Gives operations on a domain of objects which transforms, in a natural way, a theory of individuals into a theory of sets and conversely.

ŁUKASIEWICZ, JAN (1878–1956)

[1910a] 'Über den Satz des Widerspruchs bei Aristoteles', *Bulletin International de l'Académie des Sciences de Cracovie*, Classe de philologie, Classe d'histoire et de philosophie, 15–38. [1971] is an English translation.

[1910b] *O zasadzie sprzeczności u Arystotelesa. Studyum krytyczne*, Kraków, Akademia Umiejętności, 210 pp. An expanded version of [1910a].

[1921] 'Logika dwuwartościowa' (Two-Valued Logic), *Przegląd Filozoficzny* 23, 189–205. [1970b] is an English translation.

This article is referred to by Leśniewski several times. Its influence on Protothetic is obvious.

[1921b] 'O ontologji prof. Leśniewskiego' (On the Ontology of Prof. Leśniewski), *Przegląd Filozoficzny* 24, 248 and *Ruch Filozoficzny* 6, 72.

Notice of a lecture by Łukasiewicz. No abstract has been located.

[1924] 'O pewnym sposobie pojmowania teorji dedukcji' (On a Certain Way of Interpreting the Theory of Deduction), *Przegląd Filozoficzny* 28, 134–136.

The discussion following this lecture contains remarks by Leśniewski. Photostat in Leśniewski [1967].

[1928a] 'Rola definicji w systemach dedukcyjnych' (The Role of Definitions in Deductive Systems), *Ruch Filozoficzny* 11, 164.

At this lecture of March 24, 1928 Łukasiewicz mentions creative definitions and refers to them as 'hidden axioms'.

[1928b] 'O definicjach w teorii dedukcji' (On Definitions in Deductive Theories), *Ruch Filozoficzny* 11, 177–178.

This summary of a lecture given by Łukasiewicz, February 18, 1928, contains the first (known) mention of creative definitions in the literature. Mention is made of several definitions which are creative in certain systems, but no details are given in the summary. At this lecture Leśniewski affirmed his belief in creative definitions and stated that creative definitions should be used as often as possible.

[1929] *Elementy logiki matematycznej*, Warsaw, viii + 200 pp. Second edition 1958, PWN. [1963] is an English translation.

[1939] 'Der Äquivalenzenkalkül', *Collectanea Logica* 1, 145–169. Reviewed by H. Scholz, *Z* 22, 289–290. [1961b] is a Polish translation; [1967] an English translation.

This paper is important for the example of a creative definition that it contains at the end.

[1950] *Aristotle's Syllogistic, from the Standpoint of Modern Formal Logic*, Oxford.

[1951] 'On Variable Functors of Propositional Arguments', *Proceedings of the Royal Irish Academy*, sect. A, 54, 25–35. Reprinted in [1970a], 311–324. [1961c] is a Polish translation. Reviewed by Prior, *The Austral-Asian Journal of Psychology and Philosophy* 30, 33–46 and by Church, *JSL* 16, 229 (He makes some interesting comments about definitions).

[1953] 'Symposium: The Principle of Individuation I', *Proceedings of the Aristotelian Society*, Supp. vol. 27, 69–82.

Only §5–7 (pp. 77–82) concern Leśniewski. The elementary axiom is given with examples ('man is mortal' is false for it implies the false consequence 'if *C* (Sokrates) is a man and *D* (Callias) is a man then *C* is *D*'. This seems a nice way to explain things). All information is of an elementary nature. This would be a good introduction were it not for all the talk about Aristotle. §4 contains some remarks about definitions. Łukasiewicz describes Ontology: "It is an extremely rich system and built up with the utmost care and precision, so that it deserves our highest attention."

Of historical interest is the story concerning the discovery of the axiom of Ontology (pp. 77–78): "First, a personal reminiscence. It was 1921. I was dissatisfied with the inexact description of the copula 'is' given by Peano and with the vague symbol of the theory of sets. In the course of a logical discussion I asked Leśniewski what he meant by the expression '*a* is *b*'. He replied that he was using it in the sense of everyday life. I was dissatisfied, for I thought that the copula 'is' should either be defined, or described by axioms, if taken as primitive term. Some time later Leśniewski told me that he had found the required axioms sitting on a bench in the Warsaw-Saxon Park."

[1961a] *Z zagadnień logiki i filozofii, Pisma wybrane* (Problems of Logic and Philosophy, Selected Writings), PWN, Warsaw, , ed. J. Słupecki, 309 pp. These are Polish translations of some of Łukasiewicz's papers. They are reviewed individually by Rose Rand, *JSL* 33, 129–133. Included are [1961b], [1961c] and Łukasiewicz and Tarski [1961].

[1961b] 'Równoważnościowy rachunek zdań', in [1961a], 228–249. Polish translation of [1939].

[1961c] 'O zmiennych funktorach od argumentów zdaniowych', in [1961a], 250–260. Polish translation of [1951].

[1963] *Elements of Mathematical Logic*, Pergamon Press. Translation of [1929]. Reviewed by Prior, *Journal of Philosophy* 65, 152–153.

On page 32 he states his view that definitions should not be creative.

[1967] 'The Equivalential Calculus', in McCall [1967], 88–115. English translation of [1939]. Reprinted in [1970a], 250–277.

[1970a] *Jan Łukasiewicz, Selected Works*, North-Holland, Ed. L. Borkowski, xii + 405 pp. Reviewed by McCall, *Synthese* 26, 165–171 and W. Kneale, *Mind* 81, 144-147. Contains among others, an English translation of [1939], and a reprint of [1951].

There are a few scattered remarks about Leśniewski.

[1970b] 'Two-Valued Logic', in [1970a], 89–109. Translation of [1921a].

[1971] 'On the Principle of Contradiction in Aristotle', *The Review of Metaphysics* 24, 485–509. English translation of [1910a].

ŁUKASIEWICZ, JAN and TARSKI, ALFRED

[1930] 'Untersuchungen über den Aussagenkalkül', *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Classe III, 23, 30–50, [1956], [1961], and [1972] are translations.

§5 discusses the extended sentential calculus.

[1956] 'Investigations into the Sentential Calculus', in Tarski [1956], 38–59. Reprinted in Łukasiewicz [1970], 131–152. Translation of [1930].

[1961] 'Badania nad rachunkiem zdań', in Łukasiewicz [1961a], 129–143. Polish translation of [1930].

[1972] 'Recherches surle calcul propositionnel', in Tarski [1972], 45–65. Translation of [1930].

LUSCHEI, EUGENE C.

[1962] *The Logical Systems of Leśniewski*, North-Holland, vii + 361 pp. Reviewed by P. Nidditch, *Mind* 74, 142–143, L. J. Cohen, *The Philosophical Quarterly* 15, 81–82. H. Hermes, *Z* 111, 6. E. E. Dawson, *The British Journal for the Philosophy of Science* 15, 341–345, and by Ivo Thomas, *Philosophy of Science* 34, .

An excellent source of information concerning the historical and philosophical aspects of Leśniewski's systems.

LYONS, JOHN

[1966] 'Towards a 'notation' theory of the 'parts of speech' ', *Journal of Linguistics* 2, 209–236.

[1968] *Introduction to Theoretical Linguistics*, Cambridge University Press.

Uses categorical grammars (Ajdukiewicz) as a base for transformational grammar. See pp. 227–231 and 328–330.

MACHOVER, M.

[1966] 'Contextual determinancy in Leśniewski's grammar', *Studia Logica* 19, 47–57. With Polish and Russian summaries. Reviewed by K. Schütte, *MR* 34 #5640.

This interesting paper uses set theory to define Leśniewski's semantical categories. Then he provides an effective procedure for deciding if a given text is contextually determinate.

MARTIN, NORMAN M.

[1953] 'On Completeness of Decision Element Sets', *The Journal of Computing Machinery* 1, 150–154.

Leśniewski's wheel and spoke notation is used here.

MARTIN, RICHARD M.

[1953] 'On Truth and Multiple Denotation', *JSL* 18, 11–18. Reviewed *JSL* 21, 89 by G. D. W. Berry.

'Fido' *denotes* the actual dog Fido; 'dog' *denotes severally* the dogs Fido, Marni, etc. What we ordinarily call class names thus come to denote severally the members of the class, but not the class itself.

His purpose is to consider the semantics of an (arbitrary) system without enriching the metalogic with recursive definitions. He bases the system on (two-place) concatenation and logic with quantification theory, but proceeds platonistically. Also contains several definitions of truth.

[1958] *Truth and Denotation, A Study in Semantical Theory*, University of Chicago Press.

A very valuable book.

[1969] 'On Events and the Calculus of Individuals', *Proceedings of the XIVth International Congress of Philosophy* 3, 202–208.

Calls attention to the philosophic importance and usefulness of the calculus of individuals.

MARTIN, R. M. and WOODGER, J. H.

[1951] 'Toward an Inscriptional Semantics', *JSL* 16, 191–203. Reviewed by Y. Bar-Hillel, *JSL* 17, 71.

MAZURKIEWICZ, STEFAN

[1939] 'Stanisław Leśniewski (1886–1939)', *Przegląd Filozoficzny* 42, 115.
An Obituary. Includes a photograph.

MCCALL, STORRS

[1967] *Polish Logic*, Oxford University Press, viii + 406.
This anthology contains Kotarbiński [1967], Łukasiewicz [1967], Leśniewski [1967a], [1967b], Sobociński [1967a], [1967b], Ajdukiewicz [1967], and Jordan [1945].

MENNE, ALBERT, ED.

[1962] *Logico-Philosophical Studies*, D. Reidel.
Contains reprints of Bocheński [1956] and [1949b].

MEREDITH, CAREW ARTHUR (1904–1976)

[1951] 'On an Extended System of the Propositional Calculus', *Proceedings of the Royal Irish Academy* 54, Sect A, 37–47. Reviewed, *JSL* 16, 229–230, by Church and *MR* 13, 3.

MICHAŁOWSKI, WITOLD

[1955] 'Zagadnienie nazw pustych w sylogistyce w świetle ontologii Leśniewskiego' (The Problem of Non-Referential Names in Aristotle's Syllogistic from the Point of View of Leśniewski's Ontology), *Roczniki Filozoficzne* 5, 65–95, 227. Reviewed by Lejewski, *JSL* 27, 117.

[1964] 'Non-Referential Names and a Particular Quantifier', *Studia Logica* 15, 273–274.
Contains the statement that Kotarbiński and Słupecki attach existential import to Leśniewski's particular quantifier. They may, but if they do, they do so incorrectly.

MIHAILESCU, EUGENE GH.

[1937a] 'Recherches sur un sous-système du calcul des propositions', *Annales Scientifiques de l'Université de Jassy* 23, 106–124. Reviewed *JSL* 2, 51 by A. A. Bennet.
Bases the equivalential calculus on $EEpqEqp$ and $EEEpqrEpEqr$.

[1937b] 'Recherches sur la négation et l'equivalence dans le calcul des propositions', *Annales Scientifiques de l'Université de Jassy* 23, 388–403.

If a propositional formula contains only equivalence and negation then it is a tautology iff negation and each variable occurs an even number of times. This generalizes Leśniewski's theorem [1929b], which only concerned equivalential formulas.

[1969] *Logica Matematică, Elemente de Calcul cu Propozitii se Predicate*, Editura Academiei Republicie Socialista România, Bucureşti.

Chapter two is on the equivalential calculus. It seems to include a translation of the completeness of Leśniewski [1929b].

MIKOŁAJEWICZ, BOLESŁAW

[19xx] 'Zagadnienie odtwarzalności logiki tradycyjnej w pewnym elementarnym rachunku nazw' (Problem of the Reconstructability of Traditional Logic in an Elementary Calculus of Names), *Acta Universitatis Wratislaviensis*, to appear.

Shows exactly to what extent traditional logic can be reconstructed, by means of special translations, in an elementary calculus of names. This calculus is an extension of elementary ontology without definitions. It contains atomic formulas of the form 'x is y' and 'x is undoubtedly y'. One contains a weak nominal negation sign.

MORAVCSIK, JULIUS M. E.

[1973] 'Mass Terms in English', *Approaches to Natural Language*, Proc. of the 1970 Stanford Workshop on Grammar and Semantics. Ed. K. J. J. Hintikka, J. M. E. Moravcsik and P. S. Suppes, D. Reidel, 263–285. Comments by Chung-Ying Cheng (286–288) and Richard Montague (289–294) and Richard E. Grandy (295–300) and Morovcsik's Reply (301–308) follow.

This may contain some 'Mereology'.

MORAWIEC, ADELINA

[1961] 'Podstawy logiki nazw' (Foundations of the Theory of Names), *Studia Logica* 12, 145–170. With Russian and English summaries.

Develops a logic of names, which is an extension of some fragments of traditional logic. Moraviec's logic does not contain bound variables; in this respect it is close to everyday language.

MORGAN, C. G.

[1973] 'Proper Definitions in *Principia Mathematica*', *International Logic Review* 4, #7, 80–85.

Comments on Nemesszeghy [1971].

MORRISON, PAUL G.

[1970] 'An Axiom-Free Theory of the Part-Whole Relation', *JSL* 35, 358–359. (Abstract).

MORSE, ANTHONY P.

[1965] *A Theory of Sets*, Academic Press, New York and London, xxxi + 130.

Accepts the Leśniewskian view that definitions are theses of the system and can be creative. "Because of the importance we attach to definitions we formulate in the Appendix the rules we follow in making them. Earlier, S. Leśniewski worked painstakingly along these lines." (p. xxvi)

MOSTOWSKI, ANDRZEJ (1913–1974).

[1948] *Logika Matematyczna* (Mathematical Logic), Warszawa–Wrocław, Monografie Matematyczne, tom XVIII.

Contains numerous comments about Leśniewski.

MUNITZ, MILTON K.

[1974] *Existence and Logic*, New York University Press.

Contains a discussion of Mereology, and treats similar problems.

MYHILL, JOHN

[1953] 'Arithmetic With Creative Definitions by Induction', *JSL* 18, 115–118. Reviewed, *JSL* 22, 303 by G. H. Müller.

Constructs a system of arithmetic with infinitely many creative definitions. He correctly credits Leśniewski with the notion of a creative definition, but incorrectly cites Leśniewski [1930b].

[1959] Review of Suppes [1957]. *Bulletin of the American Mathematical Society* 65, 156–160.

Myhill states that he understands that Leśniewski did not require that definitions be "conservative and eliminable" as is stated by Suppes. Myhill is correct.

NEMESSZEGHY, E. Z. and NEMESSZEGHY, E. A.

[1971] 'Is $p \supset q =_{\text{Df}} \sim p \vee q$ a Proper Definition in the System of the *Principia Mathematica?*', *Mind* 80, 282–283.

This paper has been criticized by Black [1973], Dudman [1973], Morgan [1973], and Rickey [1975].

[1973] 'On the Creative Role of the Definition $(p \supset q) = (\sim p \vee q)$ Df. in the system of *Principia*: Reply to V. H. Dudman (I) and R. Black (II)', *Mind* 82, 613–616.

[1976] 'On strongly Creative Definitions: A Reply to V. F. Rickey', *Logique et Analyse* , , 111–115.

The notion of strongly creative definitions defined here has no connection with that of Sobociński (see Rickey [1978]).

NEUMANN, JOHN VON (1903–1957)

[1931] 'Bemerkungen zu den Ausführungen von Herrn St. Leśniewski über meine Arbeit "Zur Hilbertschen Beweistheorie" ', *Fundamenta Mathematicae* 17, 331–334.

ODEGÁRD, DOUGLAS

[1969] 'Classifying the Class-Membership Relation, *Logique et Analyse* 12, #47, 221–224.

ONICESCU, OCTAV and RADU, EUGEN

[1975] 'Roumanian Contributions to Logical Developments: Researches in Mathematical Logic and in the Foundations of Mathematics', *International Logic Review* 6, #11, 81–88.

Moisil "proved that the models of Leśniewski–Mihailescu fragments of classical propositional calculus are second order abelian groups", (p. 82) "constructed, by means of Gentzen's technique, a higher-order sequential propositional calculus which he denominated "elementary logic" (p. 82).

PARSONS, CHARLES

[19xx] 'A Plea for Substitutional Quantification', *The Journal of Philosophy* 68, 231–237.

PARTEE, BARBARA HALL

[1973] 'Some Transformational Extensions of Montague Grammar', *Journal of Philosophical Logic* 2, 509–531.

PASENKIEWICZ, KAZIMIERZ

[1961] *Pierwsze Systemy Semantyki Leona Chwistka* (The First Semantical Systems of Leon Chwistek).

PELLETIER, FRANCIS JEFFRY

[1974] 'On Some Proposals for the Semantics of Mass Nouns', *Journal of Philosophical Logic* 3, 87–108.

Related to Mereology.

PERREIAH, ALAN R.

[1971] 'Approaches to Supposition-Theory', *The New Scholasticism* 45, 381–408.

Section 3 (4 pages) discusses D. P. Henry's use of Ontology in Supposition-Theory.

PLANTINGA, ALVIN

[1975] 'On Mereological Essentialism', *The Review of Metaphysics* 28, 468–476.

Comments on Chisholm [1973].

POGORZELSKI, W. A.

[1969] *Klasyczny rachunek zdań: Zarys teorii*, Warszawa, Państwowe Wydawnictwo Naukowe. Translated as *Classical Propositional Calculus*. Reviewed, *SL* 33, 205.

POPPER, KARL

[1963] 'Creative and Non-Creative Definitions in the Calculus of Probability', *Synthesis* 15, 167–186 + correction, 21, 107. Also appears in: *Form and Strategy in Science; Studies dedicated to Joseph Henry Woodger on the Occasion of his Seventieth Birthday*, D. Reidel, 1964, 171–190.

The creativity is obtained here by disregarding the conditions which definitions with equality must satisfy.

POZSGAY, LAWRENCE

[1971] 'Liberal Intuitionism as a Basis for Set Theory', *Proceedings of a Symposium in Pure Math.*, Vol. XIII, *Part I: Axiomatic Set Theory*, American Mathematical Society, 321–330.

Mentions Luschei [1962] (pp. 67, 74–78) and the argument against the empty set.

PRAKEL, JUDITH M.

[1976] 'Mirroring Modalities in Leśniewski's Ontology', *XIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, pp. 22–23. (abstract of a lecture).

PRIOR, ARTHUR N. (1914–1969)

[1952] Review Article: 'Łukasiewicz's Symbolic Logic', *The Australasian Journal of Psychology and Philosophy* 30, 33–46.

Says that Łukasiewicz takes the notion of 'variable functor' from Leśniewski, but Prior makes no comment on Łukasiewicz's novel treatment of substitution. "But all in all, this paper [1951] is quite the most exciting contribution that has been made to symbolic logic in English for a very long time and put forward with a rare lucidity, brevity and elegance."

Mentions Meredith [1951] but does not seem aware that Leśniewski knew $C\delta\delta O\delta p$.

[1955a] *Formal Logic*, Oxford, 1955, revised 1962. Reviewed by Hughes Leblanc, *JSL* 27, 218–220, L. Borkowski, *Studia Logica* 15, 298–301., and by E. W. Beth, *Synthese* 11, 85–86. Also see Smart [1956].

Contains considerable introductory material on the Leśniewskian systems.

[1955b] 'English and Ontology', *British Journal for the Philosophy of Science* 6, 64–65.

[1956] 'Definitions, Rules and Axioms', *Proceedings of the Aristotelian Society* 56, 199–216 + corrections bound in the same volume.

[1959] 'Formalized Syllogistic', *Synthese* 11, 265–273.

A system of syllogistic is presented which has some vague connections with Ontology. Prior remarks about a similarity to the use of variable functors by Łukasiewicz [1951].

[1962] 'Nonentities', *Analytic Philosophy*, ed. R. J. Butler, Barnes and Noble, 129–132. Reviewed by A. R. Anderson, *JSL* 29, 140–141.

[1964] 'The Algebra of the Copula', *Studies in the Philosophy of Charles Sanders Pierce*, Second series, ed. E. C. Moore and R. S. Robin, University of Massachusetts Press, 79–94.

[1965] 'Existence in Leśniewski and in Russell', *Formal Systems and Recursive Functions*, ed. Crossley and Dummitt, North-Holland, 149–155. Abstract *JSL* 28, 262.

Prior mentions four peculiarities of Ontology: (1) compatibility with the empty universe (*Principia Mathematica* isn't. see *24.52); (2) names divided into empty, singular and plural; (3) existence can be significantly predicated ('*a* exists' is meaningful, though not always true). (4) subject and predicate of ε are of the same type. Prior's thesis is that Ontology is a broadly Russellian theory of classes deprived of variables of the lowest type. Thus an *L* name is an *R* class-name. This thesis is supported by the fact that *L* names of logically complex (*R* names are structureless — they cannot be constructed from other things). Prior's interpretation takes care of (2) and (3). Mentions that *R*'s individual variables are indispensable when considering classes (solves (4)); the inclusion of a unit class into another class (Jerzy Łoś). Prior does not accept (or perhaps does not know) Leśniewski's requirement that a language be ontologically (in the philosophical sense) neutral. This would clear up his troubles with (1). Prior seems to equate name and individual name.

[1971] *Objects of thought*, Oxford University Press, ed. P. T. Geach and A. J. P. Kenny.

There are several references to Leśniewski.

QUINE, W. V. O.

[1955] 'On Frege's Way Out', *Mind* 64, 145–159.

[1969a] 'Existence and Quantification', *Fact and Existence*, Proceedings of the Western Ontario Philosophy colloquium, Joseph Margolis, ed., Basil Blackwell — Oxford, 1–17.

[1959b] *Ontological Relativity and Other Essays*, Columbia University Press.

After writing his dissertation under Whitehead in 1932 he spent some months of informal study at the Universities of Vienna, Prague and Warsaw. He met Leśniewski in 1933 and bases his views on those contacts. See pages 63, 92, 104, 105 and 106 of this book. The views expressed here are confused.

RAND, ROSE

[1938] 'Kotarbińskis Philosophie auf Grund seines Hauptwerkes: "Elemente der Erkenntnistheorie, der Logik und der Methodologie der Wissenschaften" ', *Erkenntnis* 7, 92–120. Reviewed by Ernest Nagel, *JSL* 3, 169.

A summary of Kotarbiński's work of the title. Contains a sketch of Ontology.

RESCHER, NICHOLAS

[1955] 'Axioms for the Part Relation', *Philosophical Studies* (Minneapolis) 6, 8–11. Reviewed, *JSL* 22, 213–214 by Lejewski.

Rescher has some criticisms of Mereology but Lejewski refutes them in his review.

[1975] 'Mereology', *Encyclopedia Brittanica*, Macropedia, XI, 36–37.

RESNIK, M. D.

[1964] 'Some Observations Related to Frege's Way Out', *Logique et Analyse* 7, #27, 138–144.

RICKEY, V. FREDERICK

[1968] *An Axiomatic Theory of Syntax*, Ph.D. dissertation, University of Notre Dame, under the direction of B. Sobociński.

Most of this is published in [1972] and [1973]. The remaining sixty pages contain a formal proof that the rule of procedure presented here (and in [1973]) for Protothetic is equivalent to that in Leśniewski [1929b].

[1972] 'Axiomatic Inscriptional Syntax', Part I: 'The Syntax of Protothetic', *NDJFL* 13, 1–33. Autorreferrat, *Z* 197, 276. Reviewed *MR* 48 #1880 by R. Parikh. Abstract *JSL* 35, 361.

[1973] 'Axiomatic Inscriptional Syntax', Part II: 'The Syntax of Protothetic', *NDJFL* 14, 1–52. Autorreferrat, *Z* 226, 02027. Reviewed *MR* 50 #4257 by R. Parikh.

Simplifies the terminological explanation of Leśniewski [1929b].

[1974] 'The One-Variable Implicational Calculus', *NDJFL* 15, 478–480. Autorreferrat *Z* 262, 02012. Reviewed by K. Inoue, *MR* 51 #2856.

Computable Protothetics motivates this axiomatization of the implicational calculus involving only one variable.

[1975a] 'Creative Definitions in Propositional Calculi', *NDJFL* 16, 273–294. Autorreferrat *Z* 232, 02008. Reviewed by Curry, *MR* 52 #38.

Contains the history of creative definitions.

[1975b] 'On creative definitions in the *Principia Mathematica*', *Logique et Analyse* 18, #69–70, 175–182.

A critique of Nemesszeghy [1971].

[1976a] 'A Survey of Leśniewski's Logic', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, p. 24 (abstract of a lecture). Published as [1977].

A survey of the technical work that has been done on Leśniewski's systems of Protothetic, Ontology and Mereology. The stress in on recent work.

[1976b] 'Model Theory for Leśniewski's Logic', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, p. 24 (abstract of a lecture).

Leśniewski's very appealing intuitive semantics for Ontology is replaced by a formal semantics. The notion of model which is given here is sufficient for proving a completeness theorem. Models are given which provide proofs that certain definitions are creative.

[1977] 'A Survey of Leśniewski's Logic', *Studia Logica* 36, 405–424.

[1978] 'On Creative Definitions in First Order Functional Calculi', *NDJFL* 19.

ROSE, ALAN

[1954] 'Caractérisation, au moyen de théorie des trellis, du calcul de propositions à foncteurs variables. Applications scientifiques de la logique máthematique', *Actes du 2e Colloque International de Logique Mathématique*, Paris, 25–30 août 1952, Institut Henri Poincaré, *Collection de logique mathématique*, ser. A, no. 5, lithographed, Gauthier–Villars, Paris, and E. Nauwelaerts, Louvain, pp. 87–88. Reviewed by Segerberg, *JSL* 34, 121.

Discusses the 'Łukasiewicz–Meredith' axiom for the propositional calculus with variable functors. "Leśniewski was the first to suggest adding variables for truth-functions to propositional logic."

[1971] 'Tautologies Sans Constantes', *C. R. Acad. Sci. Paris* 272, 1617–1619.

Concerns the propositional calculus with variable functors.

ROUALT, JACQUES

[1971] *Approche formelle de problèmes liés à la sémantique des langues naturelles*, Docteur dès Sciences dissertation for the Université Scientifique et Médicale de Grenoble; Institute de Rescherces en Mathématiques Avancées.

Mereology and Ontology are used in this analysis of Natural Languages, 1971.

RVAČEV, L. A.

[1966] 'Matematika i semantika nominalizm kak interpretacija matematika' (*Mathematics and Semantics, Nominalism as an Interpretation of Mathematics*), Izdat. 'Naukova Dumka', Kiev, 88 pp. Reviewed by A.A. Mullin, *MR* 34 #3104.

"In the work an idea of domain of things as individual objects is made more exact and a formalized language in this domain is developed. In sequel a question is considered, how far one can proceed in nominalistic

interpretation of mathematics" (pp. 3–4). The author constructs a peculiar calculus of events without any references to other systems (including that of Leśniewski's). He takes into consideration a causal relation and permits existence of several worlds and possible objects.

SAGAL, PAUL THOMAS

[1973a] 'Implicit Definition', *The Monist* 57, 443–450.

[1973b] 'On how Best to Make Sense of Leśniewski's Ontology', *NDJFL* 14, 259–262. *Z* 225, 02012 (Autorreferrat).

This paper is a critical examination of Prior [1965].

[1973c] 'Predicates, Concepts, and Ontological Neutrality in Lorenzen', *Ratio* 15, 902–903.

SALAMUCHA, JAN (1903–1944)

[1930] *Pojęcie dedukcji u Arystotelesa i św. Tomasza z Akwinu. Studjum historyczno-krytyczne* (The Concept of Deduction According to Aristotle and St. Thomas Aquinas. An Historical Critical Study), Warsaw, x + 130. Reviewed by I. M. Bocheński, *Bulletin thomiste* 9, 401–405.

SCHARLE, THOMAS W.

[1962a] 'A Diagram of the Functors of the Two-Valued Propositional Calculus', *NDJFL* 3, 243–255.

The binary and n-ary functors are arranged in an array so that definitional connections between certain sets of functors are displayed. This is related to Leśniewski's wheel and spoke notation. Sheffer functions are characterized and complete sets of functors are discussed.

[1962b] 'Note to my paper: "A diagram of the functors of the two-valued propositional calculus" , *NDJFL* 3, 287–288.

He gives the exact number of n-ary Sheffer functions.

[1970] 'Are definitions eliminable in formal systems?' *JSL* 35, 182–183. (Abstract).

[1971] 'Completeness of Many-Valued Protothetic', *JSL* 36, 363–364. (Abstract).

[1976] 'Higher Epsilons in Leśniewski's Ontology', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, p. 30. (abstract of a lecture).

This important paper shows that Ontology has many inner models.

SCHEFFLER, ISRAEL

[1972] 'Ambiguity: An Inscriptional Approach', *Logic and Art: Essays in Honor of Nelson Goodman*, ed. Richard Rudner and I. Scheffler, Bobbs-Merrill, 251–272.

SCHOCK, ROLF

[1968] *Logics Without Existence Assumptions*, Almquist and Wiksell, Stockholm, *MR* 39 #3973.

SCHULDENFREI, RICHARD

[1969] 'Eberle on Nominalism in Non-Atomic Systems', *Noûs* 3, 427–430.
Comments on Eberle [1968].

SEVERENS, RICHARD HOXIE

[1960] *Ontological Commitments in Categorical Systems*, Ph.D. dissertation at Duke, directed by Romane Clark.
Section II of chapter I is based partly on Luschei [1962].

SHEPARD, PHILIP T.

[1973] 'A Finite Arithmetic', *JSL* 38, 232–248.

SIKORSKI, R.

[1970] 'The Polish Mathematical Society in the 25 Years of People's Poland', *Rev. Pol. Acad. Sci.* 15, 78–85.
An anecdotal history of the society from 1945 to 1970.

SINISI, VITO F.

[1962] 'Nominalism and Common Names', *The Philosophical Review* 71, 230–235.
This paper refutes the claim that the nominalist must interpret the 'is' in 'This is red' as identity. A very clear presentation of the axiom of Ontology is given. The primitive epsilon is interpreted both with examples and quotations from Leśniewski. The characterization of Protothetic which occurs here (p. 231) and also in [1969], p. 241, is not general enough.

[1964] 'Kotarbiński's Theory of Genuine Names', *Theoria* 30, 80–95. List of three errata enclosed with reprint.

[1965a] 'Discussion: 'ε' and common names', *Philosophy of Science* 32, 281–286.
This is Sinisi's reply to Grossman's objections [1962] to Sinisi [1962]. The first part of the paper discusses Grossman's misinterpretations of

what Sinisi says. The second part explicates the notion of a common name and gives two interpretations (Models) for Ontology.

[1965b] 'Kotarbiński's Theory of Pseudo-Names', *Theoria* 31, 218–245.

[1966] 'Leśniewski's Analysis of Whitehead's Theory of Events', *NDJFL* 7, 323–327.

[1967a] 'A Few Comments on 'A few comments on concretism'', *Theoria* 33, 72–77.

Concerns Kotarbiński's debt to Leśniewski's Ontology.

[1967b] 'Tarski on the Inconsistency of Colloquial Language', *Philosophy and Phenomenological Research* 27, 537–541.

[1969] 'Leśniewski and Frege on Collective Classes', *NDJFL* 10, 239–246.

[1976] 'Leśniewski's Analysis of Russell's Antinomy', *NDJFL* 17, 19–34.

Discusses Leśniewski's views on the Russell Antinomy which he presented in his Polish papers before the discovery of Mereology. These views are not discussed in Sobociński [1950].

SKIDMORE, ARTHUR

[1973] 'Existence and the Existential Quantifier', *International Logic Review*, #8, (vol. 4, n. 2), 280–283.

Argues in favor of "the possibility of namable objects which do not exist". Agrees with Lejewski's reading [1954b] of the existential quantifier.

SKOLIMOWSKI, HENRYK

[1967] *Polish Analytical Philosophy. A Survey and a comparison with British Analytical Philosophy*, International Library of Philosophy and Scientific Method. The Humanities Press, New York, xi + 275 pp. Also Routledge and Kegan Paul. Reviewed by Krister Segerberg, *JSL* 34, 141 and by Anton Flew in *Philo. Books* 7, no. 3, 21–22.

This book does not cover logic, but there is much of interest in the chapter entitled 'Analytical Philosophy and Marxism'.

SLESZYŃSKI, JAN

[19xx] 'O Logice Tradycyjnej' (On Traditional Logic).

A review by Łukasiewicz (reprinted Łukasiewicz [1961], 127–128) indicates that this contains some information on Ontology.

SŁUPECKI, JERZY

[1946] 'Uwagi o sylogistyce Arystotelesa' (Remarks on Aristotles's Syllogistic), *Annales Universitatis Mariae Skłodowska-Curie* (Lublin), 1, section F, 187–191. French abstract. Reviewed by R. Suszko, *JSL* 13, 166.

[1948] *Z badań nad sylogistyką Arystotelesa* (Some Investigations on the Syllogistic of Aristotle), *Travaux de la Société des Sciences et des Lettres de Wrocław, series B*, no. 6, Państwowy Instytut Wydawniczy, Wrocław 1948, 30 pp. Journal also listed as *Prace Wrocławskiego Towarzystwa Naukowego*.

[1953] 'St. Leśniewski's Protothetics', *Studia Logica* 1, 44–112. With Polish and Russian summaries. See Errata, p. 299 of the same volume. Reviewed by Lejewski, *JSL* 21, 188–191. Cf. *MR* 16, 892.

[1955a] 'S. Leśniewski's Calculus of Names', *Studia Logica* 3, 7–73, with a Polish summary. *MR* 17, 1171.

[1955b] 'A Logical System Without Operators', *Studia Logica* 3, 98–124. With Polish and Russian summaries. Reviewed by Hiż, *MR* 17, 1171.

[1958] 'Towards a Generalized Mereology of Leśniewski', *Studia Logica* 8, 131–163, with Polish and Russian summaries.

"Leśniewski's Mereology is essentially poorer than other systems of set theory; in particular, it provides no basis for developing the arithmetic of natural numbers." This paper corrects this.

[1968] 'Logic in Poland', in Klibansky [1968], 190–201.

A wide ranging survey. Papers on 'the logic of names' are discussed in one paragraph on pp. 195–196.

[1971] 'Leśniewski, Stanisław (1886–1939)', *Filozofia w Polsce: Słownik Pisarzy* (Philosophy in Poland: A Dictionary of Writers). Wrocław, Zakład Narodowy im. Ossolińskich, 221–224.

[1972] 'Leśniewski, Stanisław (1886–1939)', *Polski Słownik Biograficzny*, Wrocław, Zakład Narodowy im. Ossolińskich 17, 177–179.

SMART, J. J. C.

[1956] 'Review of Prior [1955a], *The Australasian Journal of Philosophy* 34, 118–126.

Mentions that the book contains accounts of some of the work of the Polish logicians, including a description of Protothetic. "This leads on to an extremely interesting criticism of Leśniewski's and Łukaszewicz's account of definitions as axiomatic assertions of equivalence. Prior comes down against this view, and argues for the *PM* account of definitions as abbreviatory devices. Prior has recently informed me that he has

changed his views on this matter and that he has come round to a point of view more sympathetic to Leśniewski. However, this does not impair the value of the discussion in the book which provides a very good introduction to some of the issues involved in this controversy." (p. 121)

There is also 'a discussion of Leśniewski's systems of Ontology and Mereology. These systems are intrinssically interesting but should perhaps have been treated in a different chapter. The 'classes' in mereology are not really classes at all but are concrete aggregates. For this reason a class, in Leśniewski's sense, possesses no cardinal number. It may be, say, both 10 (piles of paper), 1000 (sheets of paper) and so many million (molecules). Ontology is extremely interesting as an alternative way of securing 'individual reference' instead of Russell's apparatus of 'logical proper names' (or variables for such) plus descriptions. It also lends itself extremely well to the discussion of certain points about definitions." (pp. 124–125)

SMIRNOV, V. A.

[1965] 'Modelirovanije mira v strukture logičeskich jazykov' (The Modelling of the World in the Structure of Logical Languages.), *Logic and Methodology of Science* (Proc. 4^{th} All-Union symp., Kiev, 1965), Moscow, 117–125.

Pp. 122–124 are on Leśniewski. Using Quine's criterion, the difference between languages of Leśniewski's type and those of the Frege–Russell type is explained.

SOBOCIŃSKI, BOLESŁAW

[1932] 'Z badań nad teorją dedukcji' (An Investigation of the Theory of Deduction), *Przegląd Filozoficzny* 35, 171–193.

Contains some remarks about the equivalential calculus.

[1934] 'O kolejnych uproszczeniach aksjomatyki "Ontologji" Prof. St. Leśniewskiego', *Fragmenty Filozoficzne* 1, 143–160. [1967a] is an English translation.

[1939] 'Z badań nad prototetyką', *Collectanea Logica* 1, 171–177 (offprint). Reviewed by H. Scholz, *Z* 23, 289. [1949a] and [1967a] are English translations.

[1949a] *An Investigation of Protothetic*, Cahiers de l'Institut d'Études polonaises en Belgique, no. 5. Brussels, v + 44 pp. English translation of [1939]. Carefully reviewed by Church, *JSL* 15, 64.

This translation contains a very interesting introductory note (27 pp.), which has been omitted from [1967], and which has been discussed by Luschei [1962]. It describes the history of the ill-fated *Collectanea*

Logica, a volume which was to have included Leśniewski [1938a], [1938b], Łukasiewicz [1939], Sobociński [1939], two papers unrelated to Leśniewski's systems, and the following five papers on Protothetic which were completely destroyed when the printing house was bombed:

VII. B. Sobociński, 'O aksjomatykach prototetyki' (On the Axiomatics of Protothetic).

Contains Metatheorem L of Leśniewski (see Sobociński's proof that the four laws of logical multiplication are unnecessary hypotheses in this metatheorem), and the axiom A_l (in the notation of Sobociński [1960], p. 66. A_l is misprinted as A_1). X and XI extend the results of this paper (in an unspecified way).

VIII. S. Leśniewski, 'O pewnym jedynym aksjomacie prototetyki' (On Some Single Axioms of Protothetic).

IX. B. Sobociński, 'O różnych systemach prototetyki' (About Different Systems of Protothetic).

Discusses systems of protothetic based on terms other than equivalence and using rules other than those of $\mathfrak{S}_5$. Leśniewski's axiom A_m was to be published here also. In Sobociński [1960], A_m contains two misprints (underlined). It should be: $[pq] \colon\colon p \equiv q \,. \equiv \therefore\because [f] \therefore\because f(qf(q[u] \,.\, u)) \,. \equiv \colon\colon [r] \colon\colon f(pr) \equiv .\therefore r \equiv \dot{:} q \equiv . r \equiv p$.

X. J. Słupecki, 'Przyczynek do prototetyki' (Contributions to Protothetic).

XI. B. Sobociński, 'Uwagi w związku z pracą p. J. Słupeckiego: "Przyczynek do prototetyki"' (Remarks Concerning the Paper of Mr. J. Słupecki: "Contributions to Protothetic").

Most of the results of these papers are probably in Sobociński [1953] and [1960].

[1949b] 'L'analyse de l'antinomie russellienne par Leśniewski', *Methodos* 1, 94–107, 220–228, 308–316, and 2 (1950), 237–257. There are valuable reviews by Prior, *JSL* 18, 331–333 and Curry, *MR* 11, pp. 73, 412, 708 and *MR* 13, 199.

This very important paper contains Leśniewski's third and definitive analysis of the Russell Antinomy.

[1953a] 'Z badań nad aksjomatyką prototetyki Stanisława Leśniewskiego' (An Investigation of the Axiomatics of Stanisław Leśniewski's Protothetic), *Polskie Towarzystwo Naukowe Na Obczyźnie*, Rocznik 4 (1953–54, published 1954), 18–20. Reviewed by Lejewski, *JSL* 21, 325.

Contains Leśniewski's completeness metatheorem together with Sobociński's simplification of it. There are also remarks about single axioms.

[1953b] 'On a Universal Decision Element', *The Journal of Computing Systems* 1, 71–80.

In this paper on the propositional calculus Leśniewski is mentioned a few times.

[1954] 'Studies in Leśniewski's Mereology', *Polskie Towarzystwo Naukowe Na Obczyźnie'*, Rocznik 5 (1954–55, published 1954), 34–43. Reviewed by Prior, *JSL* 21, 325.

A nice introduction to Mereology precedes the statement of results: In 1946 Grzegorczyk found the first single axiom for Mereology. Sobociński's single axiom for element is presented, as well as a number of single axioms of Lejewski. The paper closes with some metalogical questions.

[1955] 'On Well Constructed Axiom Systems', *Polskie Towarzystwo Na Obczyźnie*, Rocznik 6 (1955–56), 54–65.

This interesting and readable paper deals with the esthetic conditions which axioms should satisfy. There is a discussion of the special conditions that axioms for Leśniewski's systems should satisfy. This paper is not as well known as it should be.

[1956] 'In Memoriam, Jan Łukasiewicz (1878–1956)', *Philosophical Studies* (Maynooth, Ireland) 6, 3–49.

Contains interesting background information about the Polish school of logic.

[1957a] 'Jan Łukasiewicz (1878–1956)', *Polskie Towarzystwo Na Obczyźnie*, Rocznik 7 (1956–57, published 1957), 3–21.

A Polish version of [1956], but less technical.

[1957b] 'La génesis de la Escuela Polaca de Logica', *Revista Oriente Europeo* (Madrid) 7, 83–95. Reviewed by J. F. Mora, *JSL* 25, 63–64.

This article discusses the contributions of Łukasiewicz, Leśniewski, and Chwistek.

[1960] 'On the Single Axioms of Protothetic, I, II, III', *NDJFL* 1, 52–73; 2, 111–126 and 129–148. Reviewed by Prior *JSL* 30, 245–246.

The most important paper on Protothetic. It contains a discussion of the history of the simplifications of the axiom of Protothetic, together with the difficult deductions from Sobociński's 1945 axiom (A_n in the notation of this paper). The proofs of the metatheorem of Leśniewski and Sobociński, which are sufficient to check the completeness of any axiom system of Protothetic, are given.

[1967a] 'Successive Simplifications of the Axiom-System of Leśniewski's Ontology', in McCall [1967], 188–200. This is a translation of [1934].

Provides the details and history of the simplifications of the axiom of Ontology, including a careful proof of the equivalence of the long axiom (A in the notation of this paper) and the short axiom (H).

[1967b] 'An Investigation of Protothetic', in McCall [1967], 201–206. English translation of [1939].

Provides a number of definitions of conjunction in terms of equivalence which are shorter and simpler than those of Tarski [1923a]. Short definitions of several other binary functors are provided; the wheel and spoke notation aids the understanding of these definitions.

[1971a] 'Lattice-Theoretical and Mereological Forms of Hauber's Law', *NDJFL* 12, 81–85.

Hauber's law is provable in Mereology.

[1971b] 'Atomistic Mereology I, II', *NDJFL* 12, 89–103 and 203–213.

The existence of atoms is not probable in Mereology. In this paper Mereology is extended by an axiom stating that every individual is a class of atoms. The system is also axiomatized using Rickey's functor '$A \in \mathrm{at}(B)$'. The definitions of atom given by Schröder and Tarski are shown to be equivalent to Atomistic Mereology. Consistency and independence are discussed in the second part.

[1971c] 'A Note on an Axiom-System of Atomistic Mereology', *NDJFL* 12, 249–251.

Provides another axiom system for Atomistic Mereology.

[1975] 'Concerning the Postulate-Systems of Subtractive Abelian Groups', *NDJFL* 16, 429–444.

Provides a single axiom for Abelian groups based on the single ternary functor $a - b = c$. Cf. Leśniewski [1929c].

SOLONIN, J. N.

[1969a] 'Teorija jazyka v rannich rabotach St. Leśniewskogo' (The Theory of Language in the Early Works of St. Leśniewski), *Problems of Philosophy and Sociology*, 1^{st} out, Leningrad university publ., 103–107.

The early philosophical views of Leśniewski are discussed. It is asserted that the goal of Leśniewski (until 1914) was to elaborate general rules for language construction and use.

[1969b] 'Glavnyje čerty logiko-matematičeskoj sistemy St. Leśniewskogo' (The Main Features of St. Leśniewski's Logical-Mathematical Systems), *Vestnik Leningradskogo universiteta, ser. Ekonomika, filosofija, pravo* 23, 93–103.

A brief description of Leśniewski's philosophical views and his three systems. There are some inaccuracies.

[1970] 'Logičeskije issledovanija St. Leśniewskogo' (Logical Investigations of St. Leśniewski), Autorreferrat of thesis, Leningrad university, 1970.

Contains a characterization of the Lwów–Warsaw school and, in particular, of views of Leśniewski. Leśniewski is called the founder of modern nominalism in mathematics. The work gives an account (supplemented by a few comments) of the principles of building up logical languages and deductive theories, worked out by Leśniewski.

[1975] 'Propositional Calculus with Variable Functors', *Contributed Papers*, to the Fifth International Congress of Logic, Methodology and Philosophy of Science, London, Ontario, Canada, 27 August – 2 September, 1975, pages XII–53 and XII–54.

A comparison of variable functors in Protothetic and in Łukasiewicz [1951].

SRZEDNICKI, JAN

[1976] 'On Being a (Material) Object', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, 31–38.

STASZEK, WALENTY

[1969] 'Z badań nad klasyczną logiką nazw' (On the Classical Logic of Names), *Studia Logica* 25, 169–188. With Russian and English summaries.

[1973] 'Elementarna ontologia Leśniewskiego jako fragment teorii mnogości Zermelo' (Leśniewski's Elementary Ontology as a Fragment of Zermelo Set Theory), *Studia Filozoficzne* 2 (87), 91–98.

Among other things, Staszek shows that by using the definition:

$x \epsilon y \equiv [\bigwedge_z (x \in y \equiv z \in \{\{z\}\} \wedge z \in y) \wedge x \in y]$

(wherein the first epsilon is the Ontological epsilon, and the remaining ones are set-theoretical), the 1920 single axiom of Ontology is derivable as a theorem of Zermelo set theory (without the axiom of foundation).

STELZNER, WERNER

[1976] 'Functor Variables, Function Variables, and Quasifunctors', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, 39–42.

STERNFIELD, ROBERT

[1966] *Frege's Logical Theory*, Southern Illinois University Press, Carbondale and Edwardsville, Illinois.

Leśniewski's views on the antinomies are discussed.

Stone, M. H.

[1937] 'Note on Formal Logic', *American Journal of Mathematics* 59, 506–514. Reviewed by Quine, *JSL* 2, 174.

Contains a proof of the Leśniewski–Mihailescu theorem.

Stonert, Henryk

[1959] *Definicje w naukach dedukcyjnych* (Definitions in Deductive Sciences), Łódź, Zakład Narodowy im. Ossolińskich we Wrocławiu.

The author discusses in detail Leśniewski's views on definitions.

Sullivan, Theodore F.

[1969] *Contributions to the Foundations of the Geometry of Solids*, Ph.D. dissertation, University of Notre Dame, under the direction of Robert E. Clay.

[1971] 'Affine Geometry Having a Solid as Primitive', *NDJFL* 12, 1–61. Part of [1969].

Generalizes Tarski's work on Euclidean Geometry to affine geometries which are equivalent to finite dimensional vector spaces over an ordered field.

[1972a] 'The Name Solid as Primitive in Projective Geometry', *NDJFL* 13, 95–97. Abstract in *Notices of the American Mathematical Society* 18, 88.

Add the name 'solid' to Mereology and define point. He gives an interpretation in ordered projective geometry such that the defined points correspond bijectively to the points of the geometry.

[1972b] 'On Certain Equivalence Classes of Spheres in L^P Spaces', *Notices of the American Mathematical Society* 19, A–29.

"It is shown that Tarski's definition of point in Leśniewski's Mereology determines, in an arbitrary L^P space, a set of equivalence classes of spheres which is in a 1–1 correspondence with the point of the L^P space."

[1973a] 'The Geometry of Solids in Hilbert Spaces', *NDJFL* 14, 575–580. This is a portion of Sullivan [1969]. Autorreferrat *Z* 232, 02023. Abstract in *Notices of the American Mathematical Society* 17, 236.

[1973b] 'Tarski's Definition of Point in Banach Spaces', *Journal of Geometry* 3, 179–189. Autorreferrat, *Z* 262, 50006. Abstract in *Notices of the American Mathematical Society* 20, A–31.

If in Tarski's definition of point [1929], 'solid' is interpretation as 'open ball' in two dimensional Banach space, then 'A is concentric to B' is equivalence relation iff the Banach space is strictly convex.

SUPPES, PATRICK

[1957] *Introduction to Logic*, D. van Nostrand. See Myhill [1959].

On p. 153 he credits, incorrectly, Leśniewski with the criteria that a definition should be eliminable and non-creative.

[1970] 'Probalistic Grammars for Natural Languages', *Syntheses* 22, 95–116.

Both Leśniewski and Ajdukiewicz are cited with respect to semantical categories.

[1973] 'Problems in the Philosophy of Space and Time', *Space, Time and Geometry*, ed. P. Suppes, D. Reidel Publ. Co., 392–395.

An extended and refined version of Noll's account of a theory of bodies which provides an adequate foundation of mechanics.

SURMA, STANISŁAW J.

[1971a] 'Method of Natural Deduction in Equivalential and Equivalential-Negational Propositional Calculus', *Universitas Iagellonica Acta Scientiarum Litterarumque, Schedae Logicae* 6, Kraków, 55–56.

NOTE: This journal is also listed as: *Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Prace z Logiki.*

[1971b] 'Przegląd wyników i metod badań nad równoważnościowym rachunkiem zdań' (Review of Results and Methods in the Equivalential Propositional Calculus), *Ruch Filozoficzny* 29, 284–290.

A survey of results and investigations on the equivalential propositional calculus.

[1972a] 'A Uniform Method of Proof of the Completeness Theorem for the Equivalential Propositional Calculus and for Some of Its Extension', *Universitas Iagellonica Acta Scientiarum Litterarumque, Schedae Logicae* 7, Kraków, 35–50. Reprinted, with some changes, in Surma [1973], 63–79. Reviewed by Canty, *MR* 54 #9973a, b.

Gives a simpler completeness proof of the equivalential calculus than that of Leśniewski [1929b] or Łukasiewicz (in Leśniewski) [1938a].

[1972b] 'A Survey of the Results and Methods of Investigations of the Equivalential Propositional Calculus, *Universitas Iagellonica Acta Scientiarum Litterarumque, Schedae Logicae* 7, Kraków, 51–75. Reprinted, with some changes, in Surma [1973], 33–61.

[1973] *Studies in the History of Mathematical Logic*, Polish Academy of Sciences, Institute of Philosophy and Sociology, Wrocław.

This collection of 17 papers was presented in 1966–1971 at the conferences of the Thematic Group for the History of Logic organized in

Cracow by the Department of Logic of the Polish Academy of Sciences. Contains reprints of Surma [1972a], [1972b].

[1976] 'On the Work and Influence of St. Leśniewski', Logic Colloquium 76. Proceedings of a conference held in Oxford in July 1976. Edited by R. O. Gandy and J. M. E. Hyland. North-Holland Publ. Co., Amsterdam 1977, 191–220.

SUSZKO, ROMAN

[1949] 'Z teorii definicji', *Poznańskie Towarzystwo Przyjaciół Nauk, Prace Komisji Filozoficznej* 7, 403–431. Reviewed, *JSL* 15, 223.

Introduces 'quasi-definitions', which "denote new axioms which contain new terms and which moreover 'determine unambiguously' the extension of these terms. The last condition plays a similar role to that played by the condition of non-creativity and of translatability in the old theory of definition".

[1958] 'Syntactic Structure and Semantical Reference I, II', *Studia Logica* 8, 213–247 and 9, 63–93. With Polish and Russian summaries.

[1976] 'The Fregean Axiom and Polish Mathematical Logic in the 1920's'. *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, 43–46.

TANAKA, SHÔTARÔ

[1966a] 'On Axiom Systems of Propositional Calculi. XVIII', *Proceedings of the Japan Academy* 42, 355–357.

The equivalential calculus can be axiomatized by $EEpqEEprErq$.

[1966b] 'On Axiom Systems of Propositional Calculi. XX', *Proceedings of the Japan Academy* 42, 361–363.

The equivalential calculus can be axiomatized using $EEpqEqp$ and $EEpEqrEEsqEsEpr$.

[1966c] 'On the Propositional Calculus With a Variable Functor', $C\delta qC\delta Np\delta q$, *Proceedings of the Japan Academy* 42, 1161–1163.

Derives the CN-calculus from the thesis mentioned in the title.

[1968a] 'On Axioms of Ontology', *Proceedings of the Japan Academy* 44, 54–55. *MR* 37 #2574.

Contains a proof that Leśniewski's 1921 axiom for Ontology (F) implies his 1920 axiom (A). A simpler proof of this result is contained in Sobociński [1967].

[1968b] 'On Theorems of Ontology', *Proceedings of the Japan Academy* 44, 231–233. *MR* 37 #2575.

Contains a proof of $[Aa] \therefore A \in a \,.\, \equiv : A \sqsubset a : [B] : B \sqsubset A \,.\, \supset .\, A \sqsubset B$. Thus $\sqsubset$ can serve as sole primitive term of Ontology.

[1969a] 'On the Proposition $C\delta CpqC\delta p\delta q$ with a Variable Functor', *Proceedings of the Japan Academy* 45, 95–96.

Derives the implicational calculus from the thesis mentioned in the title.

[1969b] 'Leśniewski's Protothetics $S1$, $S2$. I, II, III', *Proceedings of the Japan Academy* 45, 97–101, 259–262, 263–265.

Proves that every theorem of $S2$ is a theorem of $S1$. (In the notation of Słupecki [1955]).

[1970] 'On Axiom Systems of Ontology I, II', *Proceedings of the Japan Academy* 46, 255–257, and 47, 177–179.

Proof that Leśniewski's 1921 axiom for Ontology (F) is equivalent to his 1920 axiom (A). His proof that F implies A uses extensionality and accordingly is more complicated than his proof of the same result in [1968a].

TARSKI, ALFRED

[1923a] 'O wyrazie pierwotnym logistyki' (On the Primitive Term of Logistic), *Przegląd Filozoficzny* 26, 68–89. A modified French translation appeared in [1923b] and [1924]. An English translation based on all three is in [1956a]. [1972b] is another French translation.

This is Tarski's doctoral dissertation written under the direction of Leśniewski. Tarski (= Tajtelbaum) was Leśniewski's only doctor. This is the paper that made Protothetic possible. It provides a definition of conjunction in terms of equivalence (and the general quantifier).

[1923b] 'Sur le terme primitif de la Logistique', *Fundamenta Mathematicae* 4, 196–200.

[1924] 'Sur les truth-functions au sens de MM. Russell et Whitehead', *Fundamenta Mathematicae* 5, 59–74.

[1929] 'Les fondements de la géométrie des corps', *Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego*, supplement to *Annales de la Société Polonaise de Mathématique*, Kraków, 29–33. [1956a] and [1972c] are translations.

[1930] 'O pojęciu prawdy w odniesieniu do sformalizowanych nauk dedukcyjnych' (On the Notion of Truth in Reference to Formalized Deductive Sciences), *Ruch Filozoficzny* 12, 210–211.

[1933] 'Pojęcie prawdy w językach nauk dedukcyjnych' (The Concept of Truth in the Language of the Deductive Sciences), *Prace Towarzystwa Naukowego Warszawskiego, Wydział III, nauk matematyczno-fizycznych (Travaux de la Société des Sciences et des Lettres de Varsovie, Classee III, Sciences Mathématiques et Physiques)* 34, Warsaw, vii + 116 pp. [1936a] is a German translation, [1956a] is an English translation.

[1935] 'Zur Grundlegung der Booleschen Algebra I', *Fundamenta Mathematicae* 24, 177–198. English translation in [1956a]. Reviewed by A. Schmidt, *Z* 11, 2–3.

This work was influenced by Mereology, which is discussed in the footnote on pp. 333–334 of the translation.

[1936a] 'Der Wahrheitsbegriff in den formalisierten Sprachen', *Studia Philosophica* 1, 261–405. Translation of [1933]. [1972d] is a French translation.

[1936b] 'O ugruntowaniu naukowej semantyki', *Przegląd Filozoficzny* 39, 50–57.

[1936c] 'Grundlegung der wissenschaftlichen Semantik', *Actes de Congrès International de Philosophie Scientifique*,

[1939] 'On Well-Ordered Subsets of any Set', *Fundamenta Mathematicae* 32, 176–183.

[1941] *Introduction to Logic and to the Methodology of Deductive Sciences*, enlarged and revised edition. Oxford University Press, New York, xviii + 239 pp.

From a review by Church, *JSL* 6, 30–32: "The author's account of the nature of definitions (in which he follows Leśniewski) requires clarification. Apparently he rejects the view that definitions are conventions or abbreviations for expository convenience in the presentation of a formal system and thus extraneous to the system. He agrees that definitions are conventions, but conventions ascribing meaning to expressions. The reference to meaning would make definitions semantical rather than syntactical in character, but would require still that definitions be expressed in a metalanguage. Actually, Tarski expresses definitions in the object language (cf. page 150). Perhaps the intention is something like the following (which would seem to be not untenable); A definition is a primitive formula of the same character in general as an axiom, but associated with a new notation not previously occurring, and obeying certain rules of definition which so restrict its form as to insure the possibility of a proof of consistency with the axioms and previous definitions by a standard method."

O. Frink in his review, *MR* 2, 209 states: "Although emphasis is placed on the view of Leśnieski that definitions are not mere abbreviations but must conform to rules of definition, this point is not made very clear."

[1944] 'The Semantic Conception of Truth and the Foundations of Semantics', *Philosophy and Phenomenological Research* 4, 341–376.

[1956a] *Logic, Semantics, Metamathematics: Paper from 1923–1938 by Alfred Tarski*, Oxford. Translations by Woodger. Contains English translations of, among others, Tarski [1923], [1929], [1933], [1935]. Reviewed by W. A. Pogorzelski and S. J. Surma, *JSL* 34, 99–106, and by A. N. Prior, *Mind* 66, 401–410.

[1956b] 'On the Primitive Term of Logistic', in [1956a], 1–23. This translation is based on [1923a], [1923b] and [1924].

[1956c] 'Foundations of the Geometry of Solids', in [1956a], 24–29. Translation of [1929].

[1956d] 'On the Concept of Truth in Formalized Languages', in [1956a], 152–278. Translation based on [1933] and [1936].

[1956e] 'On the Foundations of Boolean Algebra', in [1956a], 320–341. Translation of [1935].

[1956f] 'The Establishment of Scientific Semantics', in [1956a], 401–408. Translation of [1936b] and [1936c].

[1972a] *Logique, Sémantique, Métamathématique 1923–1944*, vol. 1, Librairie Armand Colin, Paris, 276 pages. Translation of [1956].

[1972b] 'Sur le terme primitif de la logique', in [1972a], 1–25. Translation of [1923a].

[1972c] 'Les fondaments de la géométrie des corps', in [1972a], 26–34. Reprint of [1929] incorporating the additions contained in [1956c].

[1972d] 'Le concept de vérité dans les langages formalisés', in [1972a], 157–269. Translation of [1936a].

TRENTMAN, JOHN A.

[1966] 'Leśniewski's Ontology and Some Medieval Logicians', *NDJFL* 7, 361–364.

[1968] 'Extraordinary Language and Medieval Logic', *Dialogue* 7, 286–291.

[1976] 'On Interpretations, Leśniewski's Ontology, and the Study of Medieval Logic', *Journal of the History of Philosophy* 14, 217–222.

TREW, A

[1970] 'Nonstandard Theories of Quantification and Identity', *JSL* 35, 267–294. Reviewed by K. Inoue, *MR* 46 #3264.

Contains, among other things, a discussion of Lejewski [1967].

VACCARINO, GIUSSEPPE

[1948] 'La scuola polacca di logica', *Sigma* 2, #8–9, 527–546. Reviewed by A. Church, *JSL* 14, 127.

VANDERVEKEN, DANIEL R.

[1975] 'An Extension of Leśniewski–Curry's Formal Theory of Syntactical Categories Adequate for the Categorically Open Functors', *Bulletin de la Section de Logique*, Polish Academy of Sciences, June 1975.

[1976] 'The Leśniewski–Curry Theory of Syntactical Categories and the Categorically Open Functors', *Studia Logica* 35, 191–201.

An open functor is one whose syntactical category is not completely determined by syntactical structure of the formula where it occurs. The author enriches the notion of syntactical category so that there are no open functors. He erroneously claims that Leśniewski's systems contain open functors.

VAN FRAASSEN, BAS C.

[1966] *Foundations of the Causal Theory of Time*, University of Pittsburgh doctoral dissertation, University Microfilms, #66–13, 481.

Tarski's formulation of Mereology (pp. 236–237) is used to set up an ontology (in the philosophical sense) of events.

VUILLEMIN, JULES

[1967] *De la logique à la théologie. Cinq études sur Aristote*, Flammarion, Paris, 235 pp. Reviewed by Wilfrid Hodges, *JSL* 33, 615.

In three of these essays the author compares some of Aristotle's ideas with related notions in Cantor, Russell, and Leśniewski. The author says Aristotle's relation '*a* is said of *b*' is not the standard interpretation of the relation 'is a part of' in Mereology.

[1971] *Le Dieu D'Anselme et les Apparences de la Raison*, Paris.

WALLIS, JOHN

[1970] 'On the Frame of Reference', *Synthesis* 22, 117–150.

Repeats the statement of Quine [1969] that Leśniewski said that parentheses could form a substitution class. While I can conceive of Leśniewski making such a statement I feel that it is being misinterpreting here.

WANG, HAO

[1953] 'What is an Individual?', *The Philosophical Review* 62, 413–420. Reviewed by Boylis, *JSL* 20, 60.

Finds fault with Goodman's [1951] notion of an individual. He argues that nominalism forces finitism.

WATANABE, SYOZO

[1972(3?)] *On Many Valued Protothetics*, Ph.D. dissertation at the University of Manchester under the direction of Lejewski.

[1974] 'Many Valued Protothetics', *JSL* 39, 409–410. (Abstract).

WEINGARTNER, PAUL

[1965] 'Can One Say of Definitions that They are True or False?', *Ratio* 7, 61–93.

A very interesting paper.

[1974] 'On the Characterizations of Entities by Means of Individuals and Properties', *Journal of Philosophical Logic* 3, 323–336.

Contains a few comments about the different kinds of equality in Ontology.

[1976] 'Similarities and Differences Between the ∈ of Set-Theory and the Part-Whole-Relations', *XXIInd Conference on the History of Logic*, 5–9 July, 1976, Kraków, p. 48. (Abstract of a lecture).

[19xx] 'A Note on Aristotle's Theory of Definition and Scientific Explanation',

Discusses Aristotle's criterion of non-creativity for definitions.

WELLS,

[1951] 'Frege's Ontology', *The Review of Metaphysics* 4, 567.

WELSH, PAUL J. JR.

[1971] *Primitivity in Mereology*, Ph.D. dissertation, University of Notre Dame, under the direction of Robert E. Clay.

[1978] 'Primitivity in Mereology I, II', *NDJFL* 19, 25–62 and 355–385.

The familiar terms of Mereology are classified as primitive (by defining element) or non-primitive (by the use of models). The ternary terms on three individuals are completely classified as to primitivity. Part two presents infinitely many primitive terms for Mereology.

WHERRITT, ROBERT C.

[1971a] 'First-Order Equality Logic with Weak Existence Assumptions', *JSL* 36, 592. (Abstract).

WIEGNER, ADAM

[1948] *Elementy logiki formalnej* (Elements of Formal Logic), Księgarnia Akademicka, Poznań, 180 pp.

From the review by Mostowski, *JSL* 15, 65–66: "Scattered through the book are incidental remarks on ...the logical systems of Leśniewski."

WOJTASIEWICZ, OLGIERD

[1962] 'Towards a General Theory of Sign Systems I, II', *Studia Logica* 13, 81–101 and 21, 81–89. With Polish and Russian summaries.

Constructs a theory of language based on Mereology.

WOODGER, JOSEPH HENRY

[1931] 'Some Apparently Unavoidable Characteristics of Natural Scientific Theory', *Proceedings of the Aristotelian Society* 32, 95–120.

[1937] *The Axiomatic Method in Biology*, Cambridge University Press, 174 pp., with appendices by A. Tarski and W. F. Floyd.

On page 24 there is a diagram of the binary functors. The appendix by Tarski concerns Mereology.

[1939] 'The Technique of Theory Construction', *International Encyclopedia of Unified Science*, Vol. 2, No. 5, University of Chicago Press. Now also titled: *Foundations of the Unity of Science; toward an International Encyclopedia of Unified Science.*

The system developed in this paper has some similarities with Mereology.

[1952a] 'Science Without Properties', *The British Journal for the Philosophy of Science* 2, 193–216.

[1952b] 'From Biology to Mathematics', *The British Journal for the Philosophy of Science* 3, 1–21.

[1952c] *Biology and Language*, Cambridge, 1952.

[1960] 'Abstraction in Natural Science', *Logic, Methodology and the Philosophy of Science*, Proceedings of the 1960 International Congress, ed. Nagel, Suppes, and Tarski, 293–302.

WOODS, JOHN

[1973] 'Semantic Kinds', *Philosophia* 3, 117–152.

WOODS, JOHN and WALTON, DOUGLAS

[1977] 'Composition and Division', Studia Logica 36, 381–406

This paper uses a part-whole relationship as a theoretical basis for the fallacies of composition and division. They argue against using set theory for a reconstruction of the part-whole relationship and suggest 'something like mereology'. First Noll's work, as extended and refined by Suppes [1973] is considered, but this is rejected in favor of the part-whole relationship of Burge [1977].

YOES, M. G. JR.

[1967] 'Nominalism and Non-Atomic Systems', *Noûs* 1, 193–200.

Eberle [1968] and Schuldenfrei [1969] comment on this paper.

[1974] 'Intensional Logic and Ordinary Logic', *Noûs* 8, 165–177.

ZUBER, RYSZARD

[1973] *Logic and Semantics of Leśniewski*, Paris.

An elementary introduction for linguists.

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[1]The Index is prepared by "ALEPH - Editorial Services", Warsaw, Poland. It is only a practical device to this volume, i.e., it does not pretend to function as a thesaurus of Leśniewski's terminology — a work that would require a special research.

Comments in the square brackets are introduced by the indexer to provide the occurrence of the term in question with a context of related terms.

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