L. BIEBERBACH

Theory of Geometrical Construction

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BOOKS

TO 6f G E L E I T

For most mathematicians, dealing with geometric constructions was one of the most momentous mathematical experiences of their school days. For some, the unpredictable nature of the materials, seemingly not subject to any clear geometric methodology, was above all irritating, and they felt the reassurance of the so-called anal- ytical method as a salvation - which perhaps helped to determine a pronounced tendency in their later development. For other mathematicians, on the other hand, the joy of playing with ever new questions prevailed, and one may well assume that some geometric talents experienced their first attempts at achotistical thinking.

In this book, the Yerfaeeer expounds the whole wealth of the deeper questions of the theory of geometric constructions, in their interaction with mathematics as a whole. In the process, Boß's creative imagination has also come to the fore in his methodical penetration of the material. Thus the reader senses that this is a field of our science that is in a state of constant development.

The book thus created is probably Brzaee- azca's most mature and at the same time most youthful work. In the representation one recognizes again the temperament of the author and feels the unbreathed urge to force contact with the reader through the printed letter. The subject matter of the work is rooted in the thought processes of the School of Alexandria. We soon come across the name of Nsw'ron, who devoted more space to these matters in an *Arilhmetiea Unirerealio* than is generally known. The first achievement of the young student EnMOND **NDAQ** appears, the gcharfainnig considerations on transcendental numbers byGziarom and SIEOEß are reflected. But we also encounter some unknown names here for the first time.

Thus the work opens up to the student a view of the infinite variety of problems and methods of truly modern geometry; it gives the teacher countless opportunities to enrich and enliven the lesson - and that it has much to say to the researcher as well, no connoisseur of modern geometry needs to be assured.

November 1951.

§ 1. ¥Orders and declarations

The theory of geometric constructions teaches which construction tasks can be solved with given means of con- struction or which means of con- struction must be used to solve given construction tasks. A con- struction task is the task of finding points and lines on the drawing sheet from given points and lines using given aids. This involves drawings that are to be made on a sheet of paper with one or more pencils. It is assumed that these are ideal pencils, i.e. pencils that draw lines of zero thickness. The dots are also considered to have no extension. One of the most important ifonstruction tools is the ruler. It is used to draw straight lines with the pencil, which is guided along its straight edge. The presence of such a straight edge is the characteristic that defines the instrument. defines'). First, let us assume that the ruler has unlimited lengths and that the sheet of paper is unlimited. The problems that arise on the natural boundary of ruler and paper are put aside for the moment. But they will be dealt with very soon. This book will deal only occasionally with questions connected with the width of the line, and with questions of accuracy in drawing in general. Constructing in a limited plane with a limited ruler, on the other hand, is related to fundamental, purely mathematical questions. For this book, points, straight lines, circles and curves in general are the relevant entities of analytical geometry and ifon constructions are only means of expressing mathematical relationships. The task to be solved now consists of finding further points and lines from some given points and lines. A straight line is given by plotting. A point is given as the intersection of two lines (or any two parts of a line). The ruler is only to be used so that a straight edge is placed on two already existing, i.e. given or already constructed points and that this edge is drawn along with the pencil. New straight lines are created in the course of the construction

') Curved rulers are also known; these are rulers with curved accounts that can be used to draw curved lines on paper. Such instruments are not actually used for constructing. Rather, they are a drawing aid that makes it easier, to draw curved lines smoothly after some of their points - whether by construction or by avoidance - are already in place.

only in this way by applying the ruler to existing points. Likewise, new points are only created by intersecting existing lines. These are either given or newly constructed straight lines. We understand a *combination* with detn bittest *(alone)* to be the construction of points and lines according to this rule. Later we will look at other instruments that are closely related to the ruler, z. For example, the parallel ruler, which has two parallel straight edges, both of which can be used to draw straight lines, e.g. in such a way that one edge is placed at two existing points, while the other edge is drawn along with the J3lei. Such a construction i s then called a construction with the parallel ruler and not a construction with the ruler. It is important from the outset to clearly understand and record these definitions regarding the use of the instruments. For much misunderstanding and non-understanding, especially on the part of insufficiently educated mathematicians, who, as experience has shown, are noticeably interested in questions from the field of geometric constructions, is based precisely on the failure to understand the clear definitions and the resulting ambiguities and confusion. I repeat once again that a construction with the ruler is to be understood as the creation of new straight lines by applying the ruler to already existing points and drawing along this applied edge as well as the creation of new points as the intersection of such straight lines. Of course, we are only talking about the finite number of times the ruler is applied, i.e. constructions that end after a finite number of steps, and not about any boundary crossings from constructions that are thought to continue into infinity. But perhaps it is useful to mention this in particular.

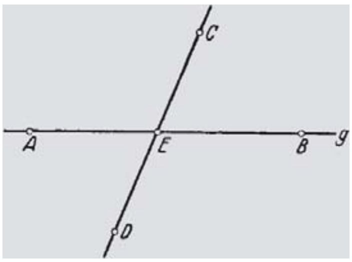
How the points and straight lines that we think of as given at the beginning of the construction got onto the paper is a question that remains completely out of consideration.

§ 2 Constructions with the ruler alone in unlimited length

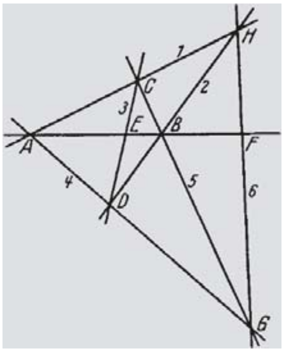
As already mentioned, we leave out all complications in this paragraph, which can result from the limitations of ruler and paper, except In other words, we assume that all the points and lines we are talking about fall on the paper and can be recorded with the ruler in a single line.

The assumptions made can also be formulated in such a way that the drawing sheet is the projective plane, the ruler is the projective straight line and that the points and straight lines are the objects of these names in projective geometry. Construction of a straight line then means the determination of a straight line through two points, and construction of a point means defining this point as the intersection of two straight lines. The purely mathematical questions of the theory of constructions are thus neatly separated from the questions of drawing technique. The same applies if the drawing sheet is defined as an affine plane, etc.

If only one point or only one straight line is g i v e n , no further point or no further straight line can be found constructively, as there are no pairs of points to which the ruler could be applied. If two points are given, the only thing that can be constructed is their connecting line.



Flg. 1



Flg. 2

if it is not also given. If two straight lines are given, then
If three points are given, their intersection can be constructed if it does not already belong to the given pieces. If three points are given, their triangle - or the line on which they all lie - can be constructed. Think analogously if three straight lines are given.

If four points are given, there are already more meaningful construction tasks. Let us take 4 points of general position'), *A, B, O, D* in Fig. 1, as given. Then, for example, *E* can be constructed as the8 intersection of the lines *AB* and *OD*. You then have three points *A, E, B* on a straight line y and two points U, *D* outside it. Now you can use the ruler alone to solve the do/paöe to find the *harmonic point* for *E* on *g* with respect to *A, B*. This is done, as is familiar from projective geometry, by drawing the straight lines numbered in Fig. 2 in the order of these numbers. N is then the fourth harmonic. This follows from the harmonic properties of the complete quadrilateral *ABCD,* in which 3 is a side and 6 is a diagonal. This cuts out the fourth harmonic to *E* on side *AB*. This is based on the theorem: On each side

') This does not mean that three of the four points should be located higher up (Fig. 1)

AB aB complete quadrilateral *ABCID* the two corners *A, B* al8 belonging to this side form a pair, the diagonal point *E* belonging to this side and the intersection with the opposite side of the diagonal triangleB £Gff as the other pair form a harmonicB quadruple')."

Dual to the task just discussed - in the sense of projective geometry - is the task of constructing aa tuftg for three straight lines a, *b, e*, the fourth harmonic / for e in relation to e, ö. In order to be able to draw, two further lines c and d, which do not belong to the bundle, must be given a. Of course, the task can also be traced back to the intersection of deB bundlea with c.

I conclude the paragraph with the proof of the theoremB: *The points and lines constructible* with the graph offin etie *given points and*
€feroden *eind exactly* theyenipen *whose coordinates in any given projecti "ng coordinatey8tRm cannot* be *expressed ratiTZHZl by the coordinates of the given* strokes.

I first prove that the coordinates of the points and lines that can be con- trued with the ruler alone can be expressed rationally by the coordinates of the given pieces. This follows from the fact that the coordinates of the lines connecting two points are expressed on the coordinates of two points defining the line. For the equation of these lines is obtained by setting the determinant to zero, which can be formed from the coordinates of the two given points and the coordinates of the current points as their three lines. Dual to this is EntBprechen- dea via the coordinates of the intersection pointsa of two straight lines°). Conversely, it can also be shown that *all points and lines whose coordinates can be rationally expressed by the coordinates of the* given *pieces are intersectable* with *the* one. *A coordinate system is to be* suprutide'*, in which* the *vertices of the coordinate triangle and the unit finct* str *belong to* the qepebeneti or Bereite £onetruierfea *points*. In the algebraic version, these are problems that only require the solution of linear equations. The constructions with the ruler alone are therefore also called linear Aonetru£tiotieti or A truHioaen erefen 6frndes. The point calculation°) must be used to prove this. There it is shown that the four basic types of arithmetic can be carried out by drawing with the ruler alone
This reference may suffice for the moment. After discussing the
Drawing parallelsa will shed further light on this question -). For the moment; lassen.
nothing more can be said than can be deduced from the reference to the above-mentioned passage°).

') Cf. E.g. L. Brzaznaaca: Projektive Geometrie, p. 78, Leipzig 1931.
•) Cf. B iaznzaca: Projective Geometry, p. 10 and ll. Leipzig 1931.
•) Cf. BißBERBACH: Introduction to Higher Geometry, p. 16, Leipzig 1933.
*) Cf. § 4, where I will prove the assertion just made.

§ 3 Ifongtruktionen in begrenxter Ebene mit begrenztem Iäneal

Konstruktion von Parallelen

I take up the construction of the fourth harmonic point just discussed with the help of the complete quadrilateral. If the four given points are located as *A, B, O, D* in Fig. 3, it is not possible to construct another point apart from point *E* if the ruler is used correctly, because the two other diagonal points of the complete quadrilateral *ABOD* fall out on the drawing sheet if they are interpreted as an affine plane or as part of it. J¥fan must then take at least one other point as given. From a **mathematical** point of view, it does not matter, which, since the harmonic to be found in relation to *A, B* is independent of which arbitrary auxiliary points or reed lines are used to find it. The choice of arbitrary auxiliary points can therefore be made entirely according to the needs of the draughtsman. Perhaps, as in Fig. 3, the added harmonic

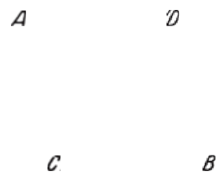


Fig. 3

point does not fall on the drawing sheet. But **then** it can be **required** to connect a point falling on the drawing sheet with this point not falling on it, i.e. to draw this connecting line at least as far as it falls on the sheet, in order to define the inaccessible fourth harmonic - as is already the c a s e with the two inaccessible diagonal points - by two straight line segments falling on the sheet. We are therefore faced with the following three basic tasks, which are posed by constructing in a limited plane:

1. To connect an accessible point, i.e. a point on the drawing sheet, with an inaccessible point, i.e. a point outside the sheet but defined by two accessible straight lines, i.e. to draw the accessible part of the straight line connecting the two points.

2. To intersect an inaccessible straight line, i.e. defined by two inaccessible points, with an accessible straight line, i.e. to specify a **further** accessible straight line section passing through the intersection point.

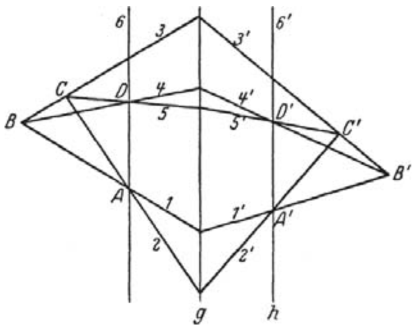
3. To intersect two inaccessible straight lines, i.e. to specify two accessible straight line segments by their point of intersection.

I emphasize some special cases, which result from the fact that the inauthentic (wrongly called infinitely remote) elements of the plane belong to the inaccessible ones in all cases.

a. two parallel (accessible) straight line segments. Draw a parallel to the two given parallels through an (accessible) point.

b., c. Given a parallelogram (and thus the two improper points a of one side and thus the improper straight line, defined by two improper points). A parallel (through a suggestible point) is to be drawn to any (b. suggestible or c. inaccessible) straight line *g*.

Ug. 4



Flg. 6

can be drawn. This last task is a special case of the second one above, if q is accessible, and a special case of the third one above, if p is inaccessible.

I turn to the solution of the tasks set.

1. q tik £ *define an inaccessible point U. The accessible point A can be* certified with II. I make use of the freedom to use arbitrary auxiliary points. I give three solutions to the problem.

The first G"'ttnp is based on the Desargue triangle theorem. Diesel states') : If two triangles are assigned to each other and lie in such a way that their sides intersect in three points of a straight line d, then the connecting lines of assigned vertices pass through a point *D* and vice versa. We choose the two triangles so that *U* is the point through which the connecting lines of associated corners pass. We assume in Fig. 4: *A,* learning *B* on p and C on £ arbitrarily. We also assume *B'* on *q* and C' on *h arbitrarily*. e and o' then intersect at a point of d. If we now assume b' arbitrarily, we have a second point of d at the intersection of *b* and ö' and thus the straight line d. If we intersect *d* with c, we still have a **point** of c' apart from *B'* and can draw c'. c' and b' intersect in *A'*. The straight line *AA'* goes through *U,* which solves the problem. (Fig. 4.)

') Cf. E.g. L. BIEBERBaca: Projektire Geometrie, p. 43, Leipzig 1931.

A second dsutip of the task results from the Zaeofittiomaaiz *about* voifstöWiife *quadrilaterals*j.* It reads: The pairs of opposite sides of a complete quadrilateral intersect any straight line in three pairs of points of an invul-lution. If five of these six points are known, the sixth is fixed. Fig. 5 shows how this can be used to solve our problem by constructing two complete quadrilaterals, in which the corresponding sides of both quadrilaterals have the same numbers, which also indicate the order in which the quadrilateral to which point *A* belongs as a corner,

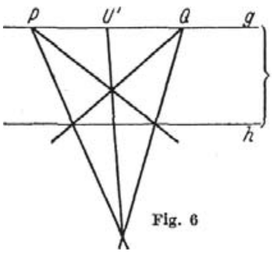


Fig. 6

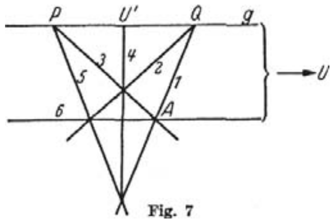


Fig. 7

is to be drawn. A brief explanation is added to Fig. 5. Two points are known on A, which may be labeled *A'* and fi'. Connect *A'* with a point given on q by the straight line 1' and draw the corresponding straight line 1 through *A* so that 1 and 1' meet on y. Similarly, draw 2 and 2'. Then take *B'* and C' arbitrarily on these lines, draw 3' and place any line generating *B* and U through its intersection with y. Then draw 4' and 5'. Then draw 4' and 5' by connecting *D'* with *B'* and C', intersect these lines with *g* and connect the intersection points with & and C by the lines 4 and 5, which intersect in *D.* This gives you 6 as straight line *AD* and 6' as straight line *A'D',* the latter as a given straight line. Both meet at q and are parallel if *q* and £ are assumed to be parallel.

A *third fermentation* of the same problem is obtained from the harmonic properties of the complete quadrilateral. First obtain a harmonic quadruple on y in which *U* is involved. *PQ* old a pair, U, *U'* as the second pair in Fig. 6 is such a quadruple.

Now try **to construct** the **fourth** harmonic **point** V in relation to *P, Q* using *A* in relation to *U'*. **To do this,** draw the straight lines I, 2, 3, 4, Ii, 6 of sig. 7 in the order of this numbering. If *U is* the improper point, i. e. if p and *h* are parallel, then *U'* is the midpoint of the line *PQ* on p. We have then solved the problem:

') Cf. s. B. L. BiEazRBzcif: Projektive Geometrie, p. 81 and 97. Leipzig **1931**.

"Given a line *PQ* on *q* with its equidistant point ff. Draw a parallel to *q* through a point *A"*. Since we previously solved the problem: "Given q and a parallel to it: Ä. Construct a line with midpoint on *q*", *it* does *not matter whether a parallel to g or a line with midpoint on 9 is given.*

The following *remark is* added to this: *Only if there is any rationally divided line on g can parallels be drawn zv g with the ruler alone.* Let the line *AB* on *g* be divided by the point U on it in the ratio m: n (m, tt integer, m > ii). In addition, two points *P, Q* outside *q* are given so that you can continue constructing. Now first determine the point *D* on *q co* with the help of *P* and Q so that *AB* as one pair and *CAD* as the other pair form a harmonic quadruple. This can be done with the help of a quadrilateral. If, as we may assume, *B lies* to the right of *A,* then *D* lies to the right of *B*. Let us see in what ratio the distance *AD* is divided by *B*. To do this, we introduce Cartesian coordinates on *g*. We may choose them so that *üaD A* has the coordinate *0, C!* the coordinate m, *B* the coordinate m -}- n. Let z be the coordinate of *D*. Then, according to the definition of *D*, the double ratio is *[A, B,- C, D)* -- - l, i.e.

s - in *n j* n

By subtracting 1 on both sides, the required partial ratio follows from this

$$\frac{m \quad a}{tm -j- n \quad n} \_ \frac{m - n z -}{}$$

From a distance divided in the ratio m: a, a distance divided in the ratio (m - a) : a was constructed. If m - n > n, we proceed analogously to (m - 2 n): tt, then to (m - 3 n) : n and so on, bi8 we find a number m - £n

n has been obtained. If m - *k n < n, co* the same procedure is applied to the inversely directed distance divided in the ratio a : (m - £a). In this way, you can obviously move on to smaller and smaller positive numbers that express the division ratio, until you finally obtain a divided in the ratio 1: I,
i.e. halved distance. This ensures that the passages are drawn to p.

2. I now turn to the second task. Consider an unconstrained *line u. A point U of this line, defined by an accessible straight line g, is to be connected* to an *accessible point A*. We obtain a harmonic quadruple on *g* in which *U* is involved. In Fig. 8, two straight lines define two inaccessible points f/ and f/" which in turn determine 'ti. *g* determines the point II on tt. Plan connects ff of Fig. 8 with *U*. The

defines a point *U′* on q. Then *P, Q* as one pair and *U, U′* of Fig. 8 as the other pair form a harmonic quadruple.

Once you have a harmonic quadruple on q, you can continue as in the third solution of problem I. Because even then8 a harmonic quadruple was constructed on p first and the (this time inapplicable) straight line fi was no longer used from then on.

Although problem 2 is fundamentally solved in this way, it is (graphically) easier to base the design on a consideration using Deaarguess' theorem. Let us assume that the shapes shown in Fig. 9 are strongly

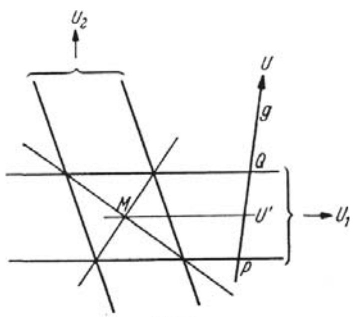


Fig. 8

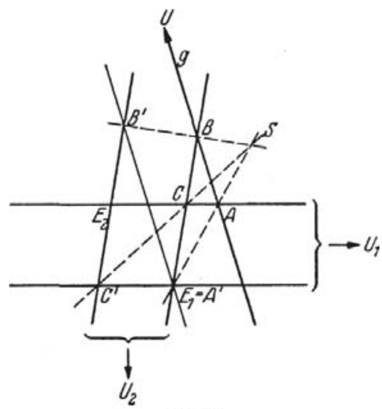


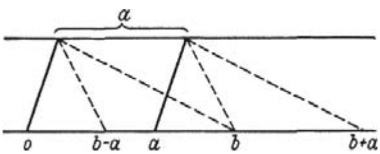
Fig. 9

The straight lines defining H_1 llnd *U* and the straight line *g* defining *U*.
This also gives the points labeled *A, B, C!, A′,* C'. We are looking for a suitable other triangle in Desargues' position for the triangle *ABC* in Fig. 9. To do this, we draw the straight lines CG' and *EFA,* which intersect at a point 'S'. Connect this with *B* and intersect the line *ISB* with *C!′E$* in a point *B′.* The triangle *E B′C!′ (E* = d') is then in perspective from S in this designation of the corners to *ABC!* Therefore, assigned sides intersect on a straight line, which can only be *U U$*. Thus *A′B′* intersects the straight line q in *U.* I t i s possible that individual points mentioned here, such as S or *B′,* are inaccessible. However, since we already know how to connect accessible points with inaccessible ones, this does not change the feasibility of the construction described in Fig. 9.

3. Finally, the 3rd problem remains. To solve it, it is necessary to connect the intersection point of two inaccessible lines twice with accessible points in order to obtain two accessible lines through the intersection point of the two inaccessible lines. This does not require a new

Consideration necessary. One only has to assume that the straight line p of Fig. 9 is itself inaccessible.

We are now also in a position to deal more precisely with the question settled by a hint at the end of § 2, namely to prove that it is possible to construct points and straight lines whose coordinates can be expressed rationally by the coordinates of the given pieces. A projective coordinate system, which was in mind at that time, is determined by four given or already constructed points of general position, namely the three corners of the coordinate triangle and the unit point. These four points can be converted by projective mapping into the four points of determination of a



Flg. 10

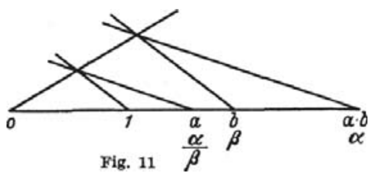


Fig. 11

any Cartesian coordinate system. If we think of the abscissas of all given points marked on the z-axis and their ordinates marked on the y-axis, we can transfer the distances of one coordinate axis congruently to the other by drawing parallels and thus determine that the abscissas and the ordinates of all congruent points form two congruent sets of points. It is now easy to see that all points whose coordinates can be expressed rationally from the abscissas and ordinates of the given points can be constructed with the ruler. One only has to remember the usual methods of graphical arithmetic (the projective generalization of which is the point calculation mentioned in § 2)*). Figs. 10 and 11 illustrate this.

We now assume again unlimited plane and unlimited ruler. The discussion in this paragraph gives us the right to do so.

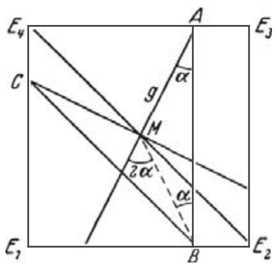
§ 4. constructions with the ruler alone according to specifications of a completely drawn regular polygon

New peculiarities arise if one not only assumes that the two pointsp and *U$* are not real, but if one further assumes that the parassel formed by the pairs of straight lines through iu$_l$ find 17 is not real.

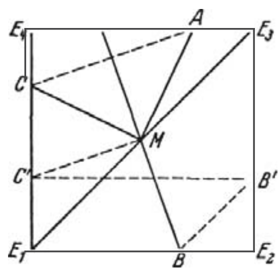
') Vg1. e.g. **Bizazszxca: Einleitung** in **die höhere Geometrie,** s. isff. **Leipzig** 1933, **where the proofs** are **also** carried out independently **of the** parallel axiom - **purely** projectively - as is **emphasized with** regard to **an** application **to be made** in § 5.

gram is a square. Then the tasks of falling a perpendicular, erecting a perpendicular, bisecting a right angle, rotating a line by u/2 etc. can be solved with the ruler alone.

Since you can draw parallels to any straight line with the ruler alone when given a parasselogram, it is only necessary to construct a perpendicular line through the center of the square to a straight line through the center of the square to solve the tasks of plumb line felling and plumb line erection. This is shown in Pig. l2. You can find a perpendicular to the line p by drawing *AB* parallel to the side of the square *E$Eq*. Then draw *BC!*



Flg. l2



E'lg. 13

parallel to the diagonal *E fIi!* . Then Uöf is perpendicular to p. This follows from the fact that the triangles *ABM* and *BC!M* are isosceles, both with the apex df. If the base angle at fi &3f is equal to et, then the base angle at *BOM* is equal to u/4 - e. The angle at the apex of this triangle is therefore uJ2 -l- 2 e. Since, according to Fig. 12, the exterior angle at the apex ff of triangle *ABM* is equal to 2 ot, *g* and U3f are perpendicular to each other. In addition, *MA* and 3fC are the same length, which also solves the problem of rotating a line by u/2. In Fig. 13, the right angle fi3fC is halved by placing a parallel *MC!'* to dC through ff and drawing *C!'B'* parallel to fi,fi and *BB'* parallel to fi S. *BM* is then perpendicular to *AC* and bisects the right angle Aufl. This is because triangle *AIIIC* is isosceles and 6 "M is perpendicular to *B f,* as we discovered when looking at Fig. 12.

Fig. 14 finally shows how to double the angle e of the two half-lines a and 6. Draw c perpendicular to e and construct the fourth harmonic d to *b* in relation to the pair n, c. The straight lines a and c are then the two bisectors of the angle formed by the straight lines *b* and *d. The* construction of the fourth harmonic straight line in the bundle of straight lines a, *b, c, d* is dual to the construction of the fourth harmonic point on the row of points shown in Fig. 2. For ease of comparison, the points to be used in Fig. 14 are numbered in the same way as the straight lines used in Fig. 2.

The assumption that a square is given not only has the effect that you can construct the perpendiculars to every straight line, but also that you c a n transfer a distance congruently to the perpendicular line, or in other words, that you can transfer every distance by uJ2

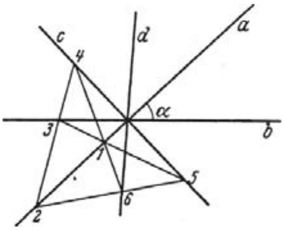


Fig. 14

can rotate. The same effect o f constructing the perpendiculars to each straight line would obviously also be achieved if two **pairs of** perpendicular straight lines were taken as given. For this defines the *perpendicularity,* and on the basis of this the perpendiculars to each line can then be constructed with the ruler alone. *However, this does* not *yet enable* you to *construct* a square. To make this clear, let us assume that, for e x a m p l e , a Rhombua with an angle **of 60° is given. This gives** you a parallelogram with two diagonals perpendicular **to each other.** By drawing parallel lines, you can construct a regular rectangle from it (Fig. l6). You can immediately recognize further rhombuses and

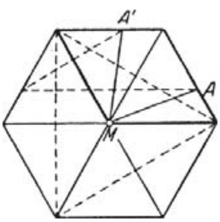


Fig. 15

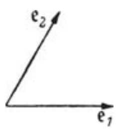


Fig. 16

thus further **vertical** pairs of straight lines. Nevertheless, it is not possible to construct a square. To see this, we first get an overview of **all** the points that can be constructed with the ruler according to a rhombus. Ala coordinate system, we define the (oblique-angled) Cartesian circle determined by the rhombus. Its both unit vectors are the ones originating from an eoke and falling on the rhombua- aeiten(Fig.16).The constructible points then have as location vectors

se,+ye, (s,yrationzl 1)

If you introduce a rectangular **coordinate system** with the equi-oriented unit vectors $B_l = e_l$ and R, then

$$e_1 = \mathfrak{E}_1, \quad e_2 = \frac{1}{2}\mathfrak{E}_1 + \frac{1}{2}\sqrt{3}\,\mathfrak{E}_2, \tag{2}$$

und es werden demnach

$$\left(x + \frac{y}{2}\right)\mathfrak{E}_1 + \frac{1}{2}\sqrt{3}\,y\,\mathfrak{E}_2 \tag{*}$$

the position vectors of the constructible points. If a square is now con-would **be atruable, then** aa and its diagonal zo-ei under u/4 correspond to position vectors inclined to each other, the length ratios is. If (z, y) = (a, *b j, [z,* y) - (c, d) are these two location vectors, the squares of their lengths are therefore given by (3).

c* -I- c d -I- *d** -- 2 *(a'* -|- *a b* -j- *b'j,* *(a,b) -t--* (0,0), (c, d) -j- (0,0) . (4)

Here e, b, c, d are rational numbers. If we reduce to common denominators and multiply by theaem, we see that a relationship (4) should also exist for rational integers a, *b, c, d.* However, theB is impossible. Otherwise, because of (4), we would have to be able to choose the integers z, y BO in any case, so that the highest pole of 2, by which

$$x^2 + x\,y + y^2 \tag{5}$$

divisible, would be an odd power. But if (5) is to be an even number a, ao obviously z and y both have to be even a. Then 2' is the highest power of 2, which is the greatest common divisor of z and y, and iet

$$x = 2^\nu x_1, \quad y = 2^\nu y_1, \tag{6}$$

this is how

$$x^2 + x\,y + y^2 = 2^{2\nu}(x_1^2 + x_1\,y_1 + y_1^2). \tag{7}$$

But since z'' y, cannot both b e even a, then

$$x_1^2 + x_1\,y_1 + y_1^2 \tag{8}$$

is an odd number, and therefore 2'' is the highest power of 2, which goes into (5). Therefore, the highest power of 2, which is included in (5), can never have an odd exponent. *Therefore, tel* with *the Nneal alone* when *given an 8eGhziggradige Rhombus or, what comes out ati| the same, a re- gular lSecUecko* keizt *square constructible*).*

For greater clarity, I would like to add the remark that the angles of the simultaneous triangles around ff in Fig. 15 can all be halved at ff with the ruler, since the perpendiculars can be erected on the diagonals of the hexagon of Fig. 15. Nevertheless, it was not possible to construct the regular dodecagon because otherwise it would obviously be possible to construct the square, which has four corners in common with the dodecagon.

We have thus recognized that the form of a *square g o e s further than the specification of two pairs of* crerades *that are perpendicular to each other.* In passing, we can also note that the use of aa fiec£tii'izi£eliiaeele instead of

') By the way, you can also prove in the same way that you cannot construct a regular hexagon with a ruler given a square.

of a ruler does not go further than the specification of two pairs of perpendicular straight lines next to the ruler. A perpendicular ruler is understood to be an instrument consisting of two mutually perpendicular ordinary rulers in rigid connection. It is used in such a way that o n e edge is applied to two existing points, while the perpendicular edge passes through an existing point. Drawing can be carried out along both edges. *This instrument only abbreviates the construction of the line perpendicular* to a *given line.* For its range in extended use, see § 9.

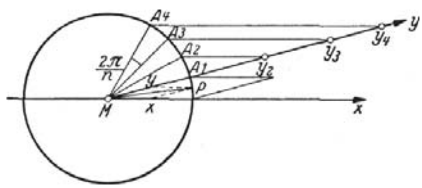
In Fig. 15, dotted lines indicate how the distance *MA* (and thus any distance) can be rotated by u/3 in *MA'* by drawing bIoßea parallel to the sides and diagonals of the rhombuses.

The considerations that we have made here for the regular hexagon and the regular quadrilateral (square) suggest the question of whether Analoger also applies to *any* "reifufor" tt-itself. This is indeed the case. If the remaining connecting lines of the corners are drawn in such a regular polygon, you will find enough parallels to the sides to be able to draw lines to any straight line with the ruler alone according to § 3. According to § 3, the sides can also be bisected with the ruler alone. **If you** then **draw** the lines connecting the side midpoints with the side midpoints or corners, you will also find e n o u g h perpendiculars on the sides to be a b l e to construct the perpendiculars to any straight line with the r u l e r alone, in accordance with a statement made in this diagram following the square. Plan can thus also construct the center of the regular tt-corner. If, in the case of an *odd n,* you connect thea center point 3f with a corner of the tt-cornera , ao you can extend this distance beyond ff according to § 3. This gives a corner of the regular 2 n-corner. In the same way, you can find the other corners of the n-corner in addition to the existing ones. Bank If *aIso, in the case of an odd n, construct* the 2 *n-Ii!ck from the* ii-Tel *using the ruler*. In the c a s e o f an even tt you can rotate any distance by 2 a/a in the same way as in the case of the hexagon (Fig. 15) by drawing parallel lines several times. This is therefore also possible in the case of an odd n. First go to the regular polygon with twice the number of corners 2 n and then rotate twice by 2 a/(2ti). Now let us answer the question suggested by the result highlighted above as to whether, in the case of an even n, it is also possible to construct a regular 2 n-corner with the ruler alone a f t e r entering a regular n-corner.

*) On a first reading, the reader can **skip** the rest of this paragraph.

can atrude. It will be proven that this is impossible. In the *case* ***of an*** *even n* ***Bonn with 'le ruler alone*** *from the* **regular** *n-fiick* **the** *regular* 2 *n-corner* **cannot** *be constructed.*

In order to prove this, we first obtain an overview of all the points that can be congruent with the ruler alone according to the specification of a regular (tt 2 m) corner. For this purpose we choose a Cartesian coordinate system whose axes include the angle 2 a/a and whose unit distances are equal to the radii of the regular n-corner (sig. 17). *Constructible are then 9enav those points whose coordinates are rational functions of* coa a/m with rafionnfeti *numerical coefficients.* This is shown in Fig. 17, from which the y-coordinates of the corners of the regular tt-corner are taken. (The *z-koozdi*



nates obviously make up the same quantity in their entirety as the y-coordinates in their entirety). From the triangle *MA y$* you have

$$y_2 = \frac{\sin 2\frac{\pi}{m}}{\sin\frac{\pi}{m}} = 2\cos\frac{\pi}{m}.$$

On the triangle *measure yq* follows

$$y_3 = \frac{\sin 3\frac{\pi}{m}}{\sin\frac{\pi}{m}} = \frac{\sin 2\frac{\pi}{m}\cos\frac{\pi}{m} + \cos 2\frac{\pi}{m}\sin\frac{\pi}{m}}{\sin\frac{\pi}{m}}$$

2 oos^ $\frac{}{m}$ -|- cos 2 $\frac{}{ot}$ t cos* $\frac{}{m}$ - I

etc. You can see that the y-coordinates of the corners are given by the g $\frac{\text{ain } k \frac{n}{}}{\text{i}}$

with integer *k* are given. Since these are rational with rational

If we express the numerical coefficients by cot u/m, we see that, given the rhombus shown in Fig. 17 and the points z = 0, y = cot u/in, we can construct the regular (n = 2 m)-vertex with the ruler alone. The specification of the regular (a - 2 in) corner is therefore synonymous with the specification of this rhombus and these points. This also proves our assertion that the points that can be congruent with the ruler alone after specifying the regular (n = 2 m) corner are precisely those whose coordinates can be expressed rationally with rational number coefficients by cos a/m. Fig. 17 also shows a corner P of a regular 2 w-vertex to which the corners of the given regular n-vertex also belong. From the triangle *MPI* it can be seen that its coordinates

*' 9 ain 2 - 2 coa - $S \cos \frac{n}{2m}$

aind. If it is possible to construct a regular 2 'i-corner with the ruler alone, ao also the one with the center ff to which this corner *P* belongs. For if the 2 n-corner is constructible, then one can rotate any distance, thus also *measure,* by 2 a/(2 a), and thus one obtains P. If now the 2 a-corner were con- atruable, then cos a/(2 m) would have to be rationally expressible by cos a/m with rational number coefficients. I will prove that this is not possible. To do so, we must first remember certain facts on the doctrine of the division of circles°)

must be remembered. The 2 m-th root of unity e '^ satisfies an irreducible equation of degree Q(2 m) in the body of rational numbers (circular division equation). Here ' p(2 m) is the Euler 'p-function, i.e. Q(2 m)
$= 2m\left(1 - \frac{1}{2}\right)$$1 $\frac{1}{p_1}$) - QI - , if 2 m = 2'p1' p°- the prime factor-

') Stan proves daa by complete induction from the formulas

$$\frac{\sin k \frac{\pi}{m}}{n} - \frac{\sin (k-1) \frac{\pi}{m}}{n} - \cos \frac{\pi}{m} + \cos (k-1) \frac{\pi}{m},$$

$$\cos \nu \frac{\pi}{m} = \cos (\nu - 1) \frac{\pi}{m} \text{ coa } \frac{n}{-} - \text{ ain (r - 1) } \quad \text{ ain -}$$

$$= \cos (\nu - 1) \frac{\pi}{m} \cos \frac{\pi}{m} - \frac{\sin (\nu - 1) \frac{\pi}{m}}{\cdot} \left(1 - \cos^2 \frac{\pi}{}\right).$$

°) ¥g1. e.g.: L. Brzazsazca-Bzrzs: Lectures on Algebra, 5th ed. Leipzig 1933.

of 2 m ifl. Accordingly, ifl g(2 m) (megen m> 1) sets even and for even m in particular g(2m) = 2 g(m) The Kreifleülgs

equation has, in addition to e '^, certain other unit roots e - - (£ integer) as solutions, and with each root also its conjugate imaginary, or what is the same iat, its reciprocal value. Therefore, the equation of the division of the circle is a so-called reciprocal equation. So if

tZ0 Z " " -[- O Z "" ' -[- - - - - --[- O (g q == 0

is the Kreia division equation, then a - ag tp, _ . If we now divide by

z and then i n t r o d u c e s $z -J- as a new unknown, we get

an equation of degree 'p (2 m)/2 whose roots are cot u/m and certain coa *k* u/m (*k* integers). (The same £ as before for the roots of the Kreig division equation.) Therefore, this equation is also irreducible in the body of rational numbers. For from a divisor of its left-hand side provided with rational coefficients, one would obtain an equation of lower ala g (a)-th degreea with rational coefficients by the specified substitution

for e - . coa u/(2 In), i.e. the quantity to be constructed from an irreducible equation of degree 'p(4 m)/2 g(2 m) with rational coefficients, is sufficient. However, if cos a/(2 m) could be expressed rationally by coa u/m with rational coefficients, then, for example

$$\cos\frac{\pi}{2m} = r\left(\cos\frac{\pi}{m}\right)$$

this rational function, e n t e r here for cos u/m the remaining roots cot £ a/tu of the irreducible equation, which satisfies cos a/m, and form

$$s\text{-}r \quad \left(\cos k\frac{n}{m}\right) \quad \text{-}\,0\,.$$

Then, according to the theorem that symmetric functions') can be expressed rationally with rational number coefficients by the elementary symmetric functions, this is an entire rational function of z of degree Q(2m)/2 with rational coefficients, one of whose zeros is cos a/(2 In). However, this contradicts the fact that cos a/(2 m) satisfies an irreducible equation of degree g (2 tu) with rational coefficients. Therefore coa a/(2 m) cannot be expressed rationally with rational number coefficients by cot a/m. Therefore, the regular 2 n-square cannot be constructed with the ruler from the regular (n - 2 m)-square alone.

') Cf. B.: Bxzazsazca Bzrzs: Lectures on Algebra, 5th ed. Leipzig 1933.

§ 6 Constructions with the ruler alone according to a firmly drawn circle with a given center. Poncelet stone constructions

If a circle periphery *K* is presented as a fixed line and the center point ff of the circle is marked, a straight line can only be drawn according to the rules given in § 1 and the considerations made in § 2 if another **point** *P* is given. The intersection of a straight line with the fixed circle is now regarded as a new process for constructing new points. The straight line *M P* intersects the circle *K* at two points. If you want to construct further lines and points, at least one point Q not located on the line *MP* must be given. According to § 3, you can then draw a parallel to *MP* through this point Q. This is because a line with center ff is known on the diameter that *MP* cuts out of *K* (§ 3). If the parallel to *M P* passes through Q at the circle *K*, connect 3f with Q and draw a parallel to *MP* through one of the intersection points of 3fQ with *K*. You can therefore construct a chord of the circle *K* parallel to *MP*. If we bisect this chord according to § 3 and connect its center with ff, we obtain the diameter of the circle perpendicular to *M P* . The case where 0fQ is already *perpendicular* to *M P* appears to be a limiting case in this construction, but it can easily be avoided by modifying the method. These two mutually perpendicular diameters determine the corners of a square whose sides are equal to the radius of the circle. According to § 4, all points can be constructed whose coordinates can be expressed rationally from the coordinates of the given points in a rectangular coordinate system whose axes fall on the two mutually perpendicular diameters of the circle and whose unit distances are equal to the radius of the circle A. Let 1 be the dimension of the radius of *K.* Let a be the dimension of the abscissa of any given or already constructed point. Divide a diameter of *K* in the ratio a : 1. This can be done in the usual way if the points on the abscissa axis are marked with the measures a and a -}- 1 (Fig. 18). It may be assumed that the diameter *AB* falling on the abscissa axis has been divided in the ratio a : 1. Now let the diameter *A B* of *K* at point N be divided in the ratio a : I. Set up the perpendicular to *AB* in C. This is possible according to § 4, since, as we have just seen, a square inscribed on the circle *K* is known. An intersection of this

perpendicular with *K* at *D.* Then, according to the height theorem, 2 JÄ/ (a + I) is the measure of the distance *CD.* Draw a triangle ff, a -}- 1, *D',* ao 'similar to *ABD*' by drawing parallel lines over the points ff, a, *a* -j- 1 of the abscissa axis.

its height *a D'* established at point a has the dimension (Fig. 19). Transfer it to the y-axis through a parallel to the z-axis and rotate aie according to § 4 by u/2 into those. Then you can see t h a t the points whose coordinates can be rationally derived from the coordinates of the given points and from]/Ä. However, since a was arbitrarily chosen from the coordinates of known and constructed points', the following result was obtained:

If a *circle K is drawn with a center point, all the points* wiil dern Direai 'iifeia *constructible, whose coordinates in a given*

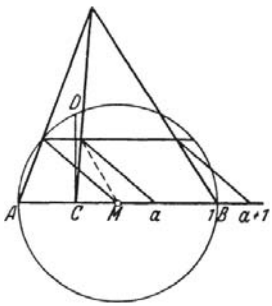
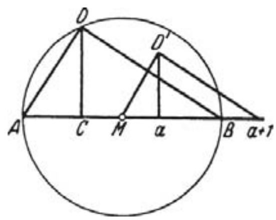


Fig. 18



Plg. 18

The equation of aoordinofeneysfeml) *e i c h are* the *coordinates of the given points by a quadrature root expression, i.e. by finally* o/mutual *application of the four orundreohnunqsarlen addition, lsublraklion, 3fultiplikalion, division and dee process of quadralunsrzelziehen. It is assumed that the* given *points* tiicAt *all lie on* theefheti *through-* mieser *deo* /eefen *circle. Should this* be the *case*, eo *no* treifer *point can be constructed with the ruler alone.*

Such constructions with the ruler alone according to the specification of a firmly drawn circle with a given center are called *Poncetel's constructions*.

The words "endlich oftmalig" occurring here correspond to the porde- (§1) that every construction comes to an end after a finite number of applications of the ruler, i.e. that it consists of a finite number of straight lines and a finite number of intersections of these with each other and with the circle A. *It is* worth *mentioning that the specification of an arbitrary* arc of *the circle K is sufficient.* It therefore does not have to be completely drawn in order to construct its intersection points with any straight line q with the ruler alone.

') In the proof, it was assumed that the unit distances of the coordinate eyetems were equal to the radians of A. However, the sats is obviously valid for any given or conatrued rectangular coordinate system, since the tran8formation from one such system to another is done by linear transformation with given or constructed coefficients.

to be able to. The following simple proof was found by **P. KÜTTs** msu. If an arc of a circle ö is given, ao it is possible, as first noted aei, to find the tangent in each aone of the points on the basis of Pascal's Conic section theorema with the ruler alone. This is shown in Fig. 20. Assume 5 points on the arc *b*. The tangent is to be found at point 1, which also has the number 2. The intersection points of the hexagon sides 23 with 56 and 34 with 61 determine the Pascal's straight line of the hexagon. Connect its intersection with hexagon 45 with 1. According to PzSGzc, this is the tangent in 1.

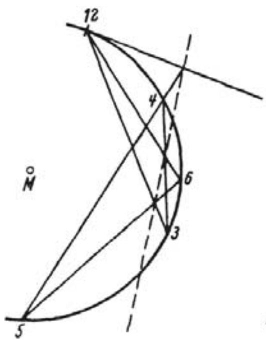


Fig. 20

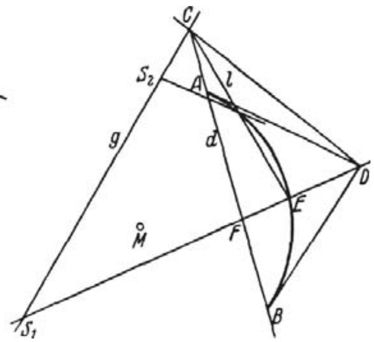


Fig. 21

You may assume that the arc *b* is not larger than a semicircle. Otherwise, connect a an end point with the center point. The intersections of these two lines with *b* determine a partial arc of h that is smaller than a semicircle.

If the points of intersection of *g* with the circle *K* (of which *b* is a partial arc) are to be determined, intersect the straight line d connecting the two end points *A* and *B* of *b with q* at a point D (Fig. 21). The tangents to *b* in *A* and *B* may intersect at a point *D* - the pole of the straight line d. The lines *AD* and *BD* may, as we want to assume, not meet q. Otherwise, as a glance at Fig. 21 shows, the question of the intersection points of *K* and *g* would not need to be discussed, as they would either - in part') - fall on t' or p would not meet the circle *K.* We now take the lines d and *CAD* as a pair of aa harmonic quadruple and construct the fourth harmonic line *1* through U. If A is one of its intersection points with ö, then the line *DE*

') If only one of the intersection points of p with the circle falls on tt, the other intersection point of p with the circle is determined in the usual way on the basis of the Pa8cal theorema with the ruler alone.

the straight line p in a point 6 of *K*. For if F is still the intersection of *DE* with d, then *(D, II',- IS,* fi) form a harmonic quadruple. If f does not meet the arc *b,* then *g* also passes *K.*

The fact that the center point is not used to find the intersection points of a straight line with the circle, even if only one arc of the circle is drawn, could lead to the assumption that one can *do without the indication of the* center point, that one can rather construct it from other given points. This is, of course, correct if there is sufficient information other than the periphery of the circle. If, for example, a parallelogram is given in addition to the periphery of the circle, then the center can of course be constructed. If, however, apart from the periphery of the circle, there are only points about which no affine or metric information is given, or, in other words, about which no relationship to the improper straight line is given - we will call them arbitrary points - then the center point cannot be constructed, i.e. distances (of more than one direction) cannot be bisected, since the center point could also be constructed. As soon as this has been proven, it is clear that in the *above-mentioned case* one cannot *dispense with the assumption that the center point of K is also given.* Let us therefore assume *that* apart from the periphery of *K there* are only a number of arbitrary points and that there is a construction of straight lines which leads to the center point of *K.* Then any construction that can be obtained from this construction by a projective mapping that sweeps *K* should also lead to the center. Let us assume a rectangular coordinate system such that the equation of *K*

$$(a - x)^2 + y^2 + 1 - a^2 = 0, \quad |a| > 1$$

becomes. The projective mapping

$$x = \frac{1}{x'}, \quad y = \frac{y'}{x'} \tag{1}$$

m e r g e s *K* into itself, but does not leave the center point alone, but m e r g e s it, i.e. the point (z, y) = (a, 0), into (z', y') = , 0$').

With D. CA fER we want to pursue these questions a little further. Let us assume that there are zicei that *are not* (in the real sense of the word) *so-so, not concen-*

*) It is very well known in projective geometry that there are collineations which featlaßa *E,* but which change the center of *K.* Think of the alternating intersections of a ßo-deep circle or of Klein's model of hyperbolic non-euclidean geometry. The proof of the test was handed down by D. Cxx:fin in an Ililbert lecture.

Ifreiee x2 and *K$ are given. Even then* you *can* tiic£t construct *the centers of these circles* from other arbitrary points, i.e., as already mentioned, without any relation to the improper straight line. For be

in a suitable coordinate system

$$(z - b)^\wedge \text{-j-} y^\wedge \text{-j-} 1 - b^* \text{--} 0 \qquad b > I$$

the equations of the two circles intersecting at z = 0, y - i, then again the projective mapping(1) merges both circles, but leaves neither center point alone.

Astrid but given to'ei concentric *circles, the* 3fiuel point *can be* con8truedf2. Since the polars of a given point are parallel with respect to two concentric circles, parallels can be drawn with the ruler alone, and therefore distances can be halved according to § 3. Since the tangents are the polars of the peri- pherie points, Fig. 22 shows another way which only fails if the two circles are the inscribed circle and the inscribed circle of the same equilateral triangle.

g

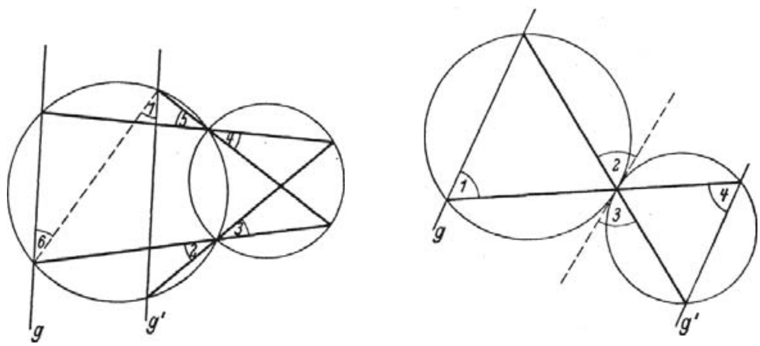
Fig. 22

If two' *circles* are bounded by *real intersection points*, their center points can also be constructed with the ruler alone. Fig. 23 shows how a parallel *g'* can be constructed to a straight line *g*. Fig. 24 shows a parallel construction for the case of two circles touching each other.

In the last two cases, eB does not c o n t a i n any real collineations of the two circles apart from reflections (or rotations) that fix the center points. This is because an ifollineation of this kind has to deform the four intersection points of the two circles. The collineation used in the case of four non-concentric circles with only imaginary intersection points exchanges the two non-real circle points with the two real intersection points of the two circles.

The proof that parallels can be drawn to many straight lines by the given specification of drawn circles teaches that the centers of the Ifreise are determined. I add the *remark that* it *is sufficient if any arc of one circle* is *drawn* and *three other points are given above the two real 5ohnill points* (or the point of contact and its tangent). This is because then the

The other intersection points required for the specified construction can be found on the basis of Pascal's theorem using the ruler alone. However, you do not yet need the arc of one circle. But this must be t a k e n as given in order t o be sure that all the constructions corresponding to the proof given at the beginning of the paragraph are feasible. I turn to the case where *three of mine* are *given without real intersection points*. In addition, we have to assume that the three circles



are linearly independent, i.e. do not belong to the same tuft. For otherwise the collineation indicated on p. 21, which transforms two of the circles individually into 8itself, but changes their centers, would also featlaase the third circle by changing a center point. It is also sufficient to assume that there is an arc of one circle, but that there a r e 5 points of each of the other two circles. In this case, 5 points of this circle are also given, since, as has already been explained, the intersection points of the same circle with any given or achon constructed straight line can be found without using the center pointsa. Then the equations of the three circles are known in the projective coordinate system determined by four linearly independent points, since only linear equations have to be solved to determine their coefficients. It will also have to be assumed that all three equations have two conjugate imaginary solutions in common (as this corresponds to the assumption that they are circles). Algthen determine in the bundle of two of the three circles that circle which p a s s e s through an arbitrarily assumed point *A* of the third circle. Let *K$, Kg* be the three given circles, assume *A* on *K* and place the circle Kg_3 of the bundles *K$, Kg* through it. Then *K* and *K q* have, in addition to *A*, a real intersection point *B,* the determination of which again only requires the solution of linear equations. Because of the intersection points of both circles known here the two conjugate imaginary circles common to all three circles and the point *A*. For example, *K is* also the circle from which an arc is drawn. From *Kg,* " which meets *K* at two now known points *A* and *B*, you only know these two points of intersection with R and its equation. However, you can immediately obtain further points of $Rq_{,3}$ by placing any straight line not meeting *B* through *A* and determining its second point of intersection 6 with *Kg*. This again only requires solving linear equations. This brings the question back to one that has already been solved. *Also in 1i'all three linearly independent circles without real lintersection points 8J CO dkP fitting points |eotgele9I and by purely linear construction*

In one of his last works, E.A. WExsg dealt with the question of the extent to which the use of the ruler can be restricted if a creia with a center point is given. His result is this: It is sufficient to specify four points in the plane, three of which lie in a straight line, and to mark with the ruler only those straight lines that pass through at least one of these four points. Then you can use the fixed ifreig with center point and the use of the ruler described in this way to s o l v e all tasks that can be solved with unlimited use of the ruler. ("Föttpende"

Lineale.)

§ 6 Constructions with compass and ruler

The addition of a fixed circle as a means of construction has extended the range of **points** that can be constructed with the ruler. Whereas **without** this **circle you can** only **construct** those points whose coordinates can be expressed rationally by the coordinates of the given points, after specifying a fixed circle all points whose **coordinates** can be represented by a square root expression of the coordinates of the given points **can be constructed.** The addition of a fixed circle to the ruler means that the compass is used only once to draw this circle. The compass is then no longer used for constructing. This **raises** the **question of** the extent to which the range of points that can be constructed is extended if the compass can be used as often as required. Constructions **with** compass and ruler are understood **to be** the finding of new points on given ones by the following **processes:**

I. Positioning the ruler at given or already con tructed points for the purpose of distorting the straight line determined by the 8e points.

2. Insert the two compass points into two given or already con- structed points, distortion of a circle around a given or already con- structed point.

constructed point as the center point with the radius taken into the compass opening (determined by two existing points).

3. Creation of new points by intersecting straight lines and circles o b t a i n e d in the way just described.

4. Straight lines and creiae occur in finite numbers.

The constructions in the way just described are called *constructions* with *compass and ruler.* It is clear that compasses and rulers can also be used in other ways. If, for example, you determine the corners of a regular heptagon on a circle periphery by probing with the compass, you make a different use of the compass than the one just described. This is not called construction with a compass (and ruler). If, by tracing with the compass on a given circle, you determine points that are equidistant from the directrix and the focal point of a parabola, you are again making a different use of the compass. This will be discussed later. (§ 21 ff.)

In this paragraph, however, we are talking about constructions with compasses and rulers in the sense of the definition just discussed in detail. It will be shown that the range of points that can be congruent by a single use of the compass is not extended by repeated use of the compass. This is taught by the theorem:

All and only those points whose coordinates in a given or constructed rectangular coordinate system c a n be represented by taking the square root of the coordinates of the given points, i . e. can be obtained from them by the four basic types of calculation and the process of taking the square root when these operations are *finally oJtmalized.*

The proof is based on the fact that, according to the rules of analytical geometry, only linear and quadratic equations can be solved to determine the intersection points of circles and lines, and that the coefficients of the equations of the lines and circles occurring can be expressed rationally by the coordinates of the given or already constructed points. This does not leave the realm of the real; in8be ondere all square roots are real. Conversely, according to the

§§ 2 and 3 can be used to construct all rational expressions in given real numbers, but also any real square root using a compass and ruler according to the height theorem. Therefore, theae instruments also allow the construction of any real points whose coordinates arise on those of the given points by means of ratio operations and any (complex) square roots. This is because complex numbers aare given by their real part and their imaginary part. Rational operations a r e also expressed rationally in these, and the extraction of square roots on complex numbers means, according to whose known theory is the augmentation of the square root of the absolute value and the bisection of the argument of the complex ZaM :

a -}- i ö - *r -oe ':f* -]- *i r* am *ipr e'*,*

yes -|- i ö]/Ücos$\frac{\varphi}{2}$ -}- i sin - *each'* *

or -- = *)/r* cos $\left(\frac{\varphi}{2} + \pi\right) + i\sqrt{r}\sin\left(\frac{\varphi}{2} + \pi\right) = -\sqrt{r}\,e^{i\frac{\varphi}{2}}$

in any case, two operations that can be performed with a compass and ruler.

Constructions using a compass and ruler are therefore also called quadratic constructions. In some of the following paragraphs, we will deal in detail with the problems that fall into the area of quadratic constructions and will define the limits of the problems that can be solved with compasses and rulers by specifying problems that do not fall into this area.

Before I go on to this, I shall speak of other means of construction which replace the compass and ruler, and also show that the ruler can be dispensed with altogether if it is only a question of constructing points.

However, it should also be emphasized that in this paragraph, just as in Poncelet Steiner's constructions, it is always a matter of *constructing* in *the* oriierial at'ene. For each given and for each constructed point it should be clear which are the precursors of its coordinates. Otherwise it is only certain that the point being searched for is among the constructed ones. See also the "Notes and additions" at the end of this book.

To conclude this paragraph, a few words should be said about the *structure of square root expressions*, which can be used to represent the coordinates of the points that can be constructed with a compass and ruler from the coordinates of the given points in accordance with the theorem proved. A square root expression Q is constructed by using the four basic arithmetic operations and the square root operation a finite number of times, starting from given numbers. The totality of numbers that can be obtained from a set of given (or already calculated) numbers by the four basic arithmetic operations is called a rationality domain or *body*. The simplest rationality domain is obtained from ZaM 1. Ea is the body of rational numbers. We now take the square root of a number from the output body *K* and add it to the output body, i.e. we add the numbers of the output area to the numbers of the output area and form ate numbers again, which can be obtained from the numbers of the output areaeg and these square root can be obtained by the four basic arithmetic operations. This is a new body *K* . We again adjoin the square root to one of the numbers. By repeating such adjunctions a finite number of times, a solid *Kq* is obtained to which the desired square root expression Q belongs.

What does such a square root expression look like 1 Let's start with one that contains only one square root, i.e. one that belongs to the body fi. Let us denote by a, h, c, d, r numbers of *K* and obtain *K* from *K*

by adjunction of - whereby we always assume that is not drawable, i.e. that j/r is not equal to a number of R -, then all numbers are

of *K* are rational functions of)/rwith coefficients aua *K*. However, since every rational function can be written as a quotient of two whole rational functions and since the even powers of *r from* powers of r

silver belong to *K*, every number of x_1 is a (fractional) linear function of *Jr,* i.e. of the form

$$\frac{a + b\sqrt{r}}{c + d\sqrt{r}}.$$

If you extend this fraction with c - d, it becomes

$$\frac{(a + b\sqrt{r})(c - d\sqrt{r})}{c^2 - d^2 r} = A + B\sqrt{r},$$

where *A, B* are again numbers aua *K*. You can write ao every number from *K as an* integer linear function of *Jr* with coefficients aua *K*. The same reasoning shows that every number from *K$* can be written as an integer linear function of

with coefficients **from FB$_1$ can be** written. Here,$_1$ *is* the **square** root of a number *r* from *K*, whose adjunction to JJ provides the ra- tionality range *K$*. Similarly, every number on *Kg*

of the form *A* + *B R,* where *A, B, R are* numbers aua x. — • ind.

A *more detailed 5'ragesIellation is* touched on at the end of this paragraph. Straight lines and circles that intersect with straight lines and circles at quite acute angles provide in praxi a quite exact determination of the intersection points. This fact raises the question of whether the construction process can be set up in such a way that only intersections of intersecting lines and circles are used to define new (constructed6r) points. This question is not yet settled. However, there are individual partial results to be mentioned in §§ 9 and 12, which suggest a negative answer to this question.

$ 7. hongtruhtions with I-tiIfe of the Zir1ieIg alone

3Iohr-Mageheronisehe Jfongtruhtions

From the point of view of §6, the result of §5 (Poncelet-Steiner constructions) can be understood to mean that the rules for using the ruler and compass given in §6 can be restricted. You only need to use the compass once to draw a circle whose center point you have to mark. Then you can use the ruler alone to find all the points that can be found with the compass and ruler (provided there are enough points). Of course, you cannot draw circles in this way. Conversely, this paragraph shows that you can d o without the ruler and construct with the compass alone. It will be proven that *all tasks that can be solved with the aid of a compass and ruler can also* be *solved with the compass alone*. These are, of course, tasks which require you to construct other points on given points and which are content with defining the straight lines you are looking for through two points. In addition, the *compass* must be used in *an extended way* compared to the compass and ruler constructions. The compass serves

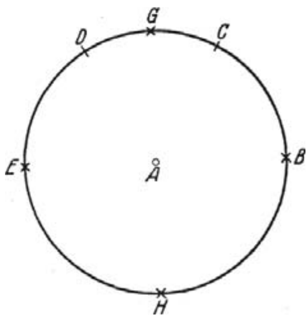
j The concentric circle calculatoris not only used to draw circles and determine their points of intersection, but also to determine which of two concentric circles has the larger radius, or, whichever comes out the same, to determine whether two circles intersect or do not intersect. (You only need i- over a radius of one of the two concentric circles as a through- line.
knife to draw another Kreia and determine whether the other one is the
The point where the two concentric circles intersect or do not intersect). While such a use of the compass in the field of circular and linear constructions, apart from special tasks, is only necessary in order to s e l e c t the correct point from among those resulting from the construction, here such an examination of relationships of arrangement with the aid of the compass is necessary in order to determine the course of the construction, as we shall see. (Cf. notes to § 6.)

It will be important to show how to find the intersections of lines given by two points with other lines and circles using only the compassa. To do this, we will first deal with some simple tasks.

1. If *AB* is the given distance, draw the **circle** *A (AB j,* i.e. the circle around *A* as the center point with the distance *AB* as the radius. Trace the radius *AB* on the periphery three times starting at *B* according to Fig. 25. This gives the points *O, D, E,* the last of which is again on the straight line *AB* in such a way t h a t the line *BE* - - *2* is *AB*. If the roles of *A* and *B are* reversed, the

Double the distance *AB* beyond *B* instead of beyond A. It is clear how a distance can b e multiplied by repeating the procedure.

2. If a line *B'E'* is given and another line *AB* **VOII** of half the length is known, the 3depth point *A' of the line B'E'* can b e found using Fig. 25. To do this, starting from the line *AB,* draw as in Fig. 25 and, starting from the points *B'E',* try to find the points O', Ö', *A'* of a figure congruent to Fig. 25. To do this, determine U' as an intersection of the circles *B' [BC! j* and A' *[EC!).* It does not matter which of the two intersection points of these two circles is taken as 6'. Then



I'1g. 25

I'tg. 36

A' is the one intersection point of the two circles *B' (BC!)* and U' *BCD).* To decide which of the two intersection points t o take as *A'*, note that of the concentric circles around A' through the two intersection points, the one going through *A'* has the smaller radius.

3. You are to construct a courtyard on a line *AB* at point *A* (i.e. construct any **point** on this line). This can be done using Fig. 25 by intersecting the creiae *B (BD)* and *E [BD j]*. This gives two points of the lotea, one of whichF c a n be seen in Fig. 25.

4. Since *AG* is *2AB*, the circle *B SAF)* meets the circle *A (AB)* at the two intersection p o i n t s D and Zf of the perpendiculars constructed in *A* on *AB* with *A [AB). B, O, III, II* aare the corners of a guo//rofea (f'ig. 25) inscribed in the *circle A (AB j).*

5. The *center of a line AB* is to be found. Plan double the line *AB* beyond *A* and *B* to the points U and *D.* An intersection of the circles U *[CSB)* and *D (AD)* is *E. II'* and Cr are the midpoints of the lines *€!E* and *DE* calculated according to 2. The circles F *[AB j* and *G [AB)* intersect (except in fi) at the center ff of the line *AB* (Fig. 26).

6. J¥fan is to construct a yard from a point C outside the straight line defined by the line *AB* and the perpendicular base point. The circles *A (A€! j* and *B (BC!)* intersect not only in U but also in the mirror image *D* of C on the straight line *AB*. The circles U *[C!A j* and *D [DA)* intersect at a further point A on the straight line *AB*. The center point of the line *AE* is the desired perpendicular base point. It can also be determined as the center point of *€!D*. These center points are constructed according to 5. (Fig. ß 7).

7. From a point C outside the circle *A (AB), the* two *tan9entets* are to be *con9rtructed* and their points of contact determined. Let p let *D* be the center of the line *AC!* The **circle** *D'(AD)* intersects the circle *A (AB)* at the two points of contact A and A'.

8. *construct* a *point C -f- A of the inoereen with respect to a circle A [A B j*. Let *r* be the radius of the circle *A (AB)*. Let U be the inverse of C with respect to *A (AB),* if firstly U lies on the half-line *AC!* and zg. az if, secondly, the product of the distances *A C! - A C! -- r** is. If initially C lies outside the circle *A (AB j,* then place from U to 7. the tangents to the circle *A (AB j,* from one of the points of contact *E* to 6. the perpendicular to the straight line *A C!* The perpendicular base point is the inverse Ü to U. It is also said to emerge from U by transformation according to reciprocal radii. However, if N lies inside *A (AB)*, double the distance *A C!* several times in succession and choose the multiple 2' so that 2 *A C!* > r o-ird. This gives you a point C" on the straight line *A CO.* Its inverse is C'. If we then form the 2' multiple of the line *A C!'*, we obtain the point C inverse to U. This is because

r° - *A C!-' A €!' 2*A C-!' A* ⁿ°' -- *A C-! A C!* .

9. *The !I'ransJormation by reciprocal radii int a circular edge sohafl ,* i.e. this mapping turns every **circle** into a **circle** or a straight line and every straight line into a circle or a straight line. **If** you introduce right-angled coordinates z, y whose starting point is the center point *A* of the circle *A (AB j of* radius *r, then*

$$\bar{x} = \frac{x r^2}{x^2 + y^2}, \quad \bar{y} = \frac{y r^2}{x^2 + y^2} \quad \text{resp. } x = \frac{\bar{x} r^2}{\bar{x}^2 + \bar{y}^2}, \quad y = \frac{\bar{y} r^2}{\bar{x}^2 + \bar{y}^2} \tag{1}$$

is the algebraic expression of the transformation by reciprocal radii. *A (ABS* is called the inversion circle. It remains **point** by point during the inversion fixed. Since straight lines and circles in the equation form

$$a_0 (x^2 + y^2) + a_1 x + a_2 y + a_3 = 0 \qquad (2)$$

are summarized, it can be seen that in figure (1) from (2) the equation

$$a_0 r^4 + a_1 \bar{x} r^2 + a_2 \bar{y} r^2 + a_3 (\bar{x}^2 + \bar{y}^2) = 0 \qquad (3)$$

which proves the assertion. In particular, a straight line (2), i.e. (2) with a - 0, turns into a circle (3) or a straight line through the JY center *A* of the inverted circle, and vice versa, a circle or a straight line (2) through *A,* i.e. (2) with n = 0, turns into a straight line (3).

I also emphasize that each diameter of the inversion circle merges into itself during the inversion and that, according to the secant tangent theorem, each circle perpendicular to the inversion circle merges into itself. This is because it is intersected by each diameter of the inversion circle in a pair of inverse points.

10. If you apply the trangformation by reciprocal radii to zthree straight *lines whose intersection point* you are *looking for,* you get two circles through *A,* whose other intersection point leads to the desired intersection point of the two straight lines by another inversion at the same circle. How to carry out this construction i If the two straight lines are each given by two points *BC!* and *DEi (B* -|- *D)*, choose the center ff of the line *BD* as the center of the inversion circlea . It does not lie on either of the two straight lines *BC!* and *DE,* e8 unless one of them is identical to the straight line *BD* ; in this case, however, *B* or *D* is already the intersection point you are looking for. As the inversion circle, choose any circle around the point ff as the center of the line, e.g. the circle ff *(MBJ2 j.* To make the inversion of the two lines convenient, fill in the perpendiculars from ff to the two lines and determine the inverses J, O of the perpendicular bases F and Cr. Then draw the cir8es over the permeaaers 3fJ and äf9. They intersect at the inverse N of the booked intersection point 6 of the two lines (Fig. 28).

11. To finally determine the step *of a straight line CAD* with a *circle A (AB)*, a perpendicular to the straight line *CAD* is drawn from *A* to 6. The base of the perpendicular is fi. After I., Stan doubles the distance *AE* over fi. This gives you a point J. The circles F *(AB)* and *A [AB j* intersect at the two intersections of *CAD* with the circle *A (AB)*. This construction *fails* if the two circles just brought to the intersection coincide, i.e. if the straight line *CAD is* a diameter of the circle *A (AB j)*. In this case, first determine any point that is not on the diameter *CAD* but outside the circle *A (AB j).* This can be done, for example, by joining the two circles U *(CAD j* and *D [DC!)*

in two points *E* and to the intersection and then double the distance AJ a sufficient number of times. From a small point *P* located outside the circle *A (AB j* and not on *CAD*, place the tangents to the circle *A AB)* and determine the two points of contact Q and B. The circle *P (PQ)* passes through Q and A and is perpendicular to *A AB j*. Choose *P (PQ)* as the inversion circle and determine the in version of the straight line *CAD*

Qp as in 10. Intersect the circle thus obtained inverse to *CAD* with the circle *A (AB j)* which remained fixed during the inversion, and calculate agree the invergences of the two intersection points. This is the intersection of *CD* with the circle *A (AB)*.

With regard to the proof of the theorem of MOaR-MsCBERO "i thus concluded, it should also be noted that in order to carry out of the construction given in 8. must decide whether the point C is on the inside, outside or on the circle *A (AB)*. To do this, use the compass as described above. Similar according to the wording of the sentence also applies to 11.

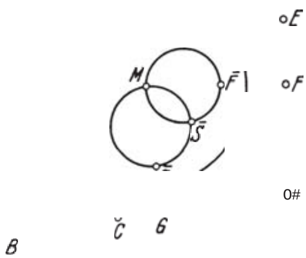


Fig. 28

§ 8 The ruler

This is a ruler with two parallel edges. Either of them may be used to draw straight lines. *The instrument must be placed on the paper so that two given or* "choti *conelruierle points* (in short, two existing points) are either on the qfeicJtea *edge or distributed over both edges*. It is clear that you cannot solve problems with this instrument that cannot also be solved with a compass and ruler'). I will xeigen that, conversely, *all points that can be c'msiruated with the compass and ruler can also be constructed with the parallel linml* 'iffetti £ötiaen. Ea therefore completely replaces the compass and ruler, apart from the fact that you cannot use it to draw circle peripheries. It is obvious that you can construct a parallelogram with the parallel ruler. You can therefore, as with the ruler, construct all points whose coordinates can be *rationally derived from the coordinates of the already existing parallelogram.

') The requirement to place a parallel ruler of width *b co* such that the two edges pass through two points at a distance d requires the construction of a right-angled triangle with the hypotenuse dand a cathetus ö, with which the determination o fÖ is connected.

of existing points with respect to a Cartesian coordinate system whose unit distances are two sides of such a parallel ruler. The proof of the above theorem is based on the fact that the parallel ruler can be used to draw tangents through an existing point to a **circle** with an existing center whose radius is equal to the distance between the two edges of the parallel ruler. In fact, you only have to place one edge of the parallel ruler at the center and the other at the point through which you want to construct the tangents. Since there are two possibilities (or none if the two points are too close to each other), you get the two tangentsl). If we can now show how to intersect a given straight line q with such a circle *K* (radius equal to ruler width), our previous findings on constructing with the ruler using only a circle with a center point have proved una theorem. The construction of these intersection points is based on the remark that the tangents to *K* in the intersection points of *g* and A *pass* through the pole G of q with respect to *K*. As soon as you have constructed 9, you have the intersection points of q and Jf as intersection points of the tangents to *K* from Cr with the straight line q. The pole Cr of q in relation to *K* can be found as follows: Construct or choose two points on q that are more than the radius away from the center of *K*. Place the two tangents to A through each of these two points *P* and *P$* and construct the fourth harmonic to p with respect to each of these pairs of tangents. These two fourth harmonics intersect at Cr, the pole of q. (Because on every straight line through G, the points of intersection with the two tangents are har- monically separated by *P* from 9 and the point of intersection with q). In a more algebraic way of speaking, one can also expose the nerve of the proof by featu- ring that the two operations of drawing a tangent to the circle and intersecting the circle with a straight line, which are dual to each other in the sense of geometry, are rationally dependent, i.e. can be traced back to each other with the ruler alone (in the presence of a parallelogram).

§ 9 The honing ruler with a winch ruler

The instrument is a generalization of the parallel ruler. Ea consists of two lines intersecting at a point at a fixed angle e. This instrument should not be confused with the right-angle ruler already mentioned in § 4. At that time, ea consisted of two rulers

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*) They coincide if the distance between **the** two points is equal to the **ruler width** matches.

aenchtechte rulers, each of which could be used to draw straight lines. It was laid down so that one **edge** passed through two existing points, the other through one existing point. We discovered at the time (§ 4) that the use of these instruments meant no more than the specification of two pairs of mutually perpendicular straight lines next to the ruler. Now, when constructing with the angle, the vertex of the angle also plays a role. *The instrument is no longer placed on the paper, dap*
the two Runteti *go through rorltandene points and there) the lscheitel on* eitter
existing straight line'). You *can* then draw along both edges to obtain straight lines that pass through existing points and intersect an existing straight line at the angle ot. The divider

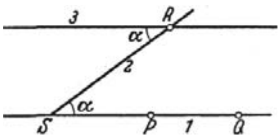


Fig. 29

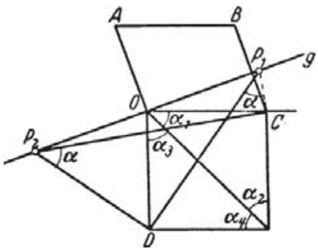
of the angle is then the newly congruent point. It is initially **clear** that this instrument can only be used to solve problems that can also be solved with a compass and ruler. This is because, according to the Periphery Angle Theorem, the geometric location of the vertex is the angle 'x whose legs p a s s through two fixed points,
a circle through the two points, which you can of course construct using a compass and ruler. You only have to draw the legs of a given angle z through the two fixed points and then construct the circle through the two fixed points and this vertex. I will now show, however, that *every point that cannot be constructed from a given* point *with compass and* diiecf *can also be constructed with the help of angles a.* offeia (without further use of ruler and compass). This is in itself probable, since the required position of the instruments (vertex on q, legs through *P* and Q) can be realized in two ways, i.e. the instrument appears to be suitable for solving quadratic tasks. In fact, I will show how t o construct the intersection points of an existing straight line q with a circle *K* with an existing center ff and an existing radius *fA*.

Fig. 29 first shows how parallels are drawn. The points P, Q, A are present. A parallel to the line *PQ* is constructed through fi b y drawing the three lines of Fig. 29 in the order of the numbers shown in Fig. 29. The angle e is first placed so that aone leg passes through T*, the other leg through fi and the intersection lies on the straight line *PQ*. Straight lines 1 and 2 are drawn to construct the point 'S. Then place the angle so t h a t one of its legs passes through ' S, the other through fi and the vertex l i e s on the straight line fi6, and now draw straight line 3. This is the

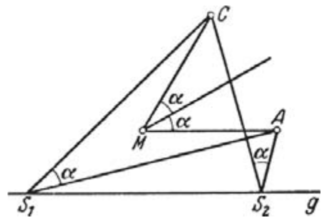
') Both legs are assumed to be unbounded straight lines.

is sought, provided that the angle z is applied to the other side of the straight line 2 at the second äfale.

Secondly, it is shown how to query a given distance *AB* from a given point 0 aua on a given straight line q through 0. Uan read off the aua Fig. 30. Draw a parallel to *AB* through *0* and then a parallel to *OA* through *B*. In this way you obtain the point C and it is élU = *AB*. Then construct the **point** *D* by placing the angle e four times in the order of the numbers e" z" z_3 , z. Then you finally create the angle e ao so that its two legs pass through U and *D* and that its segment lies on *q*. If lat ais a segment *P,* then iat *OP* -- *AB.*



Flg. 30



E'1g. 31

First of all, because of a congruence theorem of the triangle élN = *OD.* Then, according to the centriperipheral angle theorem8, the pointb *P* lies on the circle with the center 0 and the radius *OCl.* This is because the angle 2 e lies at éI and the angle e at *P*. Therefore *OP* -- *OC! AB*.

Finally, it is shown how the straight line p is intersected by the circle *K* around ff with the radius *MA*. This is shown in Fig. 31. First, the angle e is created twice in ff and the point 6' is determined on the leg 3fU by subtracting distances so *that* ff 6' = *3fA*. Then create the angle z in such a way that aa leg passes through *A* and N and that the vertex lies on q. According to the centriperipheral angle theorem, the position of the vertex on *g gives* the intersection points of q and *K*. Again, the nerve of the proof is the illuminating algebraic fact that the tasks of intersecting a circle with a straight line and placing the apex of the angle on *q zo,* that the legs of the angle pass through given points, are rationally dependent on each other, i.e. can be traced back to each other with the ruler alone.

At the end of this paragraph I will make a contribution to the question mentioned at the end of § 6. It was the demand to use only the intersection of mutually perpendicular lines and circles to define new (constructed) points when constructing with compass and ruler (because acute intersections result in bad intersection points). In the direction of this question

The following result from F. BAG ANN. Use a ruler and a perpendicular hook to construct according to the following rules: I. The ruler is only used to draw straight lines through existing points. 2. the perpendicular hook is used to construct {and draw) the perpendicular line through an existing point to an existing straight line. 3. new points are only created by the intersection of mutually perpendicular straight lines. F. BAGiiMANN has found, among other things, that *if three elements* of a *square are given and the three points are used as zero and entry points of a rectangular coordinate* system*, only those points can be constructed whose coordinates in this coordinate system belong to a* driven *ring* C{P). In algebra, a ring of numbers is understood to be a set of numbers that are not subject to addition, subtraction and multiplication in such a way that the application of these three types of calculation to numbers of the ring leads back to numbers of the ring. (A body is also abachloaaed to division). The ring G{P) mentioned here consists of the integers and those fractional rational numbers whose truncated denominator contains only prime numbers of the form 2, 4 J -t- 1, 2 integers. You can also say that the ring is generated by the numbers I/(l -j- *r*j, r* rational dea body P of rational numbers. For example, the point ('/" 0) cannot be constructed from the three given corners of the square.

(Cf. also § 12.)

That the given condition is necessary is proved as follows: The coordinates of the given points (O, 0), (I, 0), (O, 1) belong to the ring ß {P). If (, a), *[bd, öt*), (q, c) are three points with coordinates from the ring 5 (P), ao the equation of the line connecting the two points has coefficients that also belong to the ring ((P). The equation of the same may therefore be expressed in the form

"a+ y+w=O (1)

with non-divisor integer rationals ii, v and u' C R{P). For it is certainly possible to choose u, v, u' in R (P) (e.g. u = e - 6" r *b - p, u* -

- ö - eg tig), where u and r are also integers (since you can multiply by the main denominator). In this case, tt and v can even be chosen with different divisors. For if they have a prime divisor p in common, ao p aua tt, v, tr can be divided away; this is self-evident in the case of p 3 (mod 4) and follows for p 3 (mod 4) from the fact that p is also in the numerator of - u e, - v a

= tr rises.

vs-"y- +"q=0 (2)

is the equation of the perpendicular of {I) by (q, c). The coordinates of the perpendicular base point are therefore rational numbers whose denominator is in the product of u° + c-with the denominators of u', q, c. Here u and tt are non-divisible rational integers. If p i s then an odd prime divisor of tfi -j- v-, then (ti, p) = 1 and (v, p) 1 , and there is an integer rational z with (z, p) I such that u z =- tt mod p. Therefore

0 tt° -{- c° u° (z° -|- 1) mod p .

Therefore iat z° - I mod p. Therefore, according to Fermnt's theorem

$$(-I)^{\frac{p-1}{}} *1 \text{ oiod } p , (-I)^{\frac{p-1}{}} *- I .$$

Therefore f 1 0 mod 2, as should be proven.
2

The fact that the stated condition is also sufficient will not be discussed here.

§ 10 Constructing with a ruler and empty compass and with fixed **Compass opening** without ruler

tt) The ruler is used in the usual way to draw straight lines through two existing points. The compass with a fixed opening is used to draw circles with this opening as the radius around existing center points. The compass opening or, in other words, two points separated by the compass opening naturally belong to the given pieces. New points are created as intersections of straight lines and such circles with the fine circle opening as radius. The result of § 5 (ruler and fixed **circle** with center) allows the statement without further discussion that ao all constructions are feasible that can be solved with the ruler and a fixed circle with center. The result of § 6 (ruler and compass) teaches that these constructions are none other than those which can be treated with ruler and compass, i.e. all quadratic constructions and no others.

b) **J. HJEL6fSLEV** has tackled the question of the *constructions of* fair *fixed circle ö||nunq without* fiitienf. First of all, it is clear that by leaving out the ruler, the range of points that can be constructed from given points is narrowed. While you can, for example, construct the center of the distance determined by a given pair of points with a compass and ruler and therefore also with a fixed compass opening and ruler, this center cannot be found with a fixed compass opening without a ruler. If a is the fixed circle opening, i.e. only circles of radius a can be drawn around existing points and made to intersect, it is obvious that the center of *PQ* cannot be constructed from two points *P* and Q whose distance is greater than 2s. Since the two circles *P(a)* and Q(a) do not intersect, the center of PQ cannot be constructed at all.

no further point can be constructed from *P* and Q. But even i f the distance *PQ* < 2e, the center point of f*Q with a fixed circular opening cannot always be found. lat s. B. the distance *PQ* -- *a,* only the vertices of a network of equilateral triangles with side o are congruent. Pig. 32 clearly s h o w s this. Furthermore, it is assumed that there are at least two points with a distance < 2e. Even if the exact range of points that can be obtained from given points with a fixed compass opening without a ruler is not yet known, dOCh MJELJYtSLEv has shown in an intereaaant work that it is not only possible to obtain two points with a distance of < 2e.

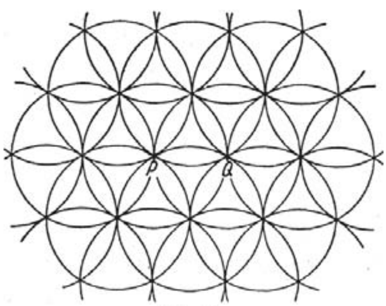


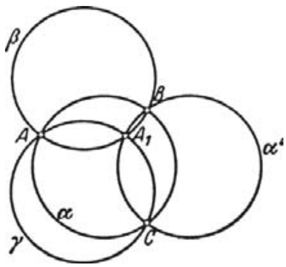
Fig. 32

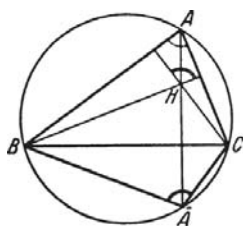
with a distance smaller than the given straight line two points of a perpendicular straight line can be constructed in a known way, but he has also shown as an essential result that any two given vectors can be added. Since you can cover the plane with a network of simultaneous triangles of edge lengtho , it is clear that you can construct any vector as the sum of vectors of the same length.
therefore only has to show that two vectors of limited length ao can be added. Since each point is represented or obtained as the intersection of two circles of radius a, it suffices to show that any **circle** of radius o can be moved in parallel by a given vector of length less than 2e'). Thea is shown in Fig. *33,* where the **circle** e is moved by

a vector AA_E , of length less than a1g 2o, is shifted. The shifted
Kreia is ot'. In Fig. 33 there are two circles Q and y, like all the others of the radius o, passing through the points *A* and B_1 , while et *passes* through *A* and *B, z'* through dv *andB*. EB It is shown that the three circles e, z' and y intersect at a point N . Here z is arbitrarily given by *A*
and the point *B* is defined as the vertex of e and Q. et' is determined by *A* and *B*. To understand the assertion, note that Q is inscribed around the triangle *AA B* and that e, y and z' can be obtained from Q by reflecting on the sides of this triangle. They therefore intersect at the vertex U of the triangles AA_i B. Read the correctness

[1]) If you want to move a point *D* by a vector *AG z*, ao place two circles of radius n through *A* and *D* - i.e. assume | *AD* | < 2 n, which is permissible - and then move both **circles** by *AAE.* The raised ciriae then intersect
in Az and *Dz,* so that *DDR* -- *AAE* iat.

the last assertion i n Fig. 34. Because of the equality of angles indicated there, the arc through *BMW is* congruent to the arc through *BAC! and* is therefore connected to it by reflection at *BC.* The angle of triangle *ABC* at *A* completes the angle at *A* to 180° (chordal quadrilateral in the circle). Similarly, the a n g l e o f the two heights at ff completes the angle of the corresponding sides at *A* to 180°. Based on these considerations, 'x' can be obtained from z by the product of the following four reflections. Reflection *IS :* Mirror z at *A B*





in Q. *A* remains fixed. Mirroring 'S : Mirror Q at the center perpendicular of AA_1 in Q. *A* changes to d_1 . Reflection 6 : A is mirrored from Q to *AA* in y. *A* remains fixed. Reflection 6t: Reflect y on *APO* in e'. This leaves

A is fixed. The product of the four reflections S_U 6 6 = S is therefore a

"' and *A into A*. However, the movement iS is now a parallel displacement. Since the reflections 6 and 'S3 occur on mutually perpendicular lines, their product is a rotation of 180°. Similarly, B_1 and St are mirror images.

succeeded on mutually perpendicular straight lines. Therefore your product a rotation of 180°. Therefore, SMS S St = 6tS'St is a parallel displacement. However, if the movement S left a point P fixed, ao the movement 'St66t would leave its image *P* fixed by 6, so it would not be a parallel displacement. Therefore, 6 is a parallel displacement, and therefore z' goes from et through ver-

shift around the vector *AA*.

§ 11 The standardized ruler

A standardized ruler is a ruler on the edge of which two points *A* and Zl are marked and which can be used as follows: I. as a ruler for drawing straight **lines** through two existing points; 2. for tracing the distance *AB* on an existing straight line from an existing point on it in each of the two possible directions; 3. for cutting **a** existing straight line *g* with a circle of radius *A B* around an existing point *P* as center. (For this purpose place the mark *A* (or *B)* on *P,* the mark *B* (or fi) on q.) The results of § 6 (ruler and compass) teach that with the normalized ruler aße quadratic constructions (pnd no further) can be carried out. For the processes 2. and 3. only mean intersections of circles and straight lines. There is of course a certain relationship to the constructions of § 10, a), with the only difference that now the drawing of the circle peripheries is avoided. Incidentally, it is also possible to dispense with drawing the peripheries in § 10 and restrict oneself to marking the points of intersection of the circles with the straight lines.

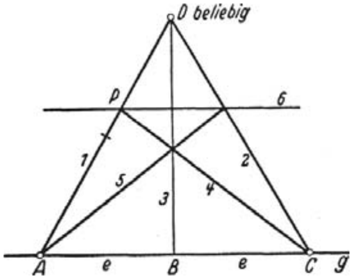
I will return to a generalization of the standardized ruler in § 16, which deals with the single-axis ruler.

§ lS. Ruler and gauge. Pilbert-like constructions. Baehmann constructions. Paper folding

The ruler is the usual one and is used to draw straight lines through two existing points. The ruler is understood to be an instrument with which operation 2. of § 11 can be carried out, i.e. an instrument with which a straight line can be traced from a point on it to both possible sides. You can use either a fixed compass opening or the standardized ruler of § 1 I , i.e. two points *AB* firmly marked on the edge of the ruler, only now with the difference that operation 3. of § 11 should not be permissible. The two marks *AB* should therefore always remain on the same existing straight line, or to put it another way, straight lines q may now only be intersected by circles whose center lies on p. It is to be expected from the outset that only some of the quadratic constructions can now be solved. This will indeed be the result, and it *will be possible to construct from existing points those whose existence can be deduced from the* first *four axiom* groups *of Euclidean geometry* (axioms of connection, arrangement, congruence and parallels in the formulation of HILBERT's foundations of geometry, but not axioms of continuity). On the basis of the aforementioned axiom groups, 1. the straight lines through existing points and the intersections of existing non-parallel straight lines exist, 2. an existing line on an existing straight line can be traced from an existing point on it in either of the two possible directions, and 3. the two straight lines exist which connect an existing straight line in an existing point under an existing point.

Intersect angle B. 4. a parallel to 9 exists to every existing straight line y through every existing point *P* not located on it. 6. a perpendicular to q exists to every existing straight line y through every existing point *P*. These are the existence statements of the first four axiom groups. (Erat from the continuity axioms results the gap logic of the lines and also, as we will see, only the existence of arbitrary quadratic irrationalities. Only the existence of certain quadratic irrationalities follows from the first four axiom groups, as our considerations will show).

I will first show how the points that exist according to these five existential outputs can be constructed using a ruler and a gauge. To I. is



Flg. ßS

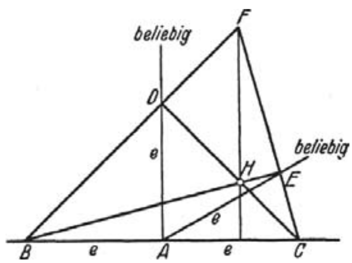
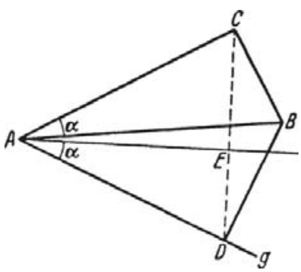


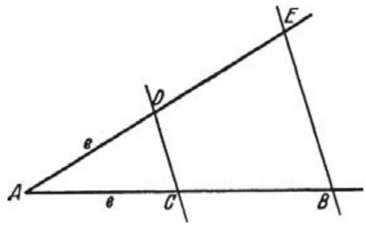
Fig. ßti

There is nothing further to note; 2. and 3. will emerge when we have discussed 4. and 5. First, then, the drawing of parallels. This is taught in Fig. 35 (here and in the following we indicate the distance to be drawn by the gauge by a written e). The auxiliary lines are to be marked in the sequence of the written numbers. The point *D* designated as arbitrary in sig. 35 is obtained by plotting an arbitrary number of parallels e on the straight line *A P* from *A*. The construction of the parallels is of course based on the harmonic properties of the complete quadrilateral, as we know from § 3.

Next, we construct a perpendicular to a given straight line p, as the 5th requires. (The perpendicular through a given point is then obtained by drawing paral1elines if we first have any perpendicular). This teaches Pig. 36. With regard to the existence of the two straight lines described as arbitrary in Fig. 36, it should be noted that it must be assumed that there i s at least one point outside the straight line p, if it is to be possible to construct further points outside *q* and different straight lines from y using a ruler and gauge. You then only have to connect e.g. *B* of Fig. 36 with such a point outside of p, then **plot** on this line twice in succession from *B* to e and the resulting points with *A*. Then you have found two straight lines through *A that* are marked as arbitrary in Fig. 36. ff in Fig. 36 is the altitude point in the triangle *BOC*. This is because the angles *BDCI* and *BIi!€l* are right angles according to Tez's theorem. ff is defined as the intersection of *BE* and *D€l.* Therefore, ZfJ is a perpendicular to q.

Now we will deal with the application of an existing angle e to an existing straight line *g* at a given point *A* (Fig. 37) as required by 3.
'x can be brought into the position of @ *BGG'* by drawing pores. On one leg of e, take a point *B at* random (by removing the gauge once or several times) and fill in from it





Flg. ß8

the perpendicular *BO* to the other leg. Similarly, fill the perpendicular *BD* from *B* onto *q*. Then draw *OD* and drop the perpendicular *AG* from *A* aua onto it. Then *f DAS -- p BAC* = z. The four points *A, B, O, D* lie on a circle according to the theorem of T Es. Therefore the angles *ABO* and *ADO* are equal as peripheral angles and therefore the angles *DAS* and *BAO are* equal as their complement angles. If you choose & on the other leg of z, ao, the same construction leads to the ablation of e on the other side of p.

Now we will carry out the removal of an arbitrary distance s (not the calibration distance) as required by 2. A distance s is to be drawn from *A* aua on an existing straight line through *A*. Since we have already dealt with the drawing of parallels, we can assume that the distance s to be drawn on p is the distance *AB* in Fig. 38. \We plot *AO -- AD -- e in Fig. 38* and draw the parallel *BE* to *CD*. Then *AE* is the distance s subtracted from *q*. This tlberlegung teaches *ba0 the generalization of the calibration map* to the *line tracer does not lead to any constructions that could not also be carried out with Nneal and Eichinap. lslreckenabtraqer* is an instrument with which any distance can be traced on an existing straight line from a point on it.

We ask how *irrationalities* can be conaŁrued by the ruler and the yardstick. W e need only note that constructing with the ruler and the standard contains the only element that goes beyond the conaŁruing with the ruler, namely the rotation of an existing distance by an existing angle at an existing point. This was mentioned last. However, it can be seen from Figs. 36 and 37 that both the setting of plumb bobs and the removal of angles can be done with the ruler alone: The ruler is an instrument for turning distances. According to the rules of analytical geometry, the point around which the rotation is to be made is chosen as the coordinate origin. Then, to carry out the rotation of a given distance e by a given angle œ, you have tg œ = eJò iat,

to intersect the straight line y= - z with the circle z* + y* = e°, i.e. ò e

- -}- *b*-

can be determined. The quadratic operation made possible by the square measure is therefore the square root of the sum of squares of already existing numbers, because o and *b* are known from the already existing points. Since, as we know, rational arithmetic operations can be performed with the ruler, we have the result: 3fit ducal "*ndlllichmap eind aus gegebenen Punkten alle und nur die Punkte konelruierbar, deren Koordinalen in bezuą auf ein øor- handenee reohluńnkłiges Koordinalensyetem etch aus den Koordinălen der gegebenen Punkle duroh endlich oftmalige Anwendvną der uier Orundrechnunąø-* arteti nut dee *lichens non Quadralunirzeln aus Quadratsvmmen ergeben.* If, starting from given real numbers, one now forms the range of all congruent numbers according to this, then this has the special characteristic of being totally real, i.e. the conjugates of all numbers of the range, which are obtained by changing the signs of the square roots, are also real and belong to the range. (After all, only square roots are taken from square sums.) Therefore, it is not possible to calculate a real square root with a ruler and a ruler.

angleda triangle with the hypotenuse 1 and the one cathetus)/2 - 1

congruent. Because then the other cathetus would also be 2 would be con- structiveŁ, and this number would therefore also belong to the range of conatruable

numbers. However, their conjugate J- 2)/2 - 2 is imaginary.

I would like to *a d d a few remarks*. 1. + *e** -- *a* $\sqrt{1+(\frac{.}{.}}$ '

is, øo is sufficient as an operation in addition to the four types of calculation

to accept the -l-. Since -l- -l- = y(+ ò°J- + -, the square roots of arbitrary sums of squares are therefore also in the range of constructible numbers. With HILBERz we now denote mil f) the body of algebraic numbers that øich can be constructed from 0 and 1 using

') And daixtit also the parallel drawing (by repeated soldering).

of the four basic arithmetic operations and the operation -{-. Here "i is a number that has already been obtained. We further denote MiG f2(i) by . HiLBEnz the body of those a1gebraic functions of variable 1 that can be obtained from 0 and l using the four basic arithmetic operations and the

Operation l -}- gain laaaen. 2 Hezzez noted in his Fundamentals of Geometry that the same regular polygons can be constructed with a ruler and a yardstick as with a straightedge and a compass. The center of the circle and a peripheral point A are taken as given. These are assumed to be the zero point and the one entry point of a right-angled coordinate system. The assertion') about the regular polygons then means that the real and imaginary parts of the coordinates of the corners of those regular polygons which are congruent with compass and ruler belong to the solid f). A corner should always be A. According to RILBER'P, however, the *A pollonioch touching problem,* i.e. the task of finding the eight circles that touch three given circles, cannot be solved with *a* ruler and compass. Furthermore, HILBERT has noted that the fef/ofttsc£e *touching problem* can be solved with a ruler and a yardstick. He did the task of inscribing three circles on a given triangle in such a way that each of the circles touches two triangle sides and the other two circles. 3. the *necessary and sufficient conditions for the constructability with ruler and* aicfimo§ are also known. As in § 6, we imagine a sequence of gauge blocks built up from the solid *K* by **successive** additions of square roots of sums of squares, starting from the solid *K,* and ask how these gauge blocks differ from general square root blocks. The concept of conjugate solids is useful for this. Plan obtains conjugate solids from a square root solid if you change the sign in one or more of the square roots that are adjoint in the construction, or in other words, if you prevent the sign in some of the adjoint square roots in the sequence of adjunctions. For example, if you change alao from *K* to

Jf, + by adjunction of + *B* r" , the result is

conjugate body, if instead of dealen *JA* - *B* or a number which is obtained by changing the sign of one or more of the roots occurring in *A"* *B* and r uaw. *It* is *necessary and sufficient for a square root body to be a gauge body* if *it* is simultaneously reeff *with* all *£onyugated* numbers, i.e. L calls *aliens its* £oayupierteti are ordered by the last *real numbers.* The one from E. **Amin**

[1]) It simply follows from the fact that the Gaussian three-partition periods satisfy equations with all real roots.

which is based on an approach by D. Hnasnz, will not be presented here.

4. I turn to the question of whether the *constructions* with *compass and ruler can be defined axiomatically in the* same way as those with standard and ruler. In fact, only one axiom has to be added to the axiom groups I, II, III, IV, which characterize the constructions with ruler and rule, in order to characterize the constructions with compass and ruler. It is the one designated by F. Senen *ab A from the circular conslruction.* It reads: *Ii¡s* exists pettnu eiti *right-angled triangle mtl given cathetus a and given hypotenuse c* > a.

Since the other cathetus of these triangles is the addition of the construction of a right-angled triangle on the cathetus and hypotenuse means the assumption of taking the square root on a difference of two squares. The constructions with ruler and standard ruler yielded the square root from a sum of two squares, in other words the construction of a right-angled triangle on two cathets. It can now be seen that both processes together can solve any quadratic equation with existing coefficients,
i.e. the square root of any existing number can be calculated. For if et z° -l- z -l- o = 0 is the quadratic equation to be 1öed with coefficients e_0 , , o on a body *K, co* the expression e2 - 4 e_0 at appears in the resolution form 1 under the square root.
But now 4 e e = (e -J- at)° - (e - et)°. Therefore

a} - 4 et e = a* -[- (ab - o)° - (o -}- e)° = ()/e{ -}- (o - a)° J° - (co -j- o)° .

In fact, the resolution of the quadratic equation is based on the extraction of square roots from the sum and the difference of squares.

If we add to the first four Ililbert axiom groups the axiooi of circular congruction, we define the geometries in which, from given points, exactly those points can be constructed with compass and ruler whose existence is required by the axioms.

5. Finally, I would like to mention a contribution to the discussion at the end of § 6 and § 9 touched on this question. F. Browser has investigated how the range of points that can be congruent with the ruler and standard gauge changes if new *points* are *only* added at *a perpendicular angle to each other.* Plan then adds the right angle hook to the ruler and the new gauge. In addition to the three construction steps mentioned at the end of § 9, there are now 4. the removal of a fixed distance on an existing straight line from a point on it in each of the two directions possible on it. Then exactly those points are congruent whose coordinates in an existing rectangular coordinate system are also congruent.

from the coordinates of the given by the operations e- *b* and $-\frac{}{1'^{\circ'} + b^2}c$ can be represented. As Boone has noted, this includes in particular the operations e -{- h= o-((ö- *b j*- *bJ* and $\sqrt{a^2+b^2} = \frac{}{\sqrt{a^2+b^2}} a + \frac{}{\sqrt{a^2+b^2}} b,$

but not e- tt and ach. However, you can use 0 and 1 with the operations

a — *b* und $\frac{a}{\sqrt{a^2+b^2}} c$ alle die Zahlen gewinnen, die man daraus mit Hilfe der four basic arithmetic operations and extracting the square root from a sum of squares. Accordingly, you can construct all those from two given points using the ruler, right angle and standard measure in Bachmann's new sense that you can construct from them using the ruler and standard measure in Hilbert's sense. It is still an open question whether it is also possible to construct from a finite number of given points with ruler, rectangular measure and standard measure according to BxcaaAxx all the points that can be obtained from them with ruler and standard measure according to HILBEST. However, there is a difference between the scope of Milbert's and Bachmann's constructions when it comes to general constructions, i.e. when there are eeriabfe among the given points and you want constructions that can be used for all values of the variables. In particular, not all p o i n t s belonging to the body N(t) can be constructed from 0, I and f in the Bachmannian manner,

i.e. which can be constructed from 0, 1 and f in Hilbert's masonry, but rather at most those elements of the body N(f) which 8are in the form

/(t) -t- t p(t) with functions *f[t)* and *g(I),* which are limited together with all their conjugates. Thus, as BzcBAlANn has noted, there is no general construction for the intersection point of an isosceles triangle, for example.

The following observations serve to prove all these statements. As Boone has remarked, the tracing of any existing {line on an existing straight line from a point on it in either of the two possible directions can be accomplished in Bachmannian masonry with a ruler, right-angle hook and standard. For the sake of brevity and in order to better emphasize the essence of the thought processes, it will be further assumed that a distance subtractor is used instead of the standard. Then we first note that the

both operations a - ö and + $ c correspond to the following Bachmann

I. Plan the midpoint 3f of the points (0, 0) and (o, 0) by drawing a square with these two points as corners and plumbing the intersection of its diagonals down to the line joining the two given points. Then draw the point

(6, 0) with the line segment at ff. This gives you the point (e - *b,* O).
2. construct the **point** (e, b) = *A* and the point (c, 0) = U. subtract the line *OR* from O to 6IA and fill a perpendicular from the resulting point U' onto the z-axis, i.e. the line OR. The base point of this perpendiculara

with respect to *0* has the coordinate -- c . We now further show that
{a- -j- 6-
can construct all numbers of the body f2 from 0 and 1 in Bachmann's manner. As we have already seen above, the two ope-

rations e - *b* andJ/p+ $ ~~c auoh the~~ operations e -}- t' and -{- - ge-

win. If '" is then an element of f), then there is a chain of bodies A , x1 , ..., *Kg* such that xy is the ifbody of the rational numbers and n' is an element of fig, so that every body x + i from K ", by adjunction

of a number " , -t- *b*, where ep and h" belong to the body Ap. However, since all numbers of the body *K*",+ can be represented in the form cp -j- d", " , -j- *b*", aua numbers ap, h"" cp, dg of the body K ", and since furthermore

d",)/o", + *b*", -- *(df a",j*+ (df b*",)*, i.e. all numbers of *Kg* + can b e represented in the form cp -l-)/ep + /p with numbers cp, ep, /p aua *Kg*, then all numbers of *K* +1 can be obtained in Bachmannian masonry if this is *the* case with the numbers of *Kg*. For the square root of a square sum and the sum of two numbers can, as we have just seen, be constructed in Bachmann's manner. It therefore only remains to show that from 0 and 1 we can construct all rational numbers in the Bachmannian manner. If then m is any positive

integer, then because of + 1 = J( )° -l- I the)/m -l- 1 can be constructed in Bachmannian masonry, if one can construct can be constructed in the Bachmannian manner. But since 2=)/1° + 1°, you can construct all for m > 0, completely, in the Bachmannian manner$. Therefore

one can also construct l/tn = J$+ (~~$)g can~~ be constructed. Since one

but every rational number can be represented as the sum and difference of such root fractions Um, all rational numbers can be obtained from 0 and 1 in Bachmannian 6fans. This result, that one can construct all numbers of the body f2 from 0 and 1 in Bachmannian manner, means that one can construct from any two given points *A* and *B* in Bachmannian manner all the points that one **can** construct from *A* and & in Hilbertian manner. For **if we take** *A* and *B* as the zero and one point of the z-axis, we can construct from *A* and *B in Hilbert's* masonry just those points whose coordinates are numbers from f), as already mentioned on p. 43.

As I said, it is an open question whether Hilbert's constructions do not go further than Bachmann's even for a finite number of given points. *A difference arises, however,*$_i$ edenJalts with the *general constructions, i.*e. if among the given points there are variables and one asks whether there are constructions which are the same for all values of the variables (cf. also § 13 on this concept of general constructions). Here it is assumed that the given points are 0, 1, t, where t is a variable. Then according to §12. 3. the body of congruent elements in Ililbert's sense is denoted by the body f2(t). We arrange it according to n "EBZ first by establishing that for any two different elements e, *b* of **J(t)** e > *b* should hold if e - *b* > 0 for large positive f. (Since a - b nla algebraic function of t has only a finite number of zero atoms or vanishes identically, for different o and *b* the difference a - *b* for large positive t is of the same sign). Then, according to BzckN aua f)(1), pick out those elements z to which a natural number m(a) belongs in such a way that -m t < z < m- t. The range of these elements z did against the two

Bachmann's basic operations o - *b* and *{a- -}- b-* c completed, contains

However, the elements o ö and eJh are always added to two of its elements a and *b*. For example, the elements I, t and UK° belong to the elements z, but not the element t°. And thea is both -l f and the quotient of l and UK°. The fact that the set of elements z of f2(t) is closed against Bachmann's basic operations can easily be deduced from the fact that with e, *b,* c the elements of highest order in t obtained by Bachmann's two operations also become infinite.

Bzciiaiznn has further noted that in a mason one can obtain at most those numbers From f)(t) on 0, l and t which are in the form
/(1) -j- t y(1), where /(t) and p(t) together with all their conjugates are bounded functions. This simply follows from the fact that 0, l and l can be represented in this form and that the application of Bachmann's operations to such numbers again leads to numbers that can be represented in this way. The liver will easily realize this. It follows, however, that there can be no general Bachmannian constructions for the altitude intersection of an equiaxed triangle. If you take the two end points of the Baais as the zero point and the entry point of the z-axis and assume the apex at (IJ2, t), then the height intersection point is at (US, l/4 t). But 1/4 l cannot be one of the numbers /(t) -)- f p(1), since none of them can grow beyond all limits for t -+ 0.

6. This paragraph should be discussed in more detail. Completely iiquivalent to the standard ruler is a ruler to be

I will show you how to use an *angle bisector,* i.e. an instrument with which you can bisect any angle. I will show that: 1. the bisection of any angle is one of the constructions that can be carried out with a ruler and a rule. 2. an existing line on an existing straight line can be bisected from an existing point on either side using a ruler and an angle bisector. For I. think of the calibration distance from the bisector of the angle to be bisected on both sides of the angle. Draw the parallels to the angle legs through the end points of these two gauge lines. They intersect at one point of the angle bisector, and obviously you can get both angle bisectors (the inner and the perpendicular outer one). Quote 2. First consider how you can draw parallels with the angle bisector and the ruler. To do this, make sure that you have the right angle involution available at every point. This is because you place any angle at the point in question and draw the two bisecting angles. These are two mutually perpendicular straight lines in the angle bisector. The two bisectors of this right angle form a second pair of perpendicular lines in the angle axis. The two pairs of perpendicular straight lines define the right angle involution, and therefore you can now find the perpendicular to each straight line at this point using the ruler alone. Since this is possible at every point, you can now calculate the perpendiculars at the two end points of a line *A, B* and draw the bisectors of the right angles obtained in *A* and *B*. *This gives you the corners of a square.* This gives you the corners of a square with edge length *AB*. Since you now have two pairs of perpendiculars in the sides, you can use the ruler alone to find perpendiculars to each straight line (§ 3). If you now want to transfer a line given on the leg of an angle to the other leg, construct an angle bisector and draw the parallels to the angle bisector through the end points of the line. If you intersect this with the other leg of the angle, you will have a line of the same length. The only thing left to do is to move this line on a straight line in such a way that any one of its end]3points has a given position. This can obviously be done by repeatedly drawing parallel lines.

Even if our theory only depends on the assumed feasibility of bisecting the angle, the reader is nevertheless entitled to ask how one should think of such a bisector. It should be noted that the angle bisector can be realized, for example, on transparent paper by means of pepier ballet. Plan fold the paper so that the two sides of the angle are congruent. If you also note that you can also produce the straight line connecting two existing points by folding the paper, you still have the result that these two operations dea **Papierfaltcns** represent a complete equivalent of the constructions with ruler and calibration measure.

I also mention a third operation by folding paper. Let there be a straight line 9 and a **pair of points** *A, B* on a transparent sheet of paper. You are to intersect the straight line y with the circle *A[AB j.* This is achieved by folding the paper in such a way that *A* remains lying, but the point & is on p. These three operations of the T*epier/eftem aare thus a *complete substitute for compass and ruler.* If we also note that the constructions of the third degree to be discussed in § 16 can also be accomplished by folding paper *in one direction*, we have an impression of the scope of this simple means of construction. It should be noted in passing that a *regular* fiia}ec£, for example, can be produced by inserting a parallel-cut strip of paper and carefully tighten it.

§ 13. the triplication of the Winliel8 and the doubling of the dice as examples of non-square constructions

Both problems lead to equations of the third degree that cannot be solved by square root expressions. The *doubling of the cube,* also known as the Delian problem, is the task of finding the cube from the edge a of a cube.
to find the edge a z = 'J/2a of the cube of twice the volume 2 a°. The equation of the third degreea therefore applies to z

$$x^3 a^3 - 2a^3 = 0 \quad (1)$$

or, what daselbo is,

$$z° - 2 = 0.$$

If an angle e is given, the two points (I, 0) and (cos et, 8in 'x) are known on the periphery of the unit circle z· -J- y° - I = 0. Wanted
becomes the point (coa zJ3, sin 'x/3). Since sin otJ3 =)/I - cost 'x/3, it is a quadratic task to find the 8in ot/3 from cos "J3. For z = 2 cot otJ3 the equation of the third degree applies

$$H - 3 z - 2 \cos 'x = 0 . \quad (2)$$

Because cos 3Q - 4 cos°Q - 3 cost, cos et = 4 cos- z/3 - 3 cos ot/3, i.e. 2 cos'x = (2 cos **z/S)°** - 3 (2 cos ot/3). By the way, 2cos « -|- 2 / 3
and 2 cos -3 ——— of the equation (2). In
particular, for et =/3 - 60° due to coa u/3 - 1/2

$$s3-3s-I-O. \quad (3)$$

This equation therefore satisfies 2 cos n/9 - 2 cos 20°.

It is easy to see that both equations (l) and (3) have no rational root. If you set z = mJn in (l) and (3) with (m, tt) = 1, i.e. with non-divisor integer rational numbers m and n, y o u have ei° - 2a° = 0 or m° - 3ma° - n° = 0 .

In the second case, every prime factor of si would have to merge into tt and every prime factor of n would have to merge into m. But since m and tt are divisors, z = 1 or z = - I. Both numbers are not sufficient (3). In the first case, every prime factor of n would have to merge into m. So tt = l or n = -1. It is sufficient to consider n - 1 further. For in, m^ - 2 = 0. But there is no integer m whose third power is 2.

Third-degree equations with rational coefficients without rational roots are irreducible'). Now the 'Solz applies: *An irreducible equation of* the third *degreec (with* rafionnfen Aoe//iziealenj Änl *no dumh a square root expression representable zero8tell8.*

Since, according to § 6, all points which can be constructed with compass and ruler have coordinates which c a n be obtained by square root expressions from the coordinates of the given points, it follows from this theorem, that the edge of the double cube cannot be constructed on the edge of the single cube with compass and ruler and that 2 coa 20° cannot be constructed on 2 cot 60° with compass and ruler, that alao the angle of 60° cannot b e divided into three equal parts by constructing with compass and ruler. This theorem meets with as much interest in non-mathematical circles as it does among economists. It says that the problem cannot be solved by using the compass and ruler in the way described and precisely defined in § 6. It does not mean that the problem cannot b e solved by using these instruments in a different way, e.g. by trial and error. The very fact that a given task is impossible to solve in a given way is a source of astonishment in lay circles. Obviously, understanding these things requires a certain familiarity with mathematical thinking, and yet nobody is surprised that you can't draw a circle with a ruler alone. This is also an impossibility. It is true that primitive, but fundamentally similar to the difficult to understand one.

The following proof is based on a method discovered by your LNDAtf in 1897. It aei

$$x^3 + a_1 x^2 + a_2 x + a_3 \tag{4}$$

an irreducible integer rational function of the third degree in a body A.

') Algebra generally **calls** an equation and the polynomial on **its left-hand side** irredusible in the body of rational numbers if the polynomial has rational coefficients and cannot be **decomposed** into factors of at least the first degree8 with rational coefficients. A rational root gives rise to a linear factor **with rational coefficients**, and a redusiblea (i.e. not irredusiblea) polynomial of the third degreea with rational coefficients has at least one linear factor with ratio- nal coefficients, alao also a rational zero.

4-

The coefficients of (4) belong to *K*. *By* definition, however, (4) cannot be decomposed into rational factors of at least the first degree with coefficients aaout *K,* or what is known to be the same: (4) has no zero *that* belongs to *K.* I now take up the explanation of the structure of square root expressions already given in § 6, in particular what was said there about square root bodies. Let us assume that a square root expression belonging to *Kg* but not yet to *K*

$$x_1 = a + b\sqrt{R} \tag{5}$$

aei zero of (4). You will immediately notice that

$$x_2 = a - b\sqrt{R} \tag{6}$$

zero of (4) iat. If you insert (5) into (4), you get a square root expression

$$A + B JR \tag{7}$$

with coefficients aua *Kg z* . Substituting (6), we obtain

$$A - B JÄ. \tag{8}$$

If (7) is to disappear, then *A* -- *B* -- 0 must abe, because aonst

$$\sqrt{R} = -\frac{A}{\cdot}$$

would be a number from N m$_l$. But this is not the case, because then *K* -- *Kg* , would be, while z is not yet in *K"* _ ;. So if (7) is zero, ao also (8). The zero z iat different from z, since sonat z, = a would be, therefore z, in Aq $_{-1}$
would have been. Now

$$a_1 + x_1 + x_2 + x_3 = 0\,,$$

if zt is the third zero of (4). a is the coefficient of z° in (4). Because of (5) and (6) iat therefore

s,- -o,-2 .

z_3 is therefore a number from i_n , dv belonging to this body a and at. According to the definition of a body, the application of the four basic types of arithmetic to numbers of a body leads to numbers of the same body. As a first result we can note awith : If an integer rational function (4) with coefficients from *K* has a zero that belongs to a body *Kg*, ao it also has a zero that belongs to N. If one repeats this conclusion n times, one recognizes that (4) must have a zero that belongs to *K* itself. But since (4) is assumed to be irreducible in *K*, this is impossible. This proves the theorem that an irreducible equation of the third degreea cannot be satisfied by any square root expression over the body of its coefficients. This also proves that the multiplication of the cube and the trisection of the angle of 60° cannot b e accomplished with compass and ruler.

There are, of course, angles that can be divided into thirds by constructing them with a compass and ruler, e.g. the right angle, as the angle of 30° can be constructed with a compass and ruler from the unit of the line drawn on its one leg. However, since the angle of 60° cannot b e divided into thirds in this way, there can *b e no standardized construction method* that would result in a tripartite division when applied to any angle. The impossibility of a general tripartite construction can also be seen directly as follows: Set 2 cos z = a in (2) and then enter z - *Z(a)|N(a)* with polynomials £(a) and *N[a)* that are not related to divisors. This leads to fi°(a) - 3Z(a) *N*(a) - a N°(a)* -- 0. According to this, *N[a)* -- const a must first be, because aonat fi(o) and *N[a)* would not be divisors. There would therefore have to be a polynomial fi(a), for which

$$1\^(o) - 3 Z(o) - o - 0$$

holds, with variable a. Therefore, a is divisible by fi(a), i.e. there is a number c such that fi(a) = a c. Then the equation applies for variable o and fixed c

$$c° a° - 3 c - I = 0 ,$$

and that is obviously nonsense.

This also touches on a more far-reaching question that has not yet been touched upon. A leading algebraist of modern times, **Mr. vN nER WzzRDEN,** instructs us geometers "that in a geometrical problem it is not a question of finding a construction for every special choice of given points, but that a general construction is required which (within certain limits) always gives the solution. Algebraically, this comes down to the fact that one and the same formula (it may contain square roots) for all fourths of a, *b,* ... within certain bounds gives a reasonable solution z which satisfies the equations of the geometric problem. Or, as we can also say, the equations by which z is determined, and the square roots etc. by which we solve the equations, must remain meaningful if the given elements o, *b,* ... are replaced by *indeterminates.* For example, if we ask whether the tripartite division 'les angles with a ruler and compass" [a problem that can be applied to the solution of equation (2) can be traced back] "the question is not whether a solution of equation (2) can be found for each specific value of z with the help of square roots; rather, the question is whether a general solution formula of equation (2) exists; in other words, a solution formula that remains meaningful for indeterminate z." So much for **VAN DER AEItDEx.** Elsewhere, the following definition is given: "An indeterminate is nothing but an arithmetic symbol."

This book takes a fundamentally different standpoint, which can be easily compared with van der Waerden's using the example of the trisection of the angle. **VAG DER WAERDEu** proves that z° - 3 z - 2 cos z at rariah/em z is an irreducible polynomial in z, i.e. that there is no square root expression over the body of coefficients *K[I ,* cot *a.j* satisfying equation (2). This result would just indicate that there is no *uniform* construction method with compass and ruler for the trisection of the angle, i.e. a method that leads to the goal when applied to every angle. However, this fact does not exclude the possibility that for individual values of ", such as the angles a/2'^ u/2 (a, m whole), trisection with compass and ruler is possible. Even more: From the van der \Vaorden's statement does not automatically follow that a three-part method with compass and ruler does not exist for every specially given angle. WERNER **\VzBER,** who has dealt with these questions of a uniform construction in two works, gives the following example, among others: The arbitrary numbers z_l , z are assigned a number y, so that the following applies: For z_l /z = a/2"',
c, m is completely rational, let y = tu z, with the smallest useful m. Other-
if y - z,. This problem can be solved in any particular case with a ruler and a compass, and yet there is no uniform construction method with compass and ruler, because the number of construction steps required to solve it depends on m. So while it follows from the proof of this book for the impossibility of the trisection of the angle of 60° with compass and ruler that there can be no uniform construction method with compass and ruler for the trisection of the angle, it does not follow from van der Waerden's result that there is even one angle for which the trisection of the angle with compass and ruler is impossible. Moreover, the difficult algebraic questions connected with the question of the uniform construction have not yet been completely clarified.

The following should be added for the tripartite division of the angle. The van der Waerden's point of view implicitly describes it as mathematically irrelevant that one can trisect the angle n/2 and that one can trisect all angles n/2" (n completely rationally) with a ruler and an equation. For not only does the theorem of the impossibility of a general, i.e. for a continuously variable angle, three-division construct apply, which has been proved in two ways, but it is also true that the angle n/2 can be trisected with a ruler and a square.

tion, but as a generalization even the theorem applies: *There is a* divided *construction* znif Dire'if *and Ziirkel,* which lie/teifficA the yrisellion *for* an ***infinite*** number of *angles.* According to **vAN DER WAERDEN,** however, the fact that all the angles a/2^ can be trisected would only be of mathematical interest if this were possible according to a uniform procedure. I prove the theorem that there is no uniform trisection method for an i n f i n i t e number of angles using a new method of function theory in this problem area. It is assumed that for an infinite number of angles ng, tt - 1, 2, ... from [0,2 a] there is a uniform tripartite construction. Then, as one w o u l d draw in exactly according to Landau's method from p. 51/52, the equation

$$4 z^3 - 3 z - \cos z, - 0$$

not only for each n a root from the body fi (cos eq), which results from the adjunction of cos "" to the body of the national numbers, but this root could also be obtained in a uniform way from the rational numbers and cos eq by means of the four basic arithmetic operations. However, the three roots of this equation are

cos (/3) , cos ((o -I- 2 n)/3) , cos ((o -|- 4 n)/3) .

For infinitely many *n,* one of these would then have to be a rational function

$$r(\cos\alpha_n) = \frac{A_0 + A_1 \cos\alpha_n + \cdots + A_k \cos^k \alpha_n}{B_0 + B_1 \cos\alpha_n + \cdots + B_l \cos^l \alpha_n}$$

with rational coefficients independent of n. However, according to the identity theorem of function theory, this means that one of the three equations

cos (z/3) r (cos z) , cos ((z -]- 2 n)/3) - *r* (cos z) , cos ((z -I- 4 a)/3} *r* (cos z)

must apply to all values of the complex variable z. However, this is not possible because the right-hand side has the period 2 a, which the left-hand side does not have.

This simple tlberlegung shows how misleading authoritative statements or assertions are, even if they emanate from an authority such as the leading algebraist mentioned above. With the same right with which von **DER** WAERDEN rejects the question of special tripartite constructions and thus also the angles that can be trisected with compass and ruler, one could also reject the question of the regular n-corners that can be constructed with compass and ruler, since there are no uniform, i.e. for all n

valid construction. This book, on the other hand, adheres to the tried and tested position that every question and opinion is permitted. Whether one finds the question or the results interesting and thus reasonable is, of course, a matter of taste refined by experience. Van der Waerden's point of view can, of course, be saved in this question if one wants to interpret his passage "within certain limits"') to mean that the construction in question should apply not only to one, but to a finite number of special cases. For the remark applies that one *can trisect* eM/icfi eiefe tnith *ruler and compass* trisectable *angles by* a petneimatne *construction.* Let there be z , k =1, 2, ..., n infinitely many angles for which the equation

$$4 z° - 3 z - \cos zt = 0 \qquad (2t)$$

is solvable by a square root expression, so that cos ('xt/3) can b e represented by a square root expression built over the body of rational numbers°). Add to the body of rational numbers all the square roots required for these finitely many angles. This gives you a square root body K in which the solutions cos (zt/3) of all n equations (2t) lie. Plan then construct a rational function *P(z)* with coefficients from K using any of the usual interpolation methods, such that *P[Zoe a.;)* - cos (zt}3), (£ = 1, 2, ..., n), i.e. equal to a given element of the body K. Thus, we have found a tripartite construction common to all n angles. This is because in *P[z)* we have a rational function of z common to all a angles, in which we only have to insert the cosine of the angle to be trisected for z, and the coefficients of this rational I'unction are square root expressions over the body of rational numbers.

Finally**,** I do not want to conceal the fact that, despite all the aversion I instinctively feel towards it, Mr. **WEaER's** attitude also has its good points. For it spurred me - in addition to Mr. **WEaER**'s thorough work on this question - to think about things.

Finally, the question of whether the three-part *angles formed* with *compass and ruler are the exception or* the rule should be addressed. First of all, it should be noted that if m and a are natural numbers, then no angle u 2"/3m = z can be divided into thirds with compass and ruler; otherwise a/3 - z m/2^ could also be divided into thirds. On the other hand, all angles a tn/2^ can be divided into three using a compass and ruler.

') But what are "limits" for indeterminates?

•) This contains the presupposition that the given cos 'xy square root expressions should also be over the body of the rational numbers.

each of the two angular locations is densely distributed everywhere'). This already shows that there can be no general tripartite conatruction. But now the following is added: If you put 2 cot e - o in (2) and if o *has* a *transcendental value, then a cannot be divided with a compass and a ruler.* To prove this assertion, according to the method of proof developed in this paragraph, it is only necessary to show that

$$H - 3 z - o = 0 \tag{2'}$$

does not have a solution that can be represented as a rational function of the trangent number o with rational ifo coefficients. This is shown as follows: If one inserts such a rational function of o into (2'), ao results in an algebraic equation with rational coefficients for a, which cannot exist because of the tranacency of o, or else the resulting equation is identical, i.e. satisfied for variable o. However, this contradicts the impossibility of a general tripartite conatruction.

Daa *result: The set of angles that cannot be divided into three parts has the power* dee Jf'mtititiiime, those *that can be divided into three parts are countable.* (They are to be counted among the angles with algebraic e = 2 cos z.) Among these, those for which coa e itself is a square root expression over the body of rational numbers should be of particular interest.

For angles with rational coainua, L. E. Dlcxsox has settled the question. His solution to the problem is: If p and g are natural numbers, (p, g) = 1 and g > I. It is impossible to trisect an angle z, deaaen Coainua p/g iat, with circle 1 and ruler, if one of the following three cases exists: 1. g iat not divisible by a third power of a natural number > 1. 2. g = c° d, c > 1, d > 2, d not divisible by a third power of a natural number > 1.
3. q = c- d, c > 1, d 1 or *d 2*, if the equation *r**- 3 *r o* 2 p|d has* no natural number but if g = c-d, c > 1, d = 1 or d = 2 and the equation mentioned under 3. has a natural number *r* < 2 c as a solution, then the angle ot can be trisected using a compass and ruler.

L.E. Dioxson also emphasizes that among the 71100 angles whose coainua p/g has a denominator g < 343, there are only 38 angles that can be trisected with compass and ruler.

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*) The fact that the angles n 2"/3 m are densely distributed everywhere can be seen as follows: Ea suffices to **show** that the numbers 2"/m on > 0 are densely distributed everywhere. lat then 0 is an arbitrary number, ao let s aua 0 s < 1 be arbitrary and determine tt ßo that
(a) z/2^ < s is. Then determine m ao that (b) l/(m -J- 1) < s/2^ N l/m. Then (c) l/(tti -{- I) < s and (d) 2"/(m -j- 1) < z 2^/m . For the length of the inter- valle, in which s i s thus inachloaaed according to (d), (e) still applies after (c) and (d)
z-/=(+ H < - a-/ < '=(+ i)J - ' Ju - Ug +i)J < ' J(i - 3'j," "ii the
The assertion proves that s can be arbitrarily specified with 0 < s < '/.

It can therefore be said that angles that cannot be divided into three parts form the rule, while those that can be divided into three parts form the receptacle (always in the area of the compass and ruler).

Sometimes the choice is also made among the means of construction *u'illkiirlioher auxiliary points*, e.g. in a known method for bisecting a distance. Here, too, the book takes the view that such arbitrary auxiliary points should always be taken from the range of the given points or the points already constructed from them in the manner always described, or that precisely defined points should be added to the given points as necessary. This explains exactly the conditions of solvability and the means required for this, including any necessary auxiliary points. In addition, the view of arbitrary auxiliary points as variables or indeterminates is of course justified and interesting.

§ 14 Regular polygons

The impossibility of constructing angles of 20° with a compass and ruler from the unit of length marked on one leg of the angle, as demonstrated in the previous paragraph, means in other words that it is impossible to construct a *regular neuneok* described on a periphery from a given circle radius. For this would mean the construction of angles of 40° from the given pieces. This formulation raises the question of the congruence of the regular siehetiecR with compass and ruler. The algebraic equation, which has no construction, is a circle division equation. lat the heptagon is inscribed in a hreia of radius 1 and complex numbers, its corners in the points

$$s = l, \quad s = \exp i \quad , \qquad i-1,2,\ldots,6$$

can be assumed. However, these are the seven roots of the equation z'- 1 = 0. If the zero z = 1 is split off, the remaining roots of the circular division equation are sufficient

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0\,. \tag{9}$$

Algebra teaches that this equation is irreducible in the body of rational numbersl). We derive from it an equation for the digits of the vertices of the regular heptagon. These digits are

$$\frac{2n}{7} \quad \text{COB}\, 6\,\frac{2n}{7}\,, \quad \text{eos}\, 2\,\frac{2\pi}{7} \quad \text{COB}\, 5\,\frac{2n}{7}\,, \quad \cos 3\,\frac{2\pi}{7} = \cos 4\,\frac{2s}{7}\,,$$

We obtain from (9) an equation for theae double coginus, each of which is an abacus of two vertices of the heptagon, by inserting in (8) as new

') See footnote -) on p. 85.

Introduce the unknown z -l- 1/z = fi. (9) is a drawn reciprocal equation. We divide by H, which gives us

$$x^3+\frac{1}{x^3}+x^2+\frac{1}{x^2}+x+\frac{1}{x}+1=0 \tag{10}$$

and note

$$x+\frac{1}{x}=Z,\quad x^2+\frac{1}{x^2}=Z^2-2,\quad x^3+\frac{1}{x^3}=Z^3-3\,Z.$$

Then (10)

$$Z^3+Z^2-2Z-1=0\,.$$

Just as above with the equation for the thirding of the angle of 60°, you can see that this equation is irreducible in the body of rational numbers. Therefore, according to the general theorem proved above, the *regular heptagon on the radius of the circumscribed circle cannot be constructed with compass and ruler.*

If we ask about the constrictability of the *regular eleven-corner* or the *regular three-corner* with compass and ruler, ao similar considerations would lead to equations of the fifth or sixth degree8 for the coordinates aof a corner. Because of the irreducibility of the circle division equation, these equations also turn out to be irreducible. The proof that they cannot be solved by square root expressions, that alao regular pentagon and regular tridiagon cannot be constructed with compass and ruler, must be based on a general theorem. It reads:

Ii'aIlo an equation irreducible in a body K can be expressed by a square root expression built over K, its €lrad must be a T oietis ron 2.

The equation to be solved aei /(z) = 0 and /(z) - z^ -i- z " 1 -l- - -l- ag a polynomial irreducible in the body *K* a- th degreeai i.e. the coefficients *of* /(z) belong to the body *K* and /(z) i8t not decomposable into factors (at least first degreea) with coefficients from *K.* } aei one of its V*roots. By adjunction of } to A, the body *K(5)* arises. It consists of all rational functions of } with coefficients aua *K.* Furthermore, we are dealing with a sequence of bodies R_0 = *K, K , K$, ... , K* such that

always K,$_1$ on *K* is obtained by adding to *K* the)/r, where *r* is a number aua *K* . The body *K* + t then consists of all rational functions of]/r, with coefficients from *K* . Falla } by a square root pressure, it must necessarily be possible to represent such a square root solid A",. Then all numbers of *K( )* also belong to *K*",. We express this by saying that R.(}) is a *lower body* of Jfp or *Kg* is an *upper body* of *K((*.

To prove the proposition stated above'), one introduces the notion of the fiefotiupradee aa body in relation to a lower body. This is understood to be the maximum number of linearly independent numbers of the upper body in relation to the lower body. This is known as the
£ numbers q, o" ... whether the upper bodya 0 is *linear* with respect to the lower body *U* tino6Äöngip, if aua the existence of the relation

$$u_1 o_1 + u_2 o_2 + \cdots + u_k o_k = 0 \tag{11}$$

for £ numbers o" u" ... , u dea lower bodyg follows uy = u =- - = u = 0. If, on the other hand, there are numbers y, u" ... , u" which do not vanish and with which (II) applies, the o" ot, ... whether *linearly dependent* with respect to the lower body *U*. If in 0 there are now n numbers o" o" ... , oq which I. are linearly independent with respect to Cf and 2. every number o of 0 is of the **form**

$$o = u_1 o_1 + \cdots + u_n o_n \tag{12}$$

with coefficients ig, ii" ... , oq sus *U*, then ii is called the aefniirqrod of *0* with respect to V, and the numbers o" ot, ... , oq are called a *series of 0* with respect to i7. So if, for example, from *K* by adjunction
of$_0$, *rg E K,* wins an upper body *K* of A, so aare 1 and
as numbers of *K* linearly independent with respect to *K,* falla does not happen to belong to the body *K*. For from the existence of *A* -}- *B Qry* -- 0 for two numbers *A* and *B* on Zf it would follow, if *B -f-* 0, that t = -*A/B* belongs to the body *K*. But furthermore, as we have already seen in § 6
any number fi of *K* in the porm *A* -{- *B Qry* -- fi with coeffi sients *A, B* aua *K*. It is alao 1, *)/r* a basis of x_1 with respect to fi and 2 the relative degree of At with respect to If. Similarly, 2 is also the relative degree of *K$* with respect to *K* etc.

The proof of our theorem is now based on the following 's'ils *about the relati "on degree: Int M an upper body of A of relati "on degree g vnd A an upper body of K before relati "on degree Ä, no* ff is *an upper body of K of relati "on degree fi I.*

That ff an upper body of *K* i8t is obvious without further ado. Is then further fi$_2$, .. . , *E p* a Baaig of ff with respect to undet, ... , e, a base of the
with respect to *K, ao* make the y J numbers fi-; et a basis of ff with respect to

') A second, shorter proof can be found towards the end of this paragraph.

on *K*. Because 1. athese y 2 numbers are linearly independent of ff with respect to *K*. From a relation

$$\sum c_{ik} E_i e_k = \sum E_i (\sum c_{ik} e_k) = 0 \tag{13}$$

with coefficients ibid from *K* follows toerat

$$\sum c_{ik} e_k = 0, \qquad i = 1, 2, \ldots, \mu, \tag{14}$$

because the numbers on the left-hand side of (14) belong to d and because the *E!* are linearly independent with respect to A. From (14), however, further follows c; = 0 for all i, *k,* because the c" belong to *K* and the e are linearly independent with respect to N. And 2. every number *E* can be derived from ff in the form

$$E = \sum c_{ik} E_i e_k \tag{15}$$

with coefficients **from** *K***. First of all,** *E* can be represented by the **Baais A₂** , ... , fig of *II* in relation to A in the **form**

$$E = \sum C_i E_i \tag{16}$$

with coefficients C; from A. Then, however, theae y numbers U; from d can be represented by the basis et, ... , e, from d in the form

$$C_i = \sum c_{ik} e_k \tag{17}$$

with coefficients c;$ from *K*. If mttn (17) is inserted into (16), ao the desired representation (15) is obtained. This proves the relative degree theorem.

The relative degree theorem first teaches that the relative degree of the square-Amin with respect to *K* is exactly 2'^ i8t, if the bodies occurring in the chain *K,* Nt, . . . , A", are all different, i.e. if the number)/r, whose adjunction produces *K* + from *K*, does not itself belong to *K.* For then as we have already seen, the relative degree of *K* + i with respect to *K* is exactly 2, and therefore, according to the law of relative degree, that of *K* with respect to *K* is 4, but that of x with respect to *K is* 8, etc.

The theorem, the proof of which is the aim of these algebraic considerations, namely that an irreducible equation which satisfies a square root equation must have a power of 2 as degree, also follows from this theorem of relative degree. For *K[ )* is, as we have seen, a lower body of a *K*". So if we succeed in showing that the relative degree of *K[ )* with respect to *K* is just tt, i.e. equal to the degree of the polynomial /(z), ao it follows from the theorem of relative degree that a is a divisor of 2"', i.e. itself a power of 2.

To do this precisely, I will first determine the relative degree of N(}) with respect to *K.* To do this, I must first show that *K(()* has a base with respect to *K.* It will turn out that not only the numbers 1, $, ... , }^ ' are linearly independent with respect to *K,* but that each number X of *K(()* is also linearly independent with respect to 1, }, ... , }^ ' in the form

$$X = \alpha_0 + \alpha_1 \xi + \cdots + \alpha_{n-1} \xi^{n-1} \qquad 18)$$

with coefficients zy from *K* can be represented completely and linearly. The linear independence of 1, $, ... , }^ ' follows from the fact that /(z) is irreducible. If 1, }, ... , }" ' were linearly dependent, there would be a relation

$$\beta_0 + \beta_1 \xi + \cdots + \beta_{n-1} \xi^{n-1} = 0 \qquad (19)$$

with non-vanishing coefficients from *K.* In addition to the equation of n-th degree /(}) 0, another equation of lower degree (19) would suffice. However, this contradicts the irreducibility of /(z), because the irreducible equation is also the equation of lowest degree with coefficients from *K*, which is sufficient. However, it follows that every number X from *K(()* can be represented in the form (18) with coefficients from *K*: We know that every number from *II(()* can be represented rationally with coefficients from *K* by 3. Such a rational function of } is the quotient of two whole rational functions of }. It may be assumed that both are at most of degree tt - 1, because d" and thus also the higher powers of can be replaced by the lower powers of } using the equation /(}) - 0. can be represented completely and linearly with coefficients from *K.* Is then

$$X = \frac{Z(\xi)}{N(\xi)} \qquad Q0)$$

such a representation of A, and if fi(}), *N(1* are polynomials of degree at most (tt - l)-, then a representation

$$\frac{Z(\xi)}{N(\xi)} = \alpha_0 + \alpha_1 \xi + \cdots + \alpha_{n-1} \xi^{n-1} \qquad (?1)$$

with coefficients z from *K,* or in other words: it is asserted that every integer rational function of at most (n - l)th degree Z(}) can also be written in the form

$$Z(\xi) = (\alpha_0 + \alpha_1 \xi + \cdots + \alpha_{n-1} \xi^{n-1})\, N(\xi) \qquad (22)$$

with suitable coefficients " from *K*, no matter what an entire rational function of at most (n - 1) -th degree with coefficients from *K* is also *N(().* To prove this, we first reduce the products *'N)* mod /(}),

i.e. we express the higher powers of } resulting from the auamultiplication of *('N[ )* by means of J(}) = 0 by the lower powers of
}, 8o that a representation

$$\xi^k N(\xi) = \sum_0 G_{ki}\, \xi^i = G_k(\xi), \quad k = 0, 1, \ldots, n-1 \qquad (23)$$

with coefficients from *K*. Then one further shows that the polynomials G (z) of the indeterminates z are linearly independent with respect to *II* as polynomials of these indeterminates. For if there were a relation

$$\sum_0 G_k\, G_k(x) = 0 \qquad 24)$$

with non-vanishing coefficients G¿ from A, then also

$$\sum_0 G_k\, G_k(\xi) = 0. \qquad (25)$$

This follows because of (23)

$$\sum_0 G_k\, \xi^k N(\xi) = 0. \qquad (26)$$

Here *N[$)* 0, because /(}) - 0 i s the equation of lowest degree with coefficients from N, which satisfies }. Therefore

$$\sum_0 G_k\, \xi^k = 0. \qquad (27)$$

But then all H - 0 for the same reason just mentioned. Since the C;(z) a r e linearly independent with respect to *K*, the powers z°, z', ... , z^ ' by these a polynomials H (z) and thus also all polynomials (a - 1)-th degree of z from *K* by the 9 (z) in the form

Cy(z) (28)

completely and linearly with coefficients from *K*. This applies in particular to the polynomial fi(z), with which the numerator in (20) is formed. Therefore

$$Z(\xi) = \sum \alpha_k\, G_k(\xi) \qquad (29)$$

with suitable coefficients zy from K. Because of (23) it follows from (29)

$$Z(\xi) = \sum_0 \alpha_k\, \xi^k N(\xi). \qquad (30)$$

This is the representation we are looking for, from which i t follows, as already stated, that the tt numbers 1, }, ... , }" ' form a basis of *K[$)* with respect to *K*. tt is therefore *the relative degree of K[5 j* iti heztiy an/ *K.* Now all that remains is to show that *K*", **has** a relative degree with respect to its lower body *K($ j.* Then it follows from the relative degree theorem that n is a divisor of 2"-, alao aelbat a power of 2. In any case, *Kg* is an upper body of *K[$)* and of *K.* The theorem of relative degree teaches in its proof') that *K*", a basia e" ... , e , p = 2-', with respect to ß. On this basis of *K*", with respect to *K,* however, a basis of *Kg* with respect to *K( )* can be selected. To do this, take the maximum number 2 of linearly independent with respect to A(}) among the 2" numbers g, ... , e . The unknown number 2 is then the relative degree of A" with respect to *K($ j.* The fact that this relative degree remains unknown, however, does not alter the fact that n is a power of 2, as already mentioned.

The condition found for the solvability of an equation by square-root expressions is a nntirendiye condition for the irreducible equation in *K* to have at least one root that can be represented as a square-root expression. Incidentally, the consideration made in the previous paragraph concerning equations of the third degree teaches that an irreducible equation of the third degree in which a root can be represented by a square root expression has at least a second root with the same property.

The fact that the condition found is not also fiinreicfietid for the solvability of an equation by a square root expression is illustrated by the example of the fourth degree irreducible equation in the body of rational numbers

@ -j- 2 9° -l- 8-j- I = 0 (31)

One immediately overlooks the fact that this equation has no rational root, klar. because only the numbers -{- 1 and - 1 come into question. Therefore, the polynomial on the left-hand side of (31) cannot be decomposed into a linear and a cubic factor. However, it cannot be decomposed into two second-degree factors either. Because if there were a decomposition°)

W+#2+*v+*=(/+-v+bO'+ v+M (32)

with whole rational coefficients a, ö, c, d, multiplication and comparison of coefficients initially teaches us that b d = 1. Only the two fits 6 = d = + 1 and h d = - 1 are possible. However, if you determine the

') Cf. p. 80.

•) First of all, the decomposition (32) has to be applied with rational coefficients, but in the present case it is easy to **recognize the** fact known from algebra that **the** coefficients o, b, c, d are then necessarily integers. **See** also **BIzazaezce- Bzrzs,** Vorlesungen über Algebra, 5th ed. p. **229** Gauaaaches Lemma.

multiplying out the coefficients of and y, w e find e + c = 0 and 8 = ed -J- bc = b (a -l- c). However, this is not compatible with each other. So (32) is irreducible. The cubic resolvent of this equation to be explained on p. 72 is

$$Z' + Z' - 1 \qquad 0. \tag{33}$$

It is also irreducible, as it has no rational root. Now **if** (31) has a square root expression for the solution, it would, as already noted, have a second square root expression for the solution. But then, according to P. 73 ouch a square-root expression exists that satisfies the cubic reaolvent. For according to the connection to be described o n p . 73, the sum of two roots of the fourth degree equation (31) divided by 2 and then squared gives a root of the cubic reaolvent. There is therefore a contradiction to the theorem from § 13.

In Galoigg's theory of equations') the sufficient condition for the solvability of an equation by a square root expression is also given. It states that the Galois group of the equation must have an index series consisting of all numbers 2. However, the condition can also be expressed using the term relative degree developed here. It then reads as follows: Plan adjoin to the body A all the roots }, }" ... , }q t of the irreducible equation of the nth degree /(z) = 0. The body thus obtained at A(}, }" ... , }q t). The necessary and sufficient condition for there to be a square root expression that is /(z) = 0 is that the relative degree of the body *K(,* }t, . - 1" - i) with respect to *K* is a power of 2.

This theorem once again shows the impossibility of doubling the cube, the tripartition of the angle, the construction of the regular 7-corner and 9-corner, but now also the new impossibility of constructing the regular 1I-corner and the regular 13-corner with compass and ruler.

Now we can also answer the question of which regular polygons can be constructed with a compass and ruler. In algebra°) it is shown that the nth root of unity exp (2 ia/a), on which the construction of the regular n-vertices depends, satisfies an equation of degree q(n) that is irreducible in the field of rational numbers°). Here q (n) is Euler's$_e$ function. Is n = p°' p{-... p)* the decomposition of n into pairs of prime numbers with different parts

') See e.g. BniazRBAGZf-BzTfEB, Vorlesungen über Algebra, 6th ed. p. 312.

•) Plan **compare e.**g. L. BizBzRBzca, Vorlesungen über Algebra, fi. **ed.** Leipzig 1933.

°) To prove a necessary condition for the number of vertices of regular polygons which are conatruable with compass and ruler, it is sufficient to consider ea, n = p and a - p-, p Primaobl, su. Because, iat the regular m-corner with **compass** and ruler constantpotencies, ao is known to be

$$\varphi(n) = p_1^{\nu_1 - 1}(p_1 - 1)\, p_2^{\nu_2 - 1}(p_2 - 1) \cdots p_k^{\nu_k - 1}(p_k - 1)\,.$$

Now use a compass and ruler to create the regular n-square from the radius of the circle.

can be congruent, then cos " , a ' and thus also e " square-

root expressions over the body of rational numbers a. According to the theorem on p. 59, the degree g(a) of the irreducible equation, which is

e - suffices to be a power of 2. Then the odd primes merging into n must appear as factors in the first potens and must all be of the form 2- -J- l. It is unknown whether there are infinitely or finitely many such primes. The ones known to date') are 3, 5, 17,

bar and is n divisors of si, ao is also the regular a-square to be found with compass and ruler. Now the primitive p-th or p-th roots of unity of the equation of the circle are sufficient

$$F(x) = (x^p - 1)/(x - 1) = x^{p-1} + x^{p-2} + \cdots + 1 = 0$$

resp.

$$F(x) = (x^{p^2} - 1)/(x^p - 1) = x^{p(p-1)} + x^{p(p-2)} + \cdots + 1 = 0.$$

The shortest proof of the irreducibility of the circle division equation for the case of prime dilpotents is probably the following one found by Pnzuzm. In both cases, F(1) = p. If J(s) were not irreducible in the body of rational numbers, ao there would be a decomposition according to the Gaussian Lensma

$$F(x) = f(x) \cdot g(x)$$

with integer polynomials /(s), g(s). Where /(z) is the irreducible integer divisor of J(z) for which /(1) = -{- p. If you think of J(s) as being decomposed into a Pmduct of irreducible integer divisors, ao is the same for exactly one of them because J(1) = p. /(1) = -j- p. Now note that the primitive a-th roots of unity are given by

$$\exp\left(\frac{2 i \pi}{n} k\right),\ (k, n) = 1,\ n = p \text{ oder } n = p^2$$

are given, and that each of them, e.g. eap ' p ——— £,) power of any other

of exp 'f ß) iat, because the ~~produ~~ct i *k* with 1 contains all to n divisors.

runs through foreign residue classes. If /(f) = 0, then g(}*) = 0 for pasaendea y. The polynomials /(s) and g(s*) aare therefore not irreducible. However, since /(s) is irreducible, it follows that

$$g(x^\mu) = f(x) \cdot h(x)$$

is with integer polynomial Ä(s). Fors = 1, then, because p(1) = 1

1 p *li(I j,*

which is obviously nonsense.

') If 2" -|- l is to be prime no, ao must obviously aelbßt be a potens of 2. The fact that 2® --]- l is not a prime number can be seen very quickly: Ea iat 6415 - 2* -]- l m = 6' -[- 2'. Aua 5- 2' - 1 mod 641 follows 5- - 2-- --1 mod 641. Alao - 2,-- = 1 mod G4l ; i.e. 2-- -}- 1 iat divisible by 641. (loose. E. Homers.)

257, 86537. Let us also write the corner numbers under 100 of such regular polygons, which can then be congruent with compass and ruler alone. They are 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, til, 60, 64, 68, 80, 85, 96.

Polygons with the corner numbers permitted here can now actually be constructed using a compass and ruler. If **you** proceed **from the tt-corner to the** 2ti-corner by bisecting angles, the proof only needs to be carried out for odd corner numbers. Such corner numbers are products of **all** the different **primes mentioned** above **of** the **form 2°** -I- 1.

the wiakel and the angle with divisor-foreign denominators and *p$* con-

structed, the angle p is also congruent. Since p_1 llld p are divisors, it is known (Euclidean divisor method!) that there are two integers **(positive** or negative) o and *b, zo* that ep, + öP = 1. Then

is tt -l- o = —— ——. The proof of conatruence requires aomit

"only" for eocene numbers that are prime numbers. It will be shown in the algebra that the division by square roots can then be solved by square root expressions. Here the proof will only be given for n = 3, n = 5, n = 17 and the corresponding constructions will be given. For a = 3, the circle division z° -J- z -J- l - 0 is always quadratic. For n = 5 the

$$x^4 + x^3 + x^2 + x + 1 = 0 . \tag{34}$$

The reciprocal equation is obtained by substituting z + Um = fi into the quadratic equation

$$Z^2 + Z + 1 = 0$$

is converted. The resolution of (34) is therefore reduced to the resolution of the two quadratic equations

$$Z' \text{ -j- } Z - I = 0, \tag{35}$$

$$x^2 - xZ + 1 = 0 . \tag{36}$$

Only (35) is important for the geometric construction. Its two roots aare niimlioh B $\frac{-1+\sqrt{5}}{2}$, $Z_2 = \frac{\text{-I-jfi}}{}$. Here fi > O, *S* < 0. In the constructive resolution of equation (35) to find the $\frac{}{2}$

Angles 2n the positive of the two roots of the two tracks

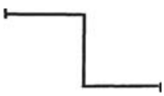
6

equation, i.e. 2 coa. To solve the equation (36) and thus

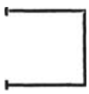
To construct the regular pentagon, you use the arithmetic *angle method,* which is explained here for any equation of the second degree. Plan first applies the ifo coefficient of the equation. Let it be:

$$a_0 x^2 + a_1 x + a_2 = 0\,. \tag{37}$$

First, the coefficient e_0 is plotted in the selected unit of length. If it then has the same sign as ab, ny is calculated at the end point of et



I"ig.39



theu is deducted at right angles to the right. That has a different sign ala e_0 , aO is deducted to the lefta . Similarly, o at the end point of to the right

'a'. 40 or to the left at right angles to an-

depending on whether a has the same or the opposite sign of . Fig. 39 shows the coefficient train of Z-° -l- fi - 1 = 0 and Fig. 40 shows the coefficient train of fi- - fi -l- 1 = 0. If you now want to determine the value of the polynomial (37) for a given value of z, ao mark a point on the line representing ob at a distance of 1 from the starting point and, starting from it perpendicular to thea line, enter the given value of

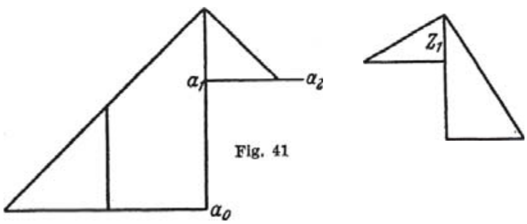


Fig. 41

of z, and swar to the left if z is positive, to the right8 if z is negative. Then connect the starting point of the coefficient train with the end point of z, intersect this straight line with the straight line on which a is plotted, establish a perpendicular at this intersection point and intersect it with the straight line on which a is plotted. The distance between the end point of the coefficient train and the intersection point just obtained represents the value of the polynomialg (37) for the selected z value. So if you want to determine x ao that equation (37) is solved, the end point of the coefficient train and the end point of the second line train, which is therefore called the solution train, must coincide. The correctness of these statements can be verified by means of Fig. 41, if one observes the similarity of the triangles that appear in this figure.

You can find the solution line by drawing the Thales circle as the diameter above the junction of the beginning and end of the coefficient line. You can also carry out the construction with a right angle if you let aleg pass through the beginning and end of the coefficient traina but place its divide on the et straight line. In particular, this results in the construction of Fig. 42 for equation (35).

The algebraic solution of equation (35) results in

$$Z_1 = \frac{-1 -j- \sqrt{5}}{2} = 2 \cos \frac{2\pi}{6} -- 2 \sin \frac{n}{10} = s_{10} .$$

In addition to the original geometric meaning, there is another meaning resulting from this. Here, r_0 denotes the length of the side of the regular decagon inscribed in the circle of radius l. However, the side of the regular pentagon can also be seen directly in figure 42.

In a right-angled triangle fi, - * is one cathetus. The other cathetus is 1. The hypotenuse is s" the side of the regular pentagon, according to of the known relationship s, = 1 + s,', , which can be easily verified.

I now turn to *the construction of the* reqtiföreti sievefinecR. To do this, we need to specify how the circular division equation

$$\frac{S''-1}{5 - 1} = fZ^* - 0 \quad k=0 \qquad (38)$$

can be solved by square root expressions, i.e. can be traced back to a chain of quadratic equations. One nets

$$2\pi \qquad \mathit{ai\alpha}$$

2 coa et + 2 cot 2 n + 2 cos 4 z -l- 2 cos 8 z = ,

2 cos 3 " -|- 2 cos 5 x -I- 2 cos 6 x -I- 2 cos '7x - u_2 .

Then tt -}- u = - 1, tq tt = -4. This can be recognized by expressing the members of u , tig by e and its powers. Then lg + o = - 1, because a zero of (38) iat. If we multiply ig ot, we see that every positive and every negative power of e with exponents 1 bia 8 occurs four times each. Therefore uy is u = -4. Further, we immediately consider that tq > 0, ti < 0. Ea therefore satisfies ii" tt of the equation

$$u^* -j- u - 4 \quad 0, \qquad (39)$$

and it is $u_1 = \frac{-1 + \sqrt{17}}{2}$, $u_2 = \frac{-1 - \sqrt{17}}{2}$, Now set

2 cos 'x -}- 2 cos 4z = v, ,

2 cos 2ot -l- 2 cos8z - r, .

Then

$$v_1 + v_2 = u_1, \quad v_1 v_2 = -1, \quad v_1 > 0, \quad v_2 < 0 .$$

So for r, and v

$$v^2 - u_1 v - 1 = 0 \tag{40}$$

mit $v_1 = \frac{u_1 + \sqrt{u_1^2 + 4}}{2}$, $v_2 = \frac{u_1 - \sqrt{u_1^2 + 4}}{2}$.

Furthermore

2 cos 3 z + 2 cos o et - m_l , 2

cos 6 x -I- 2 cos 7 x - try .

Here too

$$w_1 + w_2 = u_2, \quad w_1 w_2 = -1, \quad w_1 > 0, \; w_2 < 0 .$$

Accordingly, for tr_l and m

$$w^2 - u_2 w - 1 = 0 \tag{41}$$

mit $w_1 = \frac{u_2 + \sqrt{u_2^2 + 4}}{2}$, $w_2 = \frac{u_1 \text{j/u\{ -|- 4}}{2}$

Finally, set z, - 2 cos z, z, = 2 cos4 'x. Then z, + z,= r_l , ztz = iv" zt > z,. So for z and z applies

$$x^2 - v_1 x + w_1 = 0 \tag{42}$$

mit $x_1 = \frac{v_1 + \sqrt{v_1^2 - 4 w_1}}{2}$.

The construction of the regular heptagon is done by solving the quadratic equations (39), (40), (41), (42) using the right angle method. The earliest discovery of Cox **FRIEDRICIt** Gznss, why one can proceed in the described way when solving the circle division equation (38), can be learned in algebra in the treatment of cyclic equations. The treatment of the other circular division equations is also taught.

It is not without interest to get to know a shorte loge *oof* for the necessary condition for constructibility with compass and ruler given at the beginning of the paragraph. I mean the **theorem:**

If an irreducible equation *f(x)* -- 0 *in a body per K can be solved by a square root expression built up by K* (i.e. if it has only one such zero), then *its degree must necessarily be a polynomial of 2.*

The announced short proof goes like this: Add to *K* square-roots and thus generate a sequence of bodies from *K* - - through the re cursion Vj - *K;*1 (Jr; , r; €*l*$_i$ 1 , } - I, 2, ... , s. In the last of these bodies *K* there is a root of /(z). In fiq, /(z) is irreducible. Let *K p* be the body with the smallest number in this sequence of bodies in which / (z) is reducible. The number *k* exists, and it is *k* > 0, u'enn / (z) is at least of second degree. Let /,(z) be a divisor of the lowest possible degree (but at least first degree) of /(z) with coefficients from A . With / (z) has)+/ "(*) is also lst) - ( ) - /"(z) a divisor of /(z) from A , namely a divisor of / (z) that is different from / (z). Otherwise every common divisor of / (z) and (z) is a divisor of lower degree than / (z) of /(z), which also has coefficients from *K*. Therefore, /,(z) /,(z) is also a divisor of /(z). Now, however, / (z) /(z) has coefficients aua Ai -i since, according to the condition /(z) is irreducible in Ai-i, then /(z) = / (z) / (z) e with constant e. Therefore the degree of /(z) is divisible by 2 and twice as large as that of / . This is because /, and / have the same degree. The root of /(z) taken as the square root expression must now be the zero of one of the two polynomials / or /, be. Treat this in *K* irreducible') polynomial in the same way again. Its degree is therefore also either equal to l or divisible by 2. Repeating the conclusion leads to the proof of the stated theorem.

Of course, everything that was explained in the previous paragraph about equations of the third degree follows from it again. However, as has already been shown with the regular pol3-gons, this theorem can now also be used to *prove f/nmöyability for problems higher than the third* 9rodea. Another example will be given. The pentagonal division of an angle leads to the equation

16 coss d' - 20 cos* 0 -I- 5 cos4cos 5 # .

now zaon Taurus 2 cos ü z, then otan becomes the equation of the fifth degree

z' - 5 z* -|- o z - 2 cos5g - 0

led. If you take cos 5ß = 1/3, for example, you get the equation

3 z 15+ 15 z - 2 - 0.

') \If /,(z) were reducible in *Kz,* then every divisor of /t(z) with iioefficients from fi¿ would be a divisor of lower degree than /t(z) of /(z), while /,(z) should be a divisor of lowest degree of /(z) in *K p*.

The fact that this equation is irreducible in the body of rational numbers is proved by the conclusion given occasionally in equation (31) of this paragraph. The given angle can therefore not be fifthed with compass and ruler.

§ 16. third and fourth degree honorsa

This refers to tasks that lead to algebraic equations of the third and fourth degreea and beds of such equations. More precisely, we always understand a conatruction task to be the task of determining the coordinates of the points sought from the coordinates of given points. If the algebraic solution of such a task only requires the solution of third and fourth degree equations in addition to rational operations and quadratic equations, we call the conatruction task a third or fourth degree task. We call these t a s k s , which are also referred to as cubic and biquadratic tasks, in the same breath, because in the theory of algebraic equations it is shown that every fourth **degree** equation can be reduced to a third degree equation, the so-called cubic resolvent, by rational and quadratic operations. It is known that the equation

:r' -J- a z° -J- a z + a = 0 by (1)

the transformation

$$x = y - \frac{a_1}{4} \tag{2}$$

to an equation

-1- *by y* + b$ y* -J- *b* -- 0 {3)

(without member with third power). The cubic resolvent of the same is called the equation

$$z^3 + \frac{b_2}{2} z + \frac{b_2 - 4 b_4}{16} z - \frac{b_3^2}{64} = 0. \tag{4}$$

If q, z" are their three roots, then

$$y = \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} \tag{5}$$

the four solutions of (3), if the signs of the three square roots are chosen such that

$$\sqrt{z_1}\sqrt{z_2}\sqrt{z_3} = -\frac{b_3}{8} \tag{6}$$

i8t. Ea therefore only rational and quadratic operations are necessary to determine the solutions of the biquadratic equation (1) from the solutions of the cubic equation (4). Therefore, if the cubic equation (4) can be solved by square root expressions, the biquadratic equation (3) can also be solved by square root expressions. But also vice versa:

If the biquadratic equation (1) satisfies a square root expression, ao iat the cubic resolvents (4) can be solved by square root expressions. As already noted on p. 64, the existence of a square root expression satisfying (3) immediately implies the existence of another square root expression satisfying (3). But then the last two roots of (3) are obviously also square root expressions. On the other hand, according to (5) and (6), the four roots of (3) are

$$y_1 = \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}, \quad y_2 = \sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3},$$

Therefore did

$$2\sqrt{z_1} = y_1 + y_2, \quad 2\sqrt{z_2} = y_1 + y_3, \quad 2\sqrt{z_3} = y_1 + y_4.$$

öfit the four p-values, the three z-values are therefore also square root expressions.

The fact that in (5) square roots can occur on negative or complex zaMs is irrelevant for the execution of the corresponding constructions with the means discussed in the previous paragraphs dieaeg Buchea. If z = *r(cot* Q + i 8in 'p) is a negative or complex number and $r > 0$ is the absolute value, g is the argument, ao is = (cot 'p/2 -J- i a Q/2). Theae number is therefore determined by halving the angle 'p and taking the square root of the positive number *r*, tasks that are solved geometrically using the means discussed earlier.

Is now further

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0 \tag{7}$$

is an equation of the third degree, the substitution

$$x = y - \frac{a_1}{3} \tag{8}$$

is known to be an equation of the third degree

$$y^3 + b_2 y + b_3 = 0 \tag{9}$$

without a square element. If you then set

$$A - \sqrt[3]{-\frac{b_3}{2} + \sqrt{\frac{b_3^2}{4} + \frac{b_2^3}{27}}}, \quad B - \sqrt[3]{-\frac{b_3}{2} - \sqrt{\frac{b_3^2}{4} + \frac{b_2^3}{27}}}, \quad 3AB - -b_2 \tag{10}$$

and is p = $\frac{-1+i}{2}$ a third unit root, so are

$$A + B, \qquad qA + p^* B, \qquad p^* A + pB \tag{11}$$

the three roots of (9). Apart from rational and quadratic operations, ea alao is only necessary to solve equation (7) from two real or complex numbers to take the third roots. lat again s = *r* (coa Q -{- i sin g), r> 0, any number, so are

$$\sqrt[3]{r}\left(\cos\frac{\varphi}{3}+i\sin\frac{\varphi}{3}\right),\quad \sqrt[3]{r}\left(\cos\frac{\varphi+2\pi}{3}+i\sin\frac{\varphi+2\pi}{3}\right),$$

$$\sqrt[3]{r}\left(\cos\frac{\varphi+4\pi}{3}+i\sin\frac{\varphi+4\pi}{3}\right)$$

their third roots. *Solving the third and fourth roots with real or complete* coe|Jcients *only requires the solution of the* tasks *of dividing an arbitrary* iPin£ef *into three* peiicire *'pearls and* taking *the third root from a pooili en number, in addition to rational and* quadratic operations. These are the tasks of *the 'triaeldion dee* iPsa£efe and the vert'iel- *faohung dee cubes.* The latter name is explained as follows: By taking the third root of *r,* you are to determine the edge of those cubes whose volume is the r-faoh0 VOm volume of the cube with the edge 1. In the case of *r -- 2,* the Ferdoppfiing dea JPür/efe, this problem, known in ancient times, is called daa problem (of the oraclea) of Delos or the *Deliacke problem* (cf. § 13).

The explanations in these paragraphs, which can be found in any algebra textbook, show that in order to solve cubic and biquadratic problems, one only has to deal with the problems of multiplication of cubes and trisection of angles. This statement will be useful for the basic question of the tools that can be used to solve third and fourth degree problems. Of course, there are also direct ways of solving certain third and fourth degree problems that do not have to involve tripartition and multiplication.

§ 16. D* mäD#G61 b# #*I

A slide-in ruler is a ruler on the edge of which two points *A* and *B* are marked and which is to be used as follows: 1. as a ruler for drawing straight lines through two existing points. 2. to query the marked distance *AB* on an existing straight line from an existing point A to either direction. 3. to intersect an existing straight line y with a circle of radius *AB* around an existing point *P from* the center of the circle. 4. to insert the marked line *AB* between two existing straight lines p and *h* on a straight line through an existing point P. Or in other words: Plan the ruler through *P* and place the marker *A* (or *B) on p,* the marker *B* (or *A)* on fi. The slide-in ruler') therefore differs from the standardized ruler discussed in § 11 by the addition of the 4th use rule; as will be shown, this allows all third and fourth degree problems to be solved by sliding in. You do not need any other aid than the slide-in ruler, nor a compass. To understand this, you only have to solve the trisection of the angle and the subtraction of the cube root by means of insertion according to the results of the previous paragraphs. Before we move on to this, it should first be shown that the inset ruler opens up access to problems of the third and fourth degree. This is due to the fact that the mark *B* describes a fourth-order curve if *A is* on the line *g* moves in the same direction and the lineal curve passes atiindently through the fi8th point 0. This fourth-order curve is the *conchoid of* the basic straight line *g* with the pole €I and the interval *AB*. To find the equation of this curve, choose 0 as the origin of a right-angled coordinate system. Choose the y-axis pttrallel to the line q. The

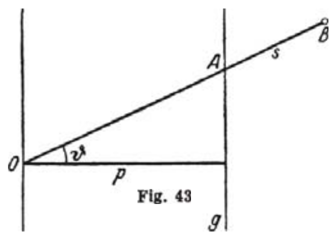


Fig. 43

Equation of p aei :c = p, and s is the distance between the marks *A* and *B*. Then, according to Fig. 43

$$x = p + s\cos\vartheta, \qquad (1)$$
$$y = p\,\mathrm{tg}\,\vartheta + s\sin\vartheta$$

a parameter representation of the conchoid. Elimination of the parameters leads to to the equation

$$(x^2 + y^2)(x - p)^2 - x^2 s^2 = 0 \qquad (2)$$

of the conchoid. The intersection of the conchoid with a straight line *h* therefore leads to an equation of the fourth degree. If you pass *h* through an already known point of the conchoid, ao this reduces to an equation of the third degree. For the construction of the third root of a number tu > 0, choose a point Q of the conchoid as such a known point

') For the practical execution of the insertion, it is oa useful to separate the drawing of the straight lines from the actual process of insertion. For the latter, either a strip of paper is used, on the straight edge of which the <9distance to be inserted has been marked, or a transparent sheet of paper8 on which a straight line with the two marks has been drawn. In both cases, the sheet of paper is brought into the correct position by moving it over the drawing sheet and then the position found is marked on the drawing sheet by piercing it, only to draw the straight line found with the ruler after removing the cover sheet at a distance of 2s from the pole fi. The straight line *is is* to be laid through such a point Q. The straight line *0 Q* then meets the Baaia straight line q of the conchoid at a distance " from the pole. Choose p and Ä so that there is a distance of length 2zris between the straight line *0 Q* and *h* on p and the intersection of p and *h* is also at a distance a from fi. To do this, you can proceed as shown in Fig. 44 by placing a circle of radius s around 0, **assuming** a point *A* on it and then placing a chord of the circle o f length 2nis through it. The straight line on which this chord falls is aei q. The **point** Q is called

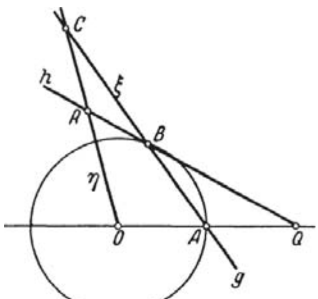


Fig. 44

then on the line *OA* ao assume that *A* is the center of the line OQ i8b. lat *B* is the other point of intersection of the chord of the line 2nie with the circle, ao iat *QB* is the line fi. The parameter of the conchoid is therefore

$$p = \sqrt{1 - m^2 s}. \tag{3}$$

It should also be noted that theae rules require that m < 1 i8t. However, this is sufficient for extracting the third root from any number, since every other case can be reduced to m < 1 by multiplication by the third power of a suitable known number. **Now** insert the distance s on a straight line through 0 **between** q and *h*. This gives you another conchoidal point A on *h* and a point U on q. The length of the line *BO* is }, the length of the line ftO is p. Then, as will be shown aoll,

$$\xi = \sqrt[3]{m}\, s\,, \quad \eta = \sqrt[3]{m^2}\, s\,. \tag{4}$$

For Bewcia, apply the MENEMoS theorem to the triangle *AOO*. Then you have

$$\frac{m\,2\,s}{\xi} \cdot \frac{s}{\eta} \cdot \frac{2\,s}{s} = 1 \tag{5}$$

or, was the same iat,

$$\frac{m\,2\,s}{\xi} \cdot \frac{2\,s}{\eta} = 1\,. \tag{6}$$

Daraua follows

m 2$ 6 (+m 2a
, —<. „*2.')

Consider further the power of the point C in relation to the circle, so you have

$$\xi(\xi + m2s) = \eta(\eta + 2s). \quad (8)$$

From this follows

$$\frac{\xi + m2s}{\eta + 2s} = \frac{\eta}{\xi}. \quad (9)$$

That is why we now have

$$\frac{m2s}{\eta} = \frac{\eta}{\xi} = \frac{\xi}{2s}. \quad (10)$$

From this

$$\xi m 2s = \eta^2, \ \eta 2s = \xi^2$$

follows and

further

$$\xi = \sqrt[3]{m}\, 2s, \quad \eta = \sqrt[3]{m^2}\, 2s.$$

The description of the procedure for carrying out the construction is clear from the above. It should also be noted that the use of the circlea can be avoided. This is because it s e r v e s t o traverse the distance s on the straight line OQ from 0 twice in succession, and furthermore to construct an equiaxed triangle with the side lengths s - twice - and 2 ms. Both constructions can be carried out with the slide-in ruler if it is used as a standardized ruler in the sense of § 11.

Perhaps even simpler is the construction for the *trisection of the angle by* illin *shifting*. If *AOB* in sig. 45 is the angle to be divided into thirds, set e = 2 r and s u b t r a c t the distance *r from R 0* on the leg OB and fill in a perpendicular from the resulting point 6' to the other leg OR. Call this perpendicular p. In addition, draw a parallel to the leg *OA* through U and call this straight line fi. Then insert the distance s on a straight line through *0* between these two straight lines q and *h*. As will be discussed later, there are four ways to do this. The one of the insertion lines that falls within the angle *AOB* performs the trisection of the same. The angle *ISOA* of Sig. 45 is namely '/ @ *AOB*. This is easy to understand elementary geometrically. At the point ff of Fig. 45, we halve the inserted distance Sf' = s = 2 r. Then C3f = r, because U, 'S, 2' lie on the periphery of a circle. The \Vinke1 6Uf' is a right angle. Furthermore, @ *C!OS q C3fO* -- @ *MMS* -t- q df S9 = 2 @ öf'SC= 2 q *SOA*. Therefore @ 'SOA = '/ @ *BOA*.

Let us now consider the conchoid with the pole fi, the base *g* and the interval 2 r. Its equation according to (2) is

(s*+ y)(s-zcosg)*-s'#r'=0. (M)

Here @ *AOB* 'p *was* set, so that p = r cos g must be entered in (1).

The straight line *li* of Fig. 45 then has the equation

y = *r* sin g . (12)

If we intersect (I 1) with (12), ao w e obtain the equation for the abscissa of the intersection points

H - 2 *H r* cot 'p - 3 *x*r** - 2 zH cos g ain°'p -}- r* ain° g cos°g = 0. (13)

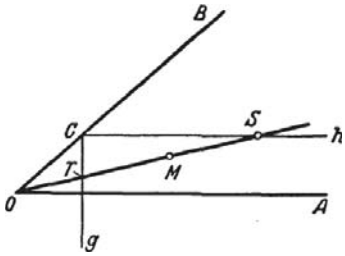


Fig. 45

According to p. 50, } = 2 cos gJ3 satisfies the equation

XS - 3} - 2 cos g0 .

Therefore, p = *r(* satisfies the equation

g'- 3gf'- 2Wcogg- O. (14)

The absciaae of 6 in Fig. 45 is

cr non ip + 2 *r* coa 'p/3 = *r* coa g -1- p .

If we e n t e r p = z - *r* cot g in (14), we obtain

$$x^3 - 3x^2r\cos\varphi - 3xr^2\sin^2\varphi + r^3\cos\varphi\sin^2\varphi = 0\,. \qquad (15)$$

If you compare (15) with (13), ao you have

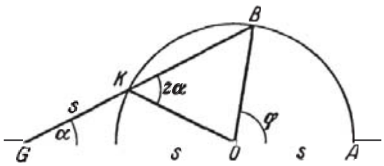
$$x^4 - 2x^3r\cos\varphi - 3x^2r^2 - 2xr^3\cos\varphi\sin^2\varphi + r^4\sin^2\varphi\cos^2\varphi$$
$$= (x + r\cos\varphi)(x^3 - 3x^2r\cos\varphi - 3xr^2\sin^2\varphi + r^3\cos\varphi\sin^2\varphi)\,.$$

This means that the three other single-axis divisions of 2r between q and *h provide* two straight lines through 6, which form the angles 120° and 240° with the trisection line, and a last straight line, which forms the \angle 180° - 'p with *OA.* This can of course also be confirmed elementary geometrically. The last-mentioned point is the conchoidal point known from the outset, through which the straight line ß t o be intersected with the conchoid is laid.

§ 17 General use of the single-use linea

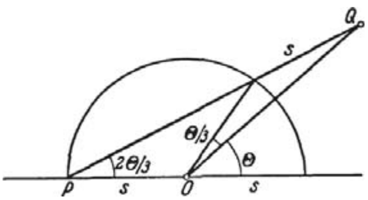
The first generalization is that the distance *AB* marked on the single-axis ruler of § 16 is not inserted between two straight lines, but between a circle and a straight line or between two circles. In this case, the ruler (single-axis ruler) is not the only means of construction, but the compass must also be used or, at least to a lesser extent, circles must b e drawn.

There is a famous *conelruklion* in this area, originating from AnciriMEDEs *for the trisection of the angle,* which is given **first.** Auf dem The distance s is again **marked** as the insertion ruler. Around the vertex *0* des of the angle 'p = *-f AOB* 8to be divided into thirds, xvie in Fig. 46,

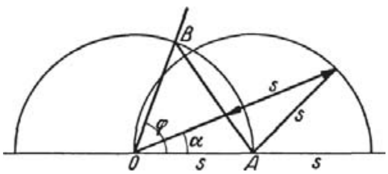


Fg. 40

a circle of radius s and then insert the distance s on a straight line through *B* between *the decade OA* and the circle of radius s around 0. You meet the straight l i n e in 9, the IReis in A. Dunn is œ = *pOGB* -- 'p}3. This looks like can be determined on the basis of Fig. 46 elementnr-geometriach like this : The triangle *ØKO* is is isosceles. Therefore, 2 œ is the exterior angle at øone vertex *K.* Since the triangle *KOB is* also isosceles, the exterior angle g = 3 z is located at Weiner's vertex.



E'lg. t7



Flg. 48

Fig. 47 shows a second arrangement of the trisectionøconatruction by inserting the distance e between a circle of radius ø and one leg of the anglea to be trisected.

M. D'OCAOnE has given a further construction by displacing a distance e x between a straight line and a circle. It can be seen in Fig. 48. The angle 'p = *BOA* to be divided into three is the center angle in a **circle** of radius e. In addition, **a circle of** radius s **is** placed **around *A.*** The distance s is inserted on a straight line through 0 between the chord N *B* and the circle of radius s around d. At *A* there are three angles 2 œ, a/2 - 'p/2 and 7r/2 - œ/2, which **together** form an elongated t 'inkel. It is therefore 2 œ + *(n|2* - ø/2) -j- (u/2 - œ/2) = a, i.e . 3 " - 'p.

All these generalized constructions by displacement, as already indicated, amount to intersecting the conchoid of the straight line of § 16 with a circle (displacement between circle and straight line) or to intersecting the conchoid of the circle with a straight line or circle, **depending on** whether one wants to displace e between **circle** and straight line or between **two** circles. A conchoid is a circle of radius r with pole *P* and the interval *s* is the geometric location of those points which are found on the lines q through *P* at a distance s from the intersection points of these lines with the circle. The equation of such a curve is obtained as follows: The starting point of a rectangular Cartesian coordinate system is placed at the pole P. Let

$$(x-p)^2+y^2-r^2=0$$

the equation of the circle *K*. By

z -- p cos ß, y = p ßinJ

polar coordinates are introduced. Then the circle A

g=pcos#+ r°-p*in*#,

and

$$\varrho = p\cos\vartheta + \sqrt{r^2-p^2\sin^2\vartheta} + s$$

is a parameterization of the conchoids of the circle. A small calculation can be made in

(*+2-* -+ '+--*EO'+vfl-4*(*+/- -f (U

recognize the equation of the conchoids. The conchoids of the circle are therefore sixth-order curves. They merge into fourth-order curves, aog. *Pae'xileche <9snails,* when the pole *P* lies on the **circle** *K*. Then p = r **can** be assumed. The **circle of** radius s aroundP then belongs **to the** conchoid. This decomposes into this **circle** and the Paacal snail. If we set p = r in (I), **we** obtain by a **short** calculation

('+ -2-sfl= (s-FQ) (2)

as the equation of Pascal's snail. Of course, you can also derive this directly by repeating the calculation made for the conchoids for p - *r,* which involves considerable simplification.

IIn the construction of the AncaII¥tEDE mentioned above for the trisection of the angle, the point *B* on the circle *K* is in **fact** the pole of a helix with s =- r, which is intersected by the straight line *OA*. It passes through point 9 of the snail. The fact that it belongs to the snail can be seen from equation (2). Therefore, a **third-degree** equation remains for the remaining intersection points of the line and **the snail,** namely **the** equation the tripartite division of the l¥inkels, as the le8er may calculate. In Fig. 47, *P is* the pole of the scroll, Alte is intersected by the straight line *0 Q*.

In general it can be said that ingenious problems of the sixth degree can be solved by intersecting the conchoids of circles with straight lines or circles. **At first** it looks as if the intersection of the Jfreis conchoids with circles must lead to problems up to **the** twelfth degree, since according to a **theorem** of algebra a C (conchoid) and a C (Ifreig) have twelve intersection points. A glance at equation (1), however, shows that the improper Kreiapoints are triple points of the conchoid, so that gecha of the twelve intersection points fall into these, leaving only six that can be considered for a congruction problem. Up to now, congruction problems higher than the fourth degree have not been dealt with much. We therefore do not pursue this opportunity any further.

It is useful to emphasize that a = r for the helix (2) used for the trisection of the angle. The auger

(z* -|- " - *2 r z)'* *r*(z'* -|- y*) (2')

has a particularly simple parameter display

$$x = r + 2r\cos\Theta\cos\frac{O}{3}, \quad y = 2r\sin\Theta\cos\frac{0}{3}, \tag{2''}$$

which makes the relationship to the tripartite division of the angle leap to the eye. If the origin of the coordinates is set to z = *r, y* - 0, ao

$$\varrho = 2r\cos\frac{\Theta}{3} \tag{2'''}$$

the equation of the worm in polar coordinates. The correctness of the plan depends directly on Sig. 47. There *P* is the pole of the snail. In Fig. 46 the pole is at *B*. In equation (2') *P,* in equation (2"') O is the coordinate starting point.

As a counterpart to the construction of the AnoBIbIEDEg for the trisection of the angle given above, I will show a modern construction - based on f-ferm lonisce - for the Fervie//ecfititip dee *Würfeie*. I choose a point D on the Paacalachian snail (2) at a distance r/2 from the pole. Such a point is the point with the coordinates

8 *r* , --y 1 $\sqrt{1 - \frac{s^2}{16 r^2}}$ *r* Compare Fig. 49, which indicates how such a point A is found. Then I cut the Paacalg screw with a

0 Beaver tree

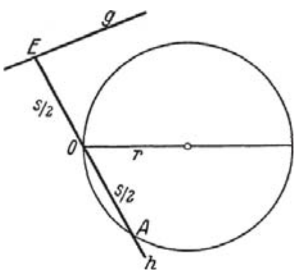
fitting line q through thea point *III.* The line at

$$x = -\frac{s}{8r} + at, \quad y = \frac{s}{2}\sqrt{1 - \frac{s^2}{16r^2}} + bt, \quad a^2 + b^2 = 1. \qquad (3)$$

If (3) is inserted into (2), ao the following equation of the third degree for t is obtained after removing the factor t:

— 2 *a r s** -J- *t* 2 *a r J* - 4 $\sqrt{1 - \frac{s^2}{16 r^2}}$
r
—21° 2 *a r* - } —4. — $\sqrt{1 - \frac{s^2}{16 r^2}}$ t° - 0. (4)
'•

If it is to be suitable for determining the third root, ao and6 so that the members t and t° are omitted. To do this, a and *b* are taken from



crack. ac

$$a\left(2r + \frac{s^2}{4r}\right) - bs\sqrt{1 - \frac{s^2}{16r^2}} = 0 \qquad (5)$$

to be determined. Because of a° -t- *b** -- 1

$$a = \frac{\sqrt{1 - \frac{s^2}{16 r^2}}}{\sqrt{4r^2 + 2s^2}}. \qquad (6)$$

Because of (6), (4) then becomes

$$= \frac{\sqrt{1 - \frac{s^2}{16r^2}}}{\sqrt{4r^2 + 2s^2}}. \qquad \text{i')}$$

If the insertion ruler, i.e. a > 0, is fixed, you will have to choose *r* appropriately in order to obtain the third root of a given number.
In any case, because of the reality of the situation

$$r > \frac{s}{4} \qquad (8)$$

can be taken. I make the approach *r* -- *k*- *o* and **take** m- e° as the number **from** which **the third** root is to be taken. Then I have for £ according to (7)

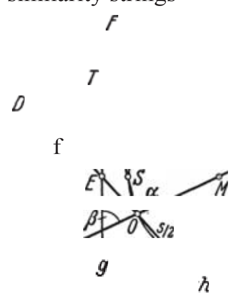
$$\frac{2\sqrt{1-\frac{1}{16k^2}}}{f4\,k-\text{-j-}} = m \text{ due to } m\,a^\circ = \frac{2\,r\,s^3\sqrt{1-\frac{a'}{16\,r^\wedge}}}{\sqrt{4\,r^2+2\,s^2}}. \tag{9}$$

2 From this one finds for £

$$k^2 = \frac{1 \text{-+-} \beta m^\wedge}{16\,(1 - \text{-}t')\,''} \tag{10}$$

Using this method, you can therefore take the third root of any positive number m s° with m 1. In addition to the one-hiebelinea1 with the two marks at a distance s from each other, Plan has to use a circle of radius £ s, where £ can be taken from (10). Other values of m are given by similarity strings formation is reduced to m < 1.

The following description of the construction described below is useful for the actual implementation. In order to I take up the designations of Fig. 50 and first determine the angle y of the straight lines q and A (Fig. 50). As in Fig. 49, *OA* -- *OE* -- s/2. In addition, *CEE* is s/2. s/2. On the triangle *OAB* is **taken from**



$$\text{co}, , \quad 4\,r\,' \quad , \quad \sqrt{1-\frac{s^2}{16\,r^2}}\,'$$

cool = a and ainQ = b can be taken from (6). Furthermore, according to Fig. 60

Therefore, according to (9)

$$\cos\gamma = \frac{2\,r\sqrt{1-\frac{\text{-}}{16\,r^2}}}{\sqrt{4\,r^2+2\,s^2}} = m\,, \quad \sin\gamma = \sqrt{1-m^2}\,.$$

The triangle *DER* in *Fig. 50* is therefore useful for constructing the straight line *q* if you take *DCH* -- m and AJ = 1. From the triangle *OAB,* take *AB** 4r° - a°J4. If you consider the ansatz *r* -- *k*g and (10), ao you get *AB* - . Since *CIA* = a, the triangle *OAB*

2 J/l - m- 2

similar to the triangle *PDE* '. The angles of the same at U and therefore complement both y to aJ2. Therefore, *CSB* is perpendicular to *EU'*. To construct the third root au8 In s°, proceed as follows: First choose the point *0* at random and see an arbitrary straight line fi through it. Then t a k e the points A, A, U on it in such a way that the line *ACH* is divided by 0 and A into three equal parts of length *8|2*. Then choose *DE* -- *m* and place *D co* so that A lies between *0* and *D*. With £J = 1 you then construct a triangle *DEIN'* that is right-angled at *D*. The straight line on which its hypotenuse falls is p. From N you fill a perpendicular to p. In *A* you construct a perpendicular to A. The intersection of both perpendiculars is at *B*. Draw a circle over *OB* -- *2 r* ala diameter. On a straight line through O, insert the distance s between this circle and the straight line p. This gives

one has a point T on p. It is ff' i =. " .

When carrying out the construction, you will notice that the insertion leads to poorly defined intersections. A glance at Fig. 50 also shows this. We come back to the construction in § 18 at Fig. 58

A special circumstance should be pointed out: While the Archimedean construction always u s e s the same s and the same r for the trisection of angles, no matter which angle you want to divide into thirds, the construction of lorisoa always requires the same s, but the radius of the circle required depends on m. Therefore, while you can divide the angle into three with a fixed ruler and a fixed ifreis, you have to use a compass to calculate the third root.

Using the Cardan formula, according to p. 74, the solution
Each task of the third degree can be reduced to the two preferred ones: trisection of the angle and multiplication of the cube. The questions of direct construction by insertion have not yet been worked through very much. As an example, let us use Fig. 61 to illustrate one of Mr.
J. E. Hoeuamr's pretty &'meiruDion of the *regular heptagon*
can be specified. The circle 0(3/4) of Fig. 51 has the equation

$$\left(x - \frac{3}{4}\right)^2 + y^2 = \frac{9}{16}, \qquad \text{(ii)}$$

In Fig. 51, the straight line z = 0 is perpendicular to a diameter *AB* in *A*. On a straight line through P(1, 0), the line 2 is inserted between the circle 0(3/4) and z = 0. This straight line meets O(3J4) at a point 6 at a distance *r* from *P*. Then S has the coordinates

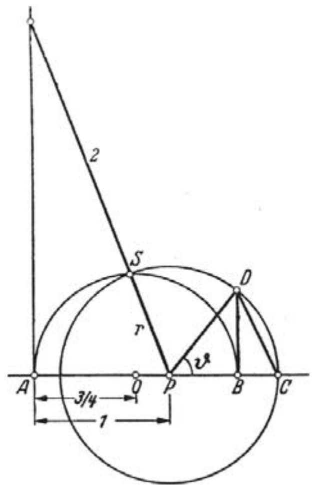
$$x = 1 - \frac{r}{r+2}, \quad y = r\sqrt{1 - \frac{1}{(r+2)^2}}, \qquad (12)$$

and inserting (12) into (I 1) gives for *r* the equation

$$r* -l- 2 r* - r - I -- 0, \tag{13}$$

which can be substituted by r = I/z into the equation

$$x* -I- s^ -2s - 1 - 0 \tag{14}$$



whose positive root according to p. 58 i s given by z = 2 cos 2 u/7. If one draws the if circle T*(r) in fig. 51 and erects a perpendicular toAB in *B*, ao iat for the angle # of fig. 51: # = 2s/7. Thus the segment *CID* of fig. 51 is the side of the regular heptagon inscribed in the circle *P(r j)*. It is left to the reader to check to what extent the compass is needed in addition to the slide-in ruler to construct the regular heptagon.

The constructions given so far in this paragraph used the intersection of the general conchoid with a straight line. Finally, a very simple construction by Newton should be mentioned.

given that the *lschnitt "onKreiokonchoide and circle*. The circles between the two mark the following points one after the other: on the z-axis z = -s, on the y-axis y = 2a , then ale auxiliaryB- point of the z-axis again z = - s 4 -l- , on the y-axis y = r 8 -H-

and then draw the two stripes through the lines marked on the y-axis.

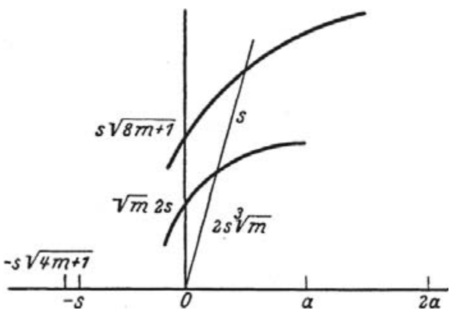
Points with centers on the z-axis in two arbitrary points z = o and z = 2a, e.g. the two circles (z - l)- + y° - *I* -t- 4m8′ *and* (z - 2)° -}- y° = 4 -}- e° (8m -J- I). If you then insert the distance s between the two ifreiae on a straight line through the origin of the coordinates, the distance between the origin and the smaller of the two circles is

28 In . Plan easily verifies the correctness of the construction using analytical geometry (Fig. 52).

J. £tJzzugzsv *gives the* following *generalization of the slide-in ruler*: A slide-in ruler is placed in such a way that no two marks fall on given circles or straight lines and that, in addition, a given circle is enclosed by it.

touches. This amounts to the same thing as using an *insertion parallel ruler,* both edges of which should have the radius of that circle *K* as the distance and which should be placed in such a way that the two marks attached to one of its edges fall on two given circles or straight lines, while the parallel edge passes through a point, the center of those circles8 *K.*

Let an *example* liven up these statements. A triangle is to be constructed from two sides a, t' and a with an incremental radius p. According to **HanzuSLEv,** this construction **p r o c e e d s** as follows: If, for example, a > 6, then a - f' is, as will be shown, the length of the distance *ID* to be walked in p. 87, Fig. 53. This distance is therefore determined by the given pieces. For the construction, a circle of radius• e is therefore placed at any of its points N



Flg. 52

a tangent c and **mark** the point Z" on it so that *'D -- a - b* did. In *D,* draw the diameter *DE* of the increigeg up to point A. Insert the line o between the lines c and *FIi¡* on a tangent of the increment to obtain the corners *B* and U of the triangle you are looking for. The empty corner *A* is then obtained by **placing** the other tangent from C' to the **inscribed circle.**

In order to prove the assertion *II'D -- a - b,* note that the point of contact of the circle of the triangle opposite the corner U is *ABC!* To see this, you only have to enlarge the sig. 53 with U as the center of similarity so that A merges into N. Then the increment changes to the starting circle. However, it is well known that *0 –* $\frac{b + c}{2}$ is the length of the tangents laid from N to this increment, while $s_a, s_b, s_c = \frac{a + b - c}{2}$ measures the tangents laid from the corners to the increment. Therefore, the tangent placed from *A* to the circle has the length e_6 . Therefore, *ID* --

yp - sg -- $\frac{-b + c}{2}$ $\frac{-a + b + c}{2}$ $= a - b.$

The construction problem just discussed is a problem of the third degree. For z = ap, a third-degree equation is obtained as follows.
Starting from

———

you get because of

$$s = x + b, \quad s_a = x + b - a, \quad s_c = s - s_a - s_b = a - x$$

/(s+*)—(s+6—°)sH-s)=0.

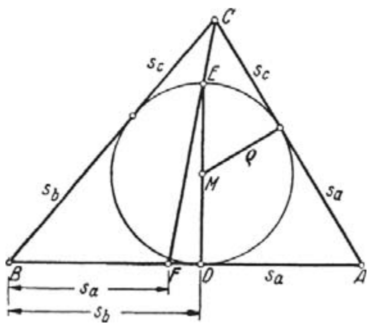
Therefore

$$x^3 + x^2(b - 2a) + x\left(\varrho^2 - a(b - a)\right) + \varrho^2 b = 0.$$

This equation is generally irreducible in the body of its coefficients. For example, according to P. **BoC&NE'-R** for a = 2, *b* -- 1,e- .* clie equation

z* - 3 z* -I- 2.09 z -I- 0.09 - 0 ,

which, as can be easily verified, is irreducible in the body of rational numbers.
P. BtfCKNER has also determined the conditions to which the n, ö, p satisfy so that it forms a triangle with these pieces.



ria. like this

In the construction described above, you can of course manage with a fixed distance n on the insertion ruler. If a - w is not a, you first construct a triangle similar to the one you are looking for with a = p. However, the *question* arises as to whether you can solve all third-degree problems (or even just the previous one) with an inset parallel ruler with a fixed distance between marks and a fixed ruler width. It is only posed, but not answered answered.

§ 18. the angle hook and the parallelogram angle hook

The remark that the *Pascal snails* are at the same time *foot-point curves of* a *circle* with respect to its pole **leads to** another version of the constructions described in the previous paragraph. The base point curve of a circle *K* with respect to a point *P* is understood to be is known to be the geometric location of the base points of the perpendiculars filled from P to the tangents of the circle.

Let us assume a circle R. of radius fi with center ff and the pole *P* at a distance p from ff. Then a glance at Fig. 54 teaches us that the geometric location of the base points of the perpendiculars drawn from *P* to the tangents of A i s a Pagcalache snail, namely a Kreia conchoid with s = fi and with the pole *P* in relation to the circle of radius r = p/2 via *IP* as diameter. This is because the point fi, at which *PC* is intersected by the straight line 6öf parallel to PC, is, according to the theorem of Taax.zs on a circle above *PCI* as the diameter. Accordingly, the geometric location of the points N is also obtained by subtracting the distance 'Sf = fi on the straight line through *P* from its second point of intersection with this circle from Rndiua *r*.

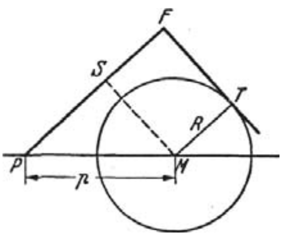
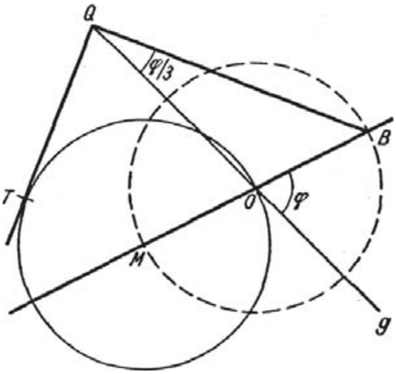


Fig. 54

In the construction shown in Fig. 46 for the tripartite division of the angles, r - s was. The point *B* was the pole of the Worm. According to the above diagram, this is the base point curve of a circle from the radius *r,* whose center ff is the peripheral point dea Kreiaea of Fig. 46 diametrically located to *B*. Q of F i g . 46 is therefore also the base of the lotea of & on a tangent through 2' to the aforementioned circle A around df as the center. This kreia R. is drawn out in Fig. 56, while the circle of Fig. 48 is only dotted. A construction for the trisection of the angles can therefore also be described as follows: Take any circle fi from the radius *r* with center ff and extend a radius *MO* deaaelben by r outwards over O to a point *B*. Place a straight line p through *0* at the angle 'p to be thirded against this extended radius. Then place a right-angle hook ao so that one of its legs *p a s s e s* through *B*, its other leg touches *K* and a is at the vertex of p. For one of the three possible positions of the hook, *p* is then *BQO* -- gJ3. This can also be seen elementarily from t h e following eg. if you consult the dotted auxiliary lines. At the same time you can see that @ 2'O f - 'p/3 iat.

The gt ttndsöfzficfie *meaning of* theaer version of the construction iat the: *The* plan has to allow only a slightly more necessary *oebraucM preficfionnlty* than at that time, achon all problems of the third - and thus also the fourth - GFade become solvable. (We will ask you about the extraction of the cube root in a moment.) At that time, § 9 required this,

that the two legs of the (right) angles pass through existing points, **while** the base lies on **an existing** straight line. **Now** aoll
the one leg deviates and touches a circle.



M$. 66

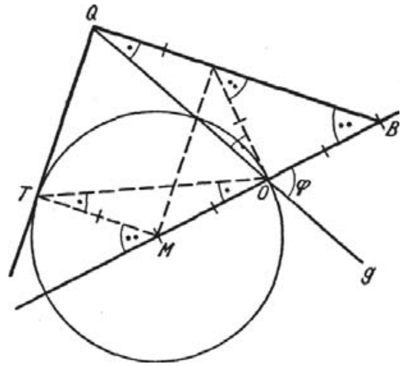


Fig. 56

The IJmfomation of the construction of Fig. 47 is shown in Fig. 57. The reader will understand it without further explanation and will also carry out the elementary proof on the basis of the dotted lines shown.

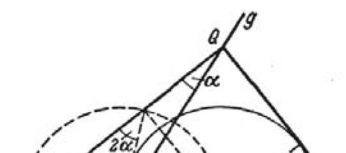
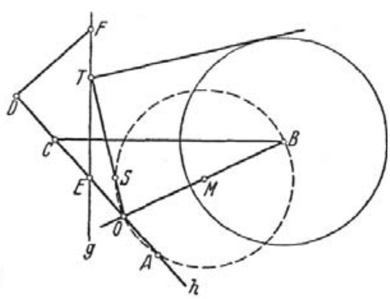


Fig. ö7



Ftß. S8

Following on from *Fig.* 50, a circle of radius *8* must be drawn around point *B* to *extend the driyets root.* The perpendicular established on OR in f' is then the tangent of this circle. The construction of the third r o o t aua m- a- then requires the same steps as in the previous paragraph as far as the construction of the straight line *q* and the point & is concerned. However, instead of the creiaea drawn at that time with a half-megger *dependent on* 'ii over *OB* ala diameter, a creia of {eefetn Rtsdius s is now to be placed around *B* as the middle point. The right-angle hook must then be attached in such a way that a one leg passes through 0, its other leg touches the aforementioned circle of radius 4 around *B* as the center, while a is a vertex on that straight line q (Fig. 58). It will be noticed that this construction can be carried out much more accurately than the insertion in Fig. o0.

Now, apart from the perpendicular hook, the compass is only needed to draw a circle with a fixed radius independent of si, or, in other words, the perpendicular hook and a fixed **circle** are sufficient. This can also be avoided if one leg, which is to touch the circle, is designed as a parallel ruler and requires that one leg does not touch a circle, but that the edge parallel to it passes through a fixed point, the center of that circle - of radius equal to the width of the ruler. This parallel right angle hook then allows the solution of all third and fourth degree problems without any other **device**. To draw the straight lines p and A on paper, you will of course still need the compass.

§ 19. the standardized right wing shark. NEo'zons Kissoidenziriiel. The carpenter's hook

A *standardized right-angle hook* should be called a right-angle hook on one leg of which a point *P* and the center Q of the line determined by *P* and the vertex 6 of the right angle are marked. The length of the line *PQ is* called the *standard* of the hook. The instrument should be used in such a way that the free leg (which bears no marks) passes continuously through a fixed point, while the lines *P* and Q lie on given straight lines. The instrument can also be used as a ruler and as a slide-in ruler. Then all third- and fourth-degree problems can be solved by the position of the 6-strong Q.

At first glance, drawing the normalized right angle hook may appear to be an embarras de richesse if you only want to solve third degree problems. After all, you have a slide-in ruler available in the standardized extended edge, with which you can solve all third-degree problems on your own. However, while you have to take the detour via the Cardan formula, extraction of the third root and trisection of the angle when using the slide-in ruler, every third degree problem is directly accessible with the standardized right angle hook, as we will see.

The theory of this standardized right-angle hook is based on the properties of the *kissoid of the* dioxi.ss. This is defined on the basis of Fig. 59 ao: A point *0 is* marked on a circle £ of radius r. Let the tangent t be drawn at the diametrically opposite point *0'*. On each straight line p the distance that is cut off on y by *k* and f is deducted from 0 in both directions through O. In Fig. 59, OQ = Zf f'. Ifissoid is the geometric location of the points Q obtained in this way. We determine its equation. The z-axis is the straight line from *0* to 0', the y-axis in *0 is* perpendicular to it. Then, according to Fig. 58, the polar coordinates p, ß of the point Q are

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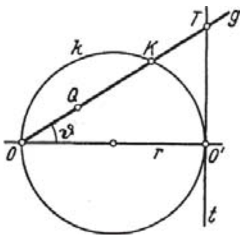
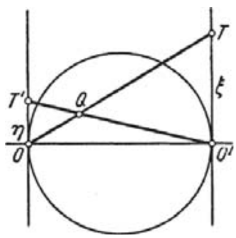


Fig. 58



Flg.60

wornua man

$$m(m+y)=2Fy° \qquad (1)$$

as the equation of the kiasoid.

If you intersect (l) with the straight line y = z/Z, ao you get the parameter display

$$x = \frac{2r}{1+\lambda^2}, \quad y = \frac{2r}{(1+\lambda^2)\lambda}. \qquad (2)$$

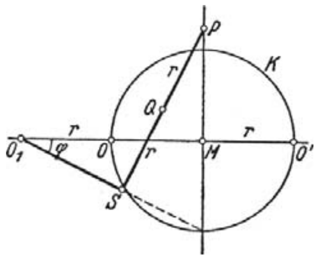
plan easily confirms that the sections labeled } and § in Fig. 60 are *O'P* and *OT'*

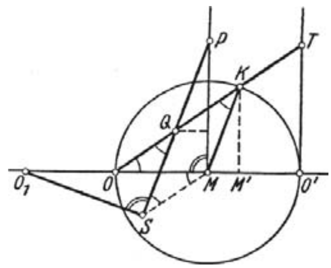
$$\xi = \frac{2r}{\lambda} \text{ and } \eta = \frac{2r}{\lambda^3} \qquad (3)$$

are. From this we can see the connection between the kisaoid and the extraction of the third root, just as the intersection of the kisaoid with any straight line leads to an equation of the third degree.

Constructively, the connection that (3) establishes between $ and p only appears to be Yvaluable for the extraction of the third root i f a kisgoid is drawn. The nortiated *reohtu'i*'tkelhaken is used for the constructive form - without a firmly drawn kiasoid. Its theory is based on the following geometric property of the kiaoid:

As illustrated in Fig. 61, extend *O'O* beyond O by *r* bia to point O and erect the perpendicular z = *r* on a diameter in the center ff of the ifirclea A. Ala radius r of the circle A is chosen as the norm *PQ* of the normalized right anglehnkena. Now place ao so that its mark *P* lies on z = *r,* while its free leg passes through O_1 . Then the mark Q of the right angle hooka describes the





Kisaoid (1). In Fig. 61, the parameterization of the geometric location of Q is shown:

$$x = r - r\sin\varphi, \quad y = \frac{2r}{\cos\varphi} - 2r\,\mathrm{tg}\,\varphi - r\cos\varphi = \frac{r(1-\sin\varphi)^2}{\text{COBA}}, \qquad (4)$$

using the same coordinate system as in (1). öfit £ith the help of the second **form** of y it is easy to confirm that **for** theae z, y the Riaaoiden-

equation (I) exists. Insert the angle # through 'p = $ - 2 #.

Of course, the property of the kisaoid just described can also be seen *in elementary geometry.* This is done at Tland of Fig. 61 as follows: Place the two lines of a straight line PQ = USQ = *r* through the kisaoid point Q*,* also take OO, = *r* and prove that the double angle at 6 is a right one. Since OQ - ff', then QQ', which is perpendicular to

IP assumed iat, equal to 3f3f'. Since further *PQ* - *fK* -- *r* iat, ao *3fK* is parallel to *PQ.* Therefore, the painted angles at A and Q are equal to each other. Since *KM* -- *OM*, ao the marked angle at O also has the same size. Since QTS = *fK* and QS is parallel fA, ao, SdfA9 is a parallelogram, the solid angle at 'S has the same size and the solid angle at df has the same size.

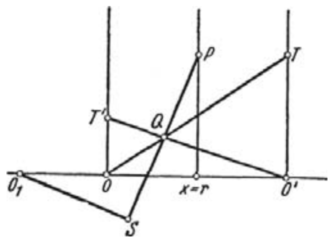
gI size. Therefore, the triangles *iSMP and* 3f6O_1 are kOngruent. For athey agree in the sides f'S, which they have in common, and in POS - *MO ,* as well as in the angles included by these sides. Therefore, the angles at ff and S are equal to each other. As the angle at ff is a right angle, there is also a right angle at 6.

The *construction of* ')/m now proceeds along the lines of Fig. 63 ao: draw the line O7" = tu 2 *r* on z = 0, draw the straight line z = r, then the straight line *P'O'* through the point 0' with the coordinates [2 *r,* 0) and place the right angle hook with the norm *r co such* that aa mark *P* lies on z = r, that its mark Q lies on the straight line

0'" and that its free leg p a s s e s through the **point** *0* with the coordinates z -- *r, y* -- 0. Then the straight line *0 Q* on *:n 2 r* intersects the line

0'T -- - 2 *r* ab.

You can also use the standardized right angle hook to solve any third-degree equations without further preparation.



solve. If you intersect the kissoid (2) with the straight line

$$ux + y + w = 0, \tag{8}$$

ao you get the equation of the third degree

tr J^ -|- J (tr -I- u - 2r) -|- 2 r 0. (7)

Comparison with (5) provides

$$w = a_0 2r.$$

According to (6) you have the straight line

(& o) + +@@2f- 0 (8)

and construct its intersection with the kiaoid using the normalized right angle hook in the same way as described for the extraction of the third root. The following method is therefore available for the trisection of the angle: Plan has the equation of the third degree

J^ - 3J - 2 **cos** x - 0 **zoit** 72 **coso/3**

to be treated. For this purpose, the straight line to intersect with the kissoid. Fig. 64 shows the course of the construction without further explanation.

Until now, we have only used the standardized right-angle hook in this way, that aa non-normalized leg ran through a point and that the points *P and* Q of the marked leg each lay on a straight line. In addition, compasses and rulers were used to draw circles and straight lines and to measure distances. No use was made of the hooks mentioned at the beginning of the paragraphs as a ruler and as a single-sided ruler. If we decide to do so, we can do without compasses and rulers. These instruments were only used to draw circles and lines.

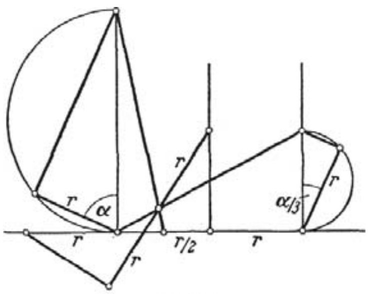


Fig. 64

lines are used to determine their intersection points. However, it has already been established in § 1I that compasses and rulers can be replaced by the standardized ruler. The marks of Hskena were placed on drawn straight lines, so that the drawing of circles is not necessary. Go through the tasks dealt with in this paragraph again with this in mind. So we have the theorem:

All problems up to and including the fourth degree can be solved with the standardized right-angle hook designed as an insertion ruler. However, since we already know from § 16 that the slide-in ruler without a supplement to the standardized r i g h t - a n g l e hook is sufficient for this purpose, we are faced with a double obstacle, which, however, has its advantages in actual construction. It should also be mentioned that the free leg of the hook must be thought of as a ruler extending to infinity on both sides, since the kissoid runs to infinity on both sides of the z-axis and only in this way can the hook cover all its points. However, the normalized leg need not be longer than *2 r* as long as the hook is not to be used as an insertion ruler. The normalized right angle hook discussed above - Nzw- Tor's kissoidal compass - is a special case of the general normalized right angle hook. It consists of marking any two points on one leg of a right angle hook and, like the free leg through a point, placing the marks on two straight lines. The generalization can also be described as extending the use of Newton's kissoid circle in such a way that the free leg is not placed t h r o u g h a point but tangentially to a circle. This is obviously associated with a displacement of the marks on the other leg of the circle.

is equivalent. Particular attention has been paid here to the case where the two marks are assumed to be symmetrical to the apex of the right-angle hook. The instrument shown in Fig. 65 below can be *u s e d* to divide the angles *ROA* into three in the way shown in the same figure. To do this, place the free leg through the apex of the angle, one mark o n one leg of the angle, while the other mark lies on a parallel to the other leg of the angle. The distance between the two marks and the right angle is the same. That in fact @ *POA -- p QOP -- 9 ROQ -- I|3 9 ROA*,

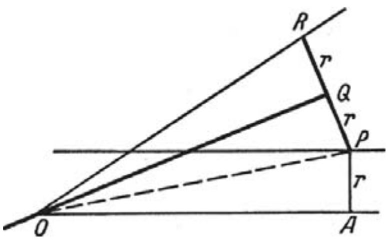


Fig. 65

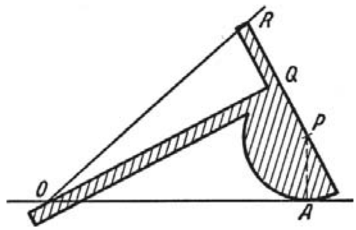


Fig. 80

can be seen without further ado. The drawing of a parallel to the leg *OA* mentioned above can be avoided by adding a semicircle of radius *PQ* around *P* as center to the instrument in the manner shown in Fig. 66 and then placing the instrument so that aa free leg passes through *0* and one mark lies on one leg of the angle, while the semicircle touches the other leg. **E. VOELLMY** has occasionally remarked that the point fi passes through a curve of the third order, the so-called triaectrix of MacmoBrs, when the free leg passes through 0 and the semicircle touches the straight line *GA*. If we take 0 as the origin of right-angled coordinates, *OA* as the z-axis and denote the marker abatnnd fi$ by *r*, its equation becomes

(3r-y)(sky)-4W=0 (9)

or, which is the same thing,

$$r = \varrho \sin \frac{\vartheta}{3}, \qquad (10)$$

if the polar coordinates of the points ff are labeled p and #.

(10) can be read off Fig. 65 without further ado, and (9) follows from this by virtue of the relationships

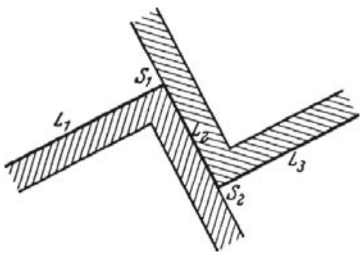
$\frac{y}{\varrho}$ = ain ß = - 4 on° 3-]- 3 on $.

Without the Ifalb circle, the instrument is also known as a *carpenter's hook.* has been labeled.

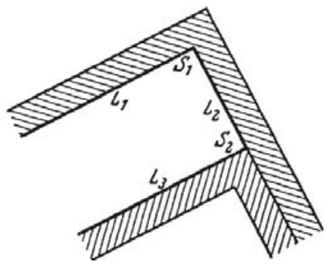
Solutions of other third-degree problems, such as the multiplication of the cube with the carpenter's hook, do not seem to have been worked out. A general theorem by F. Lonnon (1894) teaches that such solutions exist. According to this, any third-degree problem can be solved with the ruler alone if any rational (non-decomposing) curve of the third order is drawn. The question of whether curves of gender one also perform this service is still unanswered.

$ CO. Two right angle hats

Two right-angle hooks that can be moved relative to each other as shown in Figs. 67 and 68 can be used to solve any third-degree problem.



E'lg. 07



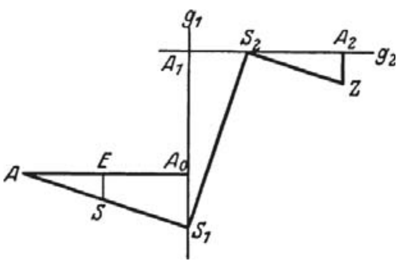
Flg. 08

The two legs fit and 6 should p a s s through given points *A* and ü, while the legs fi of both hooks can be moved against each other. The vertices 'S and St of both hooks should lie on given straight lines *gz* and q. If we t a k e *A* as the focal point and p as the secant tangent of a parabola, then fi is a tangent of this parabola. This is because the geometric location of the base points of the perpendiculars filled from the focal point to the tangents is the vertex tangent. Similarly, ß can be regarded as the focal point and q as the vertex tangent of a second parabola. Then fi is the tangent of this parabola. The described use of both right angle hooks thus amounts to the determination of common tangents of two parabolas. Since t h e r e are three common tangents of two parabolas, apart from the improper straight line, it is clear that the solution of third degree problems is possible in this way.

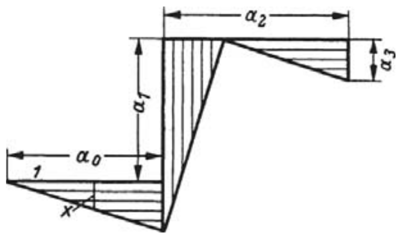
It is practical to use two ordinary right-angle hooks, the two legs of which a r e designed as parallel rulers, so that both the

The outer and inner edges each form a right angle right into the parting. These are then placed together along a matching edge. However, you can also use two right angles drawn on transparent paper instead, which you place on top of each other over the drawing sheet, or you can use such a transparent sheet instead of at least one hook.

If, for example, the vertex tangents q_i ulld qd are assumed to be perpendicular to each other and the perpendiculars from the focal points *A* and fi to these two straight lines are plotted, the result is Fig. 69 below. 2fan can deduce from this
directly, without reference to the parables, the connection with Öer



E'lg. 69



Flg. 70

resolution of a third-degree equation. This is illustrated in Fig. 70. There, the coefficients of the equation

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0 \tag{1}$$

and the unit distance are entered. Here e 0, e_i < 0, e, < 0, a, < 0 and z < 0 are assumed. From the similar lines shaded in Pig. 70 hatched, similar to each other triangles you can read that *ISBN* a_0 z and that 'S$_i$ fi$_i$ - e z + tat. Then tat further *SIA* -- (e•
+ °) *iSA (a$:r + a j:n -}- a$.* Therefore iat really
MA = [(a_0 z -]-) z + et] z= - e ulld z a solution of equation (I). In Fig. 69, there are two right-angled straight lines, the coefficient line *AA gA A$S* and the solution line *ABS Stk.* They both lead from the starting point *A* to the end point S. The following rule applies to the formation of the coefficient line. First plot a line *AA$_0$* of length e, a > 0. Then, at right angles to this line, plot a line of length ! e, to the left of dea
vector *AAE,* if a, < 0, to the right, if a > 0. Al8then again at right angles to the just plotted *A pA,* e is plotted to the left, if a_l e < 0, and to the right, if alu > 0. According to the same rule
a is finally applied. Then use the two right angles to form a
hook the solution move ao that the leg 6 of the first hook through *A*

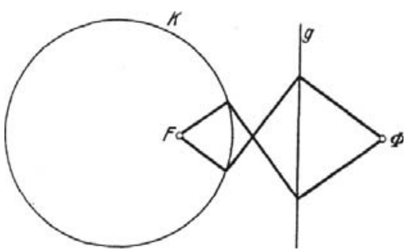
while its part 6 lies on qd. Then attach a leg fit of the second hook to another leg At so that a vertex 6 falls on p, and also change the position of the two hooks until the last leg fiq of the second hook p a s s e s through fi. On the straight line *Ei8*

z is then derived, i.e. $z < 0$, if S lies to the right of *AAE*. If 'S

linka of *AA* , then $z > 0$.

The method described in this way is known as the Lillachea right angle method. Plan, it can also be used to calculate the polynomials on the left-hand side of (1) according to the Hornerach scheme

use.



Fla. 71

The explanations at the beginning of this paragraph suggest a generalization. It consists in placing the intersections of the right angle hooks on creiae instead of straight lines. Since the base points of the perpendiculars, which are dropped from a focal point of an ellipse or hyperbola onto the tangents, fulfill the large vertex creia, we are now dealing with the

Solution of the **fourth** degree problem of *determining the common* tangent cos ztrei fiegefec6tiiiteti, which can be solved directly with the two right angle hooks. **Fig. 71** shows the **common** tangents of an ellipse **with** the focal point and the large vertex circle *K* as well as a parabola with the focal point & and the vertex tangent p (see also § 25).

5 S1. The standing circle

This refers to a compass with two points, which should be used as follows: If circles, straight lines and points a r e drawn, ao a point is to be determined by tracing with the compass on one of these straight lines or circles, the distance of which from a given point is equal to the greatest or smallest distance from the points of another given circle or another given straight line. For example, you solve the problem of determining a point *P* on a circle R. in such a way that its distance from another given point is equal to the (shortest) distance of the point P from a straight line p, so that one point of the compassa is inserted into an arbitrarily assumed point *P* of *K*, with the other point and by rotating the compassg around P determine whether the circle with the radius PC around P touches the straight line p as the center. Keep changingP until you have succeeded. Fix the position found of P on K by piercing it and then use the compass and ruler to draw a perpendicular from P to p in order to determine the point on p that serves to solve the problem more precisely than is possible simply by touching it with the rotating compass.

Perhaps this procedure may hardly differ from a trial and error, which is familiar to every practical draughtsman. *Conceptually*, however, this *is* a *mathematical operation* which *is assumed to* be *executable,* i.e. the *determination of a point P on K co that*

$$\text{Distance } PS' \text{ -- Distance } Pg.$$

Plan can of course also measure the distance Pq with the compass. However, as this can be achieved by plumb bobs, this does not extend the range of pieces that can be constructed with compass and ruler in the sense of § 6. The same applies when determining the greatest and smallest distance of a point from a circle.

However, the newly added operation extends the range of conatiuable pieces beyond what can be conatiuated with compass and ruler in the sense of § 6, namely by including all cubic conatiuctions. The geometric location of the points *P,* which are equidistant from and y is known to be a parabola with the focal point and the directrix q. The *operaliott k8I* assumed to be *aue|iilsrbar is* therefore *equivalent to the* ar "iitfftitip *of the lschnillpunktle of a circle K.* tind *der Parabel mit Brennfninkl M* vwd Zeitlinie p. lat aber in einem passend gewählten rechtwinkligen Roordinatengyatem

$$y' - 2pm - 0 \tag{I}$$

the equation of the parabola and

$$z° -t- y° -l- 2a\,z -I- 2\,b\,y -|- c = 0 \tag{2}$$

the equation of any circle, the ordinates of the intersection points of the two have the equation of the fourth degree

$$y^* -|- y^* (4\,p° -|- 4\,ep) -|- 8\,b\,p°\,y -j- 4\,c\,p' -- 0. \tag{3}$$

Comparison with any fourth degree equation

$$y + '',2- + '', + '', \quad o \tag{4}$$

leads to

$$a = \frac{a_2 - 4p^2}{4p}, \quad b = \frac{a_3}{8p°}, \quad , \quad \frac{a_4}{4p-}. \tag{5}$$

Accordingly, (3) did the generalized equation of the fourth degreea with real coefficients and without a member with y° (cf. § 15, p. 73 because of complex coefficients and complex roots). This proves that every equation of the fourth (and therefore also of the third) degree, alao every cubic and biquadratic problem, can be solved if, in addition to the compass and ruler described in § 6, the Steoh compass is added to carry out the intersection of circle and parabola in the manner described, and that with a fixed parabola but a variable circle all these problems are already exhausted.

At the beginning, other *operations* with the *lithotripsy circle* were taken into consideration. We shall see in a moment that each of them performs the same service as the one just discussed. The task now is to determine a point *P* on a **circle** A so that it is equidistant from another circle Cr and a point. This can be done with the divider compass by first assuming T* on *K* as desired, taking the distance *PC* and by rotating it around *P,* finding out whether the circle around *P* touches the circle O with the radius T*J. Keep trying until you have found such a position of *P* on *K*, then insert the tip of the compass, connect in a straight line with the center of Cr and intersect the straight line with 9. This operation results in the intersection of the circle *K* with an ellipse or hyperbola. This is because the geometric location of the points *P,* which **are** equidistant from a point and a **circle** Cr**, is** an ellipse or a hyperbola. It obviously consists of all points whose distances from two fixed points (F and the center of G) have a fixed sum or difference (equal to the Rodiug of Cr), and this is one of the usual definitions of ellipse and isyperbola.

The intersection of a given ellipse or hyperbola with a given creia is carried out as follows also with the divider as follows: The two focal points of an ellipse and the sum 2e of the itadiene vectors are given. The two focal points and ae are marked and the circle with which the ellipse is to be cut is set (Fig. 72). We draw a circle q with radius 2a around one of the focal points and determine a point *P* on fi with the compass so that a is a distance equal to a distance from y. If the circle with the radius *Plz* around *P* touches the circle *g at* a point f, then Jt, *P* and 2' lie in a straight line (a radius of q) and therefore 2e = *F P* -l- *PC* -- *F P* -l- *F$P*.

The same procedure **is used** for the *hyperbola.* The focal points B_l , and the difference of the radius vectors 2o are given. Again, a circle q of radius 2o is drawn around lt as the center.

Ellipse case 9 encloses the other focal point, q now excludes it. Again, a point *P* on *K* is equidistant from Jg and *g*. If a circle with the radius *Pkt* touches the circle *g* in 2', ao are again J" P, f' on a radius of p. But now iat

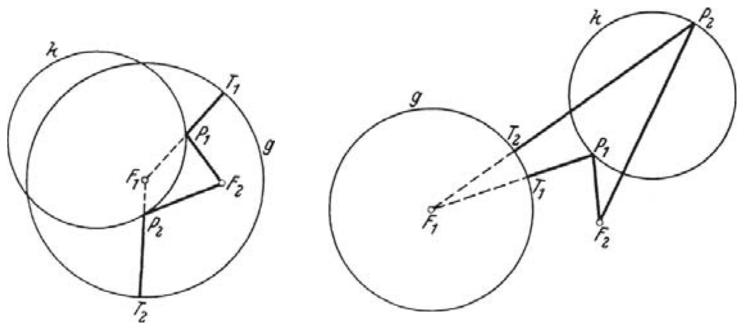
$$2a = F_1P - PT = F_1P - F_2P.$$

Fig. 73 illustrates this.

Or also: Let the equation of q

$$x^2 + y^2 - r^2 = 0$$

and ifP has the coordinates (/, 0), / > 0, then the condition for the



geometric location of *P(x,* y)

$$\sqrt{x^2 + y^2} - r = \sqrt{(x - f)^2 + y^2}. \tag{6}$$

If we remove the roots, we obtain the equation of the geometric Ortea

$$4\,x^2\,(f^2 - r^2) - 4\,y^2\,r^2 + 4\,x\,f\,(r^2 - f^2) + (r^2 - f^2)^2 = 0. \qquad \text{(T)}$$

Daa is a hyperbola for } > r and an ellipse for < *r* an ellipse. *The operation assumed to be aue|iihrbar* means *atso* actually the artnittfunp *of the sonitt*- puzi£te eiziee *circle* tnwith *an ellipse or a hyperbola.* We now show again that every task of the third or fourth degree can be traced back to those just mentioned. However, as we shall see shortly, this is already proven b y what we have just shown for the intersection of parabola and circle. The cluster of second-order curves determined by (I) and (2) also includes the 6-intercept cones for any value of the parameter 2 -1- 0.

s'+(l+X)y+2/-) +2A + -0, (8)

which are ellipses or hyperbolas depending on the choice of parameter. Since (8) and (2) have the same intersection points as (1) and (2), it has therefore already been proven that the general equation of the fourth degree (4) can be traced back to the intersection of (2) with (8). However, in contrast to the parabola (1), the **curve** (8) is not independent of the specially presented fourth degree equation (4), but is connected to it by (5) like the creia (2). However, since the axis ratio is determined by the parameter 2 alone, ea is independent of the problem (4) presented. The conic section (8) is therefore, if it does not decay, similar for the right 2 to a fixed one independent of the task - or in the hyperbolic case deasen conjugated, i.e. equally asymptotic. However, since this similarity transformation can be carried out constructively with a compass and ruler, we can see in close proximity to the result') that any *third and fourth ode can* be used*, if one adds to the compass and ruler in the sense of § 6* the operation of *cutting a* given *elliptic* or *perpendicular with a circle that has been* deprecated from *the ellipse, whereby the ellipse is assumed to be noncircular.*

$ SS. Drawn cone and compass and ruler

A result announced in § 21 may be proven because of its fundamental importance. IPenti a *non-creio|iform non-sinqular* second-order *curre9 "or9eleqt, every* third *and* fourth *ordee task can be* solved with *compass and mnml if* the intersection of Jfreisen tittzf t7erndea with the qezeicAzteteti U fii'isutiinont *to* the feruendiitigamöqficfi£eiten dieeer Jnefrumetite in the sense of § *6.* This result is reminiscent of the one in

§ 5: Constructions with the ruler alone, if a drawn circle with its center pointsa is available. There is no need for such an additional specification here, because the center of the drawn cone section, for example, can be constructed from the known periphery, since distances can be bisected and parallels drawn with compass and ruler. It was also noted in § 5 that it is not necessary to have the full periphery of the circle. It is sufficient to know an arbitrary peripheral arc. The fact that an analogous result applies here can be seen without further ado in the parabola case. If there is only a parabolic arc covering a certain y-interval, the roots of equation {4) § 21 which fall within this interval are covered without further ado. A transformation y - e y does not change the form of equation (4) § 21. However, such a transformation can be used to convert any root into a root belonging to the interval. Since you do not need to leave the domain of rational arithmetic to do this, *you can use a compass and*

') See also the notes and .§ 2i*

') One may restrict oneself to the case a, -J= 0.

Nn' :d each task of the fourth degree t o deti *ischnitl a stroke with an* öefieöiq *fixed parabolic arc. The* ***same*** applies *in the case of the ellipse and the hyperbola.*

If it is a question of the intersection of the Ng with a straight line, the consideration of § 5 can be transferred because of its projective character. However, it is not worth pursuing thisa because, as is well known, the intersection of a straight line and a line can be constructed using a compass and ruler. ' achon can be constructed if only 5 points of the C' are given. If it were known that any fourth-degree equation can be obtained by inserting the rational parameter representation of the second-order curve into the equation of a paaaing circle'), it would be possible by parameter transformation to transform an interval i containing a real cube of a fourth-degree equation into a partial interval of that interval that corresponds to the given conic averaging arc. The transformed equation is then solved by intersecting a suitable circle with the given conic arc and then applying the parameter transformation in reverse to the solution found in order to solve the original fourth degree equation. This will be explained in more detail at the end of this paragraph. First of all, it is important to recognize whether the most general equation of the fourth degree can be obtained by inserting the rational parameter representation of a fine noncircular ellipse or hyperbola into a variable circular equation. This will prove to be correct with the addition that one can obtain all those fourth-degree equations which cannot be decomposed into second-degree factors by solving quadratic equations. and can therefore be solved. If the parameter representation

$$x = a\,\frac{1-t^2}{1+t^2}, \quad y = b\,\frac{2t}{1+t^2} \tag{I}$$

the *illllysis*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad a \neq b \tag{2}$$

into the circular equation

z° -j- y° -j- 2 e z -j- 2Q y -J- y = 0, (3)

we obtain the equation

f^ (a° - 2 a a -j- y) -j- P 4 Q 6 -J- i- (- 2 a° -j- 4 *b** -J- 2 y) -j-
-l- *t 4 Q b* -J- e° -j- 2 ota -j- y = 0, **(4)**

i.e. a fourth-degree equation in which the coefficients of P and t ') The proof given in the previous paragraph does not suffice.

are equal to each other. You have to ask alao to what extent you can bring any fourth-degree equation to this particular form using a compass and ruler. If you do this in

$$x^4 + a' x^3 + b' x^2 + c' x + d' = 0 \tag{5}$$

the substitution z - - + 2, the coefficients of zJ and zt become

$$a_1 = 4\lambda + a', \quad c_1 = 4\lambda^3 + 3a'\lambda^2 + 2b'\lambda + c'. \tag{6}$$

For a sufficiently *large* 2, both have the same sign.

the substitution z = $\sqrt{\frac{c_1}{a_1}}$ 1, the result is $\sqrt{\frac{c_1}{a_1}}$ = q $\sqrt{\frac{c_1}{a_1}}$ one equation of the fourth degree

t^ -J- *A I** -l- *B t** -J- *A t* -J- II = 0, (7)

where 1° and l have the same coefficients. If we compare any equation (7) with (4), we obtain the linear equations for e, Q, y

a 2a *A* -j- *Q 4* *b- y A* - *a* A* -0,

- *y (J3 - 2) - a*B* - 2o^ + 46* 0, (8)

" 2o (C' -|- 1) - y (C' - 1) - o*6' -|- o^ 0,

whose determinant does not vanish if 1 - & -l- N -J- 0 did. All slopes (7) with the additional condition 1 - *B* -}- *CO -f-* 0 can be solved as an intersection of circle and ellipse. However, if 1 - *B* -|- f7 = 0, you have

$$t^4 + At^3 + Bt^2 + At + C = (t^2 + 1)(t^2 + At + C),$$

and therefore the fourth degree equation is now solvable by square root expressions. In order that (7) really follows from the equation (4) established with the solutions ", Q, y of (16), it must still be shown that e° - 2 e o -j- y -t- 0 iat. However, you can see from the last two equations (8) that 2 z e =- a° + y can only be if o° = b°, which is not the case.

However, if you use the parameter representation

$$x = a\,\frac{1 + t^2}{1 - t^2}, \quad y = b\,\frac{2t}{1 - t^2} \tag{9}$$

dec I::I yperbel

, - t, - 1 - 0 (10)

in (3), the result is

i* (e°- 2 et o-l- *y j- I*4-* |*3 b* -J- *I** (2e° -l- 4h°-2 y) -j- 14 Q *b* + e° -t- 2 et e -l- y= 0. (11)

Now the coefficients of t° and t are equal except for daa YorEeichen. If

(4 J -|- o') (4 2° + 3 "' 2- + 2 ö'2 -]- c') = 0 has at least two real zero measures, one can now choose 2 ao that in (6) < 0). öfold the sub-

g tuti on z, $\sqrt{=-q^{c,}}$ t, then you get a lt $\sqrt{q^{-ci}}$ ° = -q $\sqrt{q\frac{-cr_1}{a_1}}$ ei ne because of a

Fourth degree equation

$$\iota' - A I^* + B t^* + A t + O -- 0\,, \qquad (12)$$

in which the coefficients of t- and f differ only in sign.

Comparison of any (20) with (19} leads to the linear equations

a a 2 *A*	-l- 4 Q ö - *y*	*A- a- A*	-0,	
" o *2 B*	- *(B* -t-	*2- a- B* -J- *2a** -l- 4b° = 0,		(13)
et e 2 (N -J- 1	)- y (N - l)	- a° C -J- e°	= 0	

for e, Q, y, whose determinant does not vanish when 1 -}- *B* -J- *CO -f-* 0. **In order that** (12) really follows from the equation (4) established **with** the solutions e, Q, y of (13), it must still be shown that a° - 2 n e -J- y -{- 0. But you can see from the last two equations (l3) that from 2 e o = a° + y would follow e° -I- ö° = 0, which is impossible. All equations (12) with the additional condition 1 + *B* -j- G' -1- 0 can therefore be obtained by intersecting the circle and the hyperbola.

But if l -l- *B* + D - 0, ao iat

$$l\^{} - At^* + Bt^* -l- At -l- U = \{1^\circ - 1)\,(t^\circ - A\,I - U)\,.$$

The equation of the fourth degree can alao be solved in the case of 1 -}- & + U = 0 by square root expressions. All in all, therefore, a sweep-drawn arbitrary non-aingular non-circular C'" is sufficient to solve all fourth-degree equations with compass and ruler.

Finally, it should be shown that one does not need the full U, but that with *any Jesu drawn* bogeti *of the same* Ott8kO *ml.* Ea at alao any arc

$$x = a\,\frac{1-\varepsilon t^2}{1+\varepsilon t^2}\,, \quad y — b\,\frac{2\,t}{1 -\}- s\,t-}\,, \quad t' < t < t''\,, \qquad e = -l- 1\text{ , ellipse} \qquad (14)$$

$$e = - I\text{ , hyperbola}$$

is given as a fixed point. The given equation of the fourth degree (5) is given by

') But if the equation just given has only one real root for Z, then for the same 4 2 -{- a' 0 a must be a. Since for ßie also the third degree factor disappears, this means according to (6) that in the fourth degree equation obtained by the substitution z , - o'/4 from (5) the members with z and s{ are missing. As the fourth degree polynomial then becomes a second degree polynomial in s*, the corresponding equation can be solved by square root expressions.

eine Substitution

$$x = \varkappa t + \lambda_1 \qquad \text{mit } \varkappa = \sqrt{4\lambda_1^3 + 3a'\lambda_1^2 + 2b'\lambda_1 + c'} \tag{15}$$

and pasaendem rational 2, ao can be transformed so that a given square root of the same falls into the interval given in (14). To do this, consider the cubic equation in 2

(zt - Z)° (o' + 4 2) e = z° (c' -j- 2 *b' I* -J- 3 e' 2° -j- 4 Z°) . (16)

Let z = z¢ -{- -t- I on (t', t") be assumed. (18) has at least one real zero @. I first assume that this zero is -{- z, and that for aie is also c' -t- 4 2_0 0. Then take a sufficiently close the rational approximate value 2, -{- z_1 of this root with 't' -)- 4 2_1 -b 0 and choose the value rt ao, that for z - z, and 2 = the equation (is) is fulfilled. If 2 is sufficiently close to the root of the equation (16) belonging to r_0 , then the value z, belonging to 2, also falls within the interval (t', t").

But then the value belonging to theaem B_l and l = z_l to be taken from (15) is 8 - (z_l - 2t)/z, real and -J- 0. Therefore z = x z -|- B_l and so there is

a root z_1 = ' 1 of t~~he fourt~~h equation tranaformed by (15)

degree in (t', t"). Now the two exceptional cases are to be examined. lat first a' -l- 4 2p = 0 for the root of the cubic equation (16), then the coefficients , q, of the equation transformed by z - t - a'/4 are both zero and the case of a fourth degree equationa is present, which can be solved with compass and ruler. lat however for the zero degrees J_0 of (16) @ = y, then one uses again that according to (24) also

4 z' -J- 3 n' z* + 2 *b'* zt + c' = 0 (17)

must be. Daa is called a root z, the biquadratic equation (5) also satisfies the equation of the third degree (17). From this it can be seen that

e' z} + *2 b' :nj + 3* c' z, + 4 *d'* -- 0 (18)

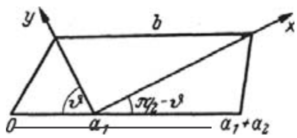
must be. Pa11g now the two equations (17) and (18) have mutually proportional coefficients, one checks that for suitable y the equation

(5) must have the form (z -t- y)" = 0, i.e. it can be solved with a compass and ruler. However, if the two equations (17) and (18) have non-proportional coefficients, the polynomials on the left-hand side must have a common divisor of at most the second degreea, and thus zt can be determined by solving an equation of the second degreea.

§ 23. Hjelmslevs Stechzirkelversuche

Io nEg HJEcMgLuv, to whom the explanations in § 21 go back, refers to these and the

Congruction methods are geometric experiments because, as already mentioned, they are more closely related to practical drawing. They differ, of course, as already mentioned, in that they are certain mathematical operations which are assumed to be feasible, whereas in the case of practical drawing experiments they are only attempts to achieve a usable drawing accuracy without an equivalent in a mathematical operation which in itself means an exact solution to the task. Thus, in the previous paragraph, it was the intersection of a circle and a straight line with a second-order curve that was assumed to be conatruable. The divider is the instrument with which, given a certain



use of the same, the cut becomes conatruable. This paragraph will deal with another use of the compass, of which many more can be thought up. This paragraph will among many possible examples grasp. Mark the points 61, , et -{- e on a number line. At the points, attach two further straight lines q and fi at the angles a - # and aJ2 - ß against the positive direction of the number line. These are the z-axis and the y-axis (Fig. 74). It should now be possible to perform the following operation: construct a polygon line from O to a point on the y-axis, from there to a point on the z-axis and from there to the point e -j- a . The first and third of these lines should have the same length (not a given length, but only the same length), while the line between the y-axis and the z-axis should have a given length *b -f-* 0. According to the Coainuaaatz or the Pythagorag, the conditions of the task are

$$y^2 + a_1^2 - 2a_1 y \cos \vartheta = x^2 + a_2^2 - 2a_2 x \sin \vartheta, \qquad (I)$$

$$°+y'=b. \qquad (2)$$

The construction can be carried out with the dividers by inserting an arbitrary opening r in 0 and marking the point on the y-axis that has this distance s from *0.* Then take the given distance ö and enter the distance from the point found on the y-axis to the z-axis; then see if this point of the z-axis of 'it -l- e has the correct distance s (equal to the distance assumed between *0* and the y-axis). This experiment is carried out with ever new assumptions of the first distance s until the correct polygon course is found. In algebraic terms this means: It is assumed that you can construct the intersection points of the courses (1) with the circle (2),

(I) i8t a hyperbola if e* sin° ß - e* coa° ß is 0. From a distance, this formulation is reminiscent of the intersection of a hyperbola with a circle discussed in § 22. There, however, it was a matter of a hyperbola that was fixed once and for all and of the intersection with an arbitrary circle. Every third and fourth degree problem with compass and ruler can be traced back to this. Here we are dealing with a hyperbola that depends on the task and a circle that depends on the task, which are to be intersected. Nevertheless, the question of the extent to which each third- and fourth-degree task can be reduced to
(I) and (2) with the completion of the previous paragraph. In the event that

$$a_1^2 \sin^2\vartheta - a_2^2 \cos^2\vartheta > 0 \tag{3}$$

the similarity transformation leads to

s= # info-P o sin#,

$$y = \eta \sqrt{a_1^2 \sin^2\vartheta - a_2^2 \cos^2\vartheta} + a_1 \cos\vartheta \tag{4}$$

Abe Byperbel (I) in

$$\xi^2 - \eta^2 = 1 \tag{5}$$

transitions. The circle (2) merges into

$$\sqrt{a_1^2 \sin^2\vartheta - a_2^2 \cos^2\vartheta} \qquad \sqrt{a_1^2 \sin^2\vartheta - a_2^2 \cos^2\vartheta} \qquad \frac{b^2}{\text{0 8iD' -0j CO8'}} \tag{6}$$

about. We compare this with the equation

$$(\xi + \alpha)^2 + (\eta + \beta)^2 = r^2 \tag{7}$$

of any circle. Can it be shown that a" a" ö, ß can be chosen in such a way that

$$r = \sqrt{a_1^2 \sin^2\vartheta - a_2^2 \cos^2\vartheta} \tag{8}$$

and sinceEu still fulfills (3), we have a connection to the previous para- graph, since ea is then the intersection of the fixed hyperbola (Ii) with an arbitrary circle (7). In fact, we can see from (8)

$$a_1 = \frac{b}{r} \cdot \frac{\beta}{\cos\vartheta}, \quad a_2 = \frac{b}{r} \cdot \frac{\alpha}{\sin\vartheta}. \tag{9}$$

If this is inserted into the first equation (8), it becomes

$$\sqrt{\beta^2 \operatorname{tg}^2 \vartheta - \alpha^2 \operatorname{cotg}^2 \vartheta} \qquad (10)$$

or for e -J= 0

$$\beta^2 \operatorname{tg}^2 \vartheta - \alpha^2 \operatorname{ctg}^2 \vartheta = 1, \quad \text{d. i.} \quad \beta^2 \operatorname{tg}^4 \vartheta - \operatorname{tg}^2 \vartheta - \alpha^2 = 0, \qquad (11)$$

which means that (3) is fulfilled by itself. From (11) we can see for -}- 0

$$\operatorname{tg}^2 \vartheta = \frac{1 + \sqrt{1 + 4\alpha^2 \beta^2}}{2\beta^2} .$$

If ß is determined from this, whereby 0 < # < u/2 can still be assumed, ao returns (8)

tg 0 , -sin # . (12)

These equations then also prove to be sufficient for (8) without further ado. In the case et = 0, Q -l- 0, which is still excluded, the equations (8) are satisfied if we now use

e = 0, cotgß = *b* -- *r* ainß .

In the still closed case Q = 0 including z = Q = 0 one takes to satisfy (8)

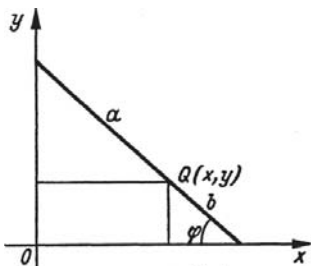
$$\vartheta = \frac{\pi}{2}, \quad \frac{a_2}{a_1} = a, \quad --= r.$$

In fact, as in § 22, any third or fourth degree problem can be solved with a compass and ruler and this divider test.

§ 24 Ellipse circle

Since, according to one of the results of § 22, every third and fourth degree problem can be reduced to the conetruction of the intersection points of a fixed ellipse - independent of the problem - with a circle adapted to the **problem**, it is obvious to use an ellipse circle **to carry out** this construction. **The best known** of these instruments is based on the obvious fact that a fixed **point** Q **of a line of fixed length** moving with its two end points *A* and *B* on the two right-angled coordinate axes describes **an** ellipse. In particular, if the line has the length 'i -l- *b* and is described by Q in the

If the ratio a : *b is* divided in such a way that the section a lies on the y-axis, Q describes the ellipse -J-$\frac{y^2}{b^2}$ = 1, as a glance at Fig. 75 shows. Such an ellipse circle1 can be produced in a simple way in a form adapted to the intended con- structions by using a similar method.
as in the case of insertionIineaI at the edge of aa straight-lined trimmed or cut folded strip of paper or on the edge of a ruler the dots

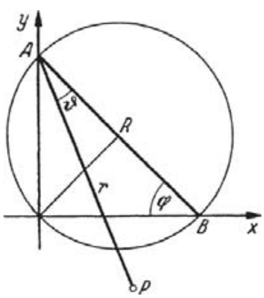


$x = a \cos\varphi$ Fig. 75 $b \sin\varphi$

AQB and then position the ruler so that *öa0 A* and *B* fall on the coordinate axes, but Q comes to lie on the circle with which you wish to intersect the ellipse.

Other applications of the same idea arise from the remark that every point connected in the plane with the line *AB*, not necessarily lying on it or its extension, describes an ellipse, unless it lies on the **circle with the diameter** *AB*. **Then** it describes a **straight line through the** origin (like *A* and *B* themselves).

According to Fig. 76, we have for the geometric location of the points r(z, y), which **with respect to** *A* ala origin **and**



the vector *AB* may have the polar coordinates *r, P* as the direction of the output, the parameter representation

$$x = r\cos(\varphi + \vartheta),$$

y = r a ('p -j- #) - 2A ain 'p,

whereupon, by solving the equations (1), which are linear in cos 'p and a g, for cos 'p and sin 'p and by squaring and adding as

"

Equation of the geometric Ortea of *P*

$$x^2(r^2 - 4rR\cos\vartheta + 4R^2) - 4xyrR\sin\vartheta + y^2r^2 = (r^2 - 2rR\cos\vartheta)^2 \quad (2)$$

finds. The discriminant is

$$(r - 2R\cos\vartheta)^2. \quad (3)$$

It is aIso really an ellipse ea aei that (3) disappears. In this case, however, *P* lies on the circle of radius fi above the through-of my *AB*. Then (2) exists on the double-counting straight line

$$x \sin\vartheta - y \cos\vartheta = 0 \tag{4}$$

through the origin. The aforementioned fixed connection of the point *P* with the line *AB* can be established, for example, by choosing *P* as the third corner of a fixed triangle whose other two corners are *A* and a feater point Q of the line *AB*, while T* lies on the periphery of the circle through the three points *A, 0, B* (Fig. 77). Q divides the distance *AB* by the length e -{- *b* in the ratio e : *b*. The movement which is the same as that of Q

described ellipse + $_{b-}$ = 1 $^{y^2}$can be generated by moving *A* and *B* on the coordinates-

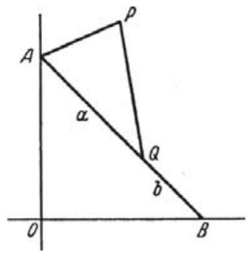


Fig. 77

can also be caused by the fact that *A* and T- move on straight lines through the origin. This formulation suggests the question as to which curve describes a corner Q of a geometrically fixed triangle *A QP* if the corners *A* and *P* are allowed to move on two straight lines through the origin (so that the sides of the triangle remain fixed during the movement). The answer can be found by looking at the circle *K* through the three points €I, *A, P*. This circle has the same shape for all positions of the triangle *A QP* have the same radius. This is because the angle *AOP* at *0* is the same for all positions of triangle *A QP* (because *A* and P are always assumed to lie on the same straight line through *0*). For all circles *K,* which correspond to the different positions of triangle *A QP*, there is always a constant peripheral angle to the constant chord. It is therefore always a circle with the same radius p. *A* and *P* (and therefore also Q) are firmly connected to it. It is the circle of radius p through the two points *A* and *P,* the center of which is the apex of an equilateral triangle with two equal sides e erected above the arc *A P*.

The movement of the triangle *A QP* can also be ale movement of this circle can be described. *A* moves on a straight line through the origin. The diameter of *A* meets the perpendicular established to *OA* in *0* at the point *B* diametrical to *A* (**TBaLES** theorem**).** This point *B* **is** fixed to the circle. *A* and *B* move on two mutually perpendicular straight lines through 0, which may be taken as coordinate axes. Every point firmly connected to this circle, such as Q, which is not located on its periphery, describes an ellipse, as shown above. Accordingly, you can also use a triangle drawn on transparent paper, for example, as an ellipse circle, the corners *A* and T* of which are drawn along two lines drawn on the drawing sheet can be moved through the origin, while the third corner Q describes an ellipse (or, exceptionally, a straight line). The intersection of thea ellipse with a circle8 of the drawing sheeta can easily be seen when moving the transparent paper over the drawing sheet and can b e indicated by drawing in. **Instead of** the transparent sheet, you can also use a three-pointed compass - also according to **IlJELbtSLFv** - which must be constructed in such a way that its three points can be set in the corners of a triangle and held in this position. Then slide two points along two intersecting straight lines until the third point meets the **circle** that you intersect with the ellipse.

§ 56. moving transparent cover sheet and standing circle

The constructions of § 20 with the double right-angle hook can also be carried out with the *right-angle hook* and the *plumb bob*. The right-angle hook is used both as a ruler and for plumb bobs, while the compass is also used to measure distances. In this way, a third degree equationa

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

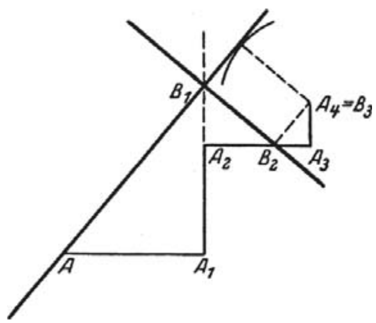
as given in § 20 by its right-angled coefficient train ala visible in Fig. 78. To solve this, first place the right-angle hook in such a way that no edge *passes* through *A*, while aa part at *B* lies on the straight line *A Ab* used to represent the coefficient. Its second leg then intersects the straight l i n e of the ifo coefficient a at *B*. Instead of applying the first leg of the second right-angle hakena to B_1 B$, vertex at *B* , *find* the point *Bq* on *AOA* with the divider in such a way that the circle drawn around *Bq* with the radius *B B$* touches the straight line *AB*. If the point *Bq falls* after *A$* , the equation of the third degree is solved by the chosen position of the right angle hook, i.e . the z-value corresponding to this position. The position of the right angle hook must be changed until thea effect - *Bz* falls on *A* - is achieved (Fig. 78). To do this, it is convenient to d r a w two straight lines crossing at right angles on a transparent sheet of paper and move this cover sheet over the drawing sheet until the test with the compass shows the correct position'). Once the c o v e r sheet has been placed on the drawing sheet in such a way that the line to be inserted has the correct position, the position of points *B* and *B$* can be determined by

---

1) ("brigena one can also remain with the version of § 20 and also use the Replace the **second** rowing angle hook **with** a second aolchea cover sheet.

Use the compass to transfer the markings onto the drawing sheet and then connect the transferred points on the drawing sheet with the ruler.

We will now briefly discuss how the right angle method of § 20 can also be modified for fourth and *higher degree oleohunqs* with the help of a right angle hook and the dividers to make it easier to use. Imagine again the right-angle graph of the coefficients of an equation. Let it start at *A* and end at Ag+ (left the degree of the equation). Now imagine, as at the beginning of the paragraph, a pair of straight lines crossing at right angles drawn on a transparent sheet. Now place thea sheet first ao on the sheet of the coefficient additiona so that one of the two straight lines crossing at right angles passes through *A* and that the vertex *B* of the right angleg lies on A_1 lg. Then determine, as before with the equation third degree, the point *B* on *Az A z* with the dividers so that it is at a distance B_1 B$ from the line AB_1 wil in Fig. 78, and then *B* on A A so that *Bz* from *BMBF* is at the distance *BB* has. If the presented equation, for example, is one of the fourth degree,



I'lg. 78

then the criterion for the solution found is that *the sheet A* collapses, and the position of the transparent sheet has to be changed at a mild right angle until this effect occurs. However, if the equation presented is of higher than the fourth degree, place the transparent sheet 8o so that aa straight line on *BGB* Grillt, while the vertex of the right angles now lies at *Bq,* and then, without having to draw the straight line *BMBF* on the drawing sheet, determine the point *B$* with the compass on *A$A$ ao* that aa distance from *B Bq* becomes equal to *BqB$*. If *B$* then coincides with *A g* for equations of the fifth degree, the equation is solved. Otherwise you have to repeat the test until this effect occurs. How to proceed with equations of even higher degree is obvious. It is important and has also been emphasized by **HJELxgLEY,** who invented thea procedure**,** that a *transparent cover sheet* with *a course, e.g. a pair of right-angled intersecting €!erads, "ereeheneo transparent cover sheet in conjunction with the isleoh circle is sufficient to solve#.* • *i-de on an algebraic G/eicfiuirgendeiner Orades leading constructionoau|9abe.*

A few *examples* may further illustrate the combined use of transparent sheet and compass. First, I will deal with the

Task of \V. K. B. Holz, to construct a *triangle* with *the upper cusps*. The orthocenter, i.e. the height intersection -f of a triangle, determines two lines on each side, one of which extends from Zf to a corner, the other from ff to the opposite side. A aei is the height belonging to corner *A*, *lsq Mix* upper **section**, i.e. the section adjacent to corner *A*. *The* mean perpendiculars of a triangle are also understood to be the distances *l'' lt, 1* of the center pointsa *U* from the sides. Then iat, as will be shown first, *kq -- 21q, hp 21d, k -- 21* .

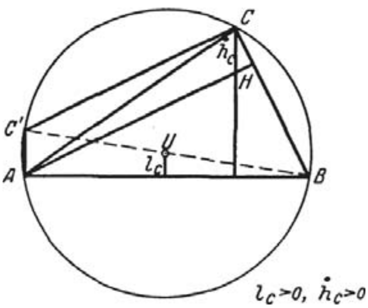
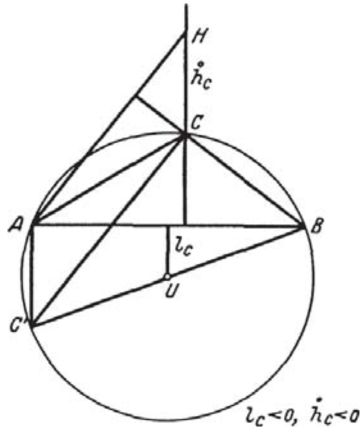


Fig. 79



Flg- 80

the signs of these distances must be taken into account. To do this, orient the triangle by orbiting ea in the positive sense, i.e. in such a way that an interior lies to the left, and take the distance of a point from a side to be positive if the point lies to the left of the oriented side. Then the f'' f'' f are the signed distances of the point *U* from the sides, while is the difference of the distance of the point *A* from the side e reduced by the distance of the orthocenter ff from the side o, uaw. Then iat é = 2 f'' *kp* *21d, k* -- 21, also correct with sign. This can be read from Figs. 79 and 80. There iat the diameter *UB is extended* to the intersection D' with the circumcircle of the triangle, ao that *ACl'* becomes parallel to the height *h* and UU' to the height A. Therefore iat e.g.

AC' -- HC -- Ü'' AC' -- 2f

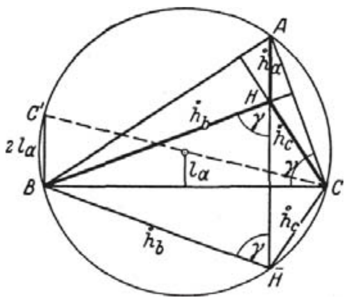
immediately apparent.

If you now carry a chord on the circumference with the chord lengths *h'' kp, h*, with the sign ao, as shown in Fig. 81

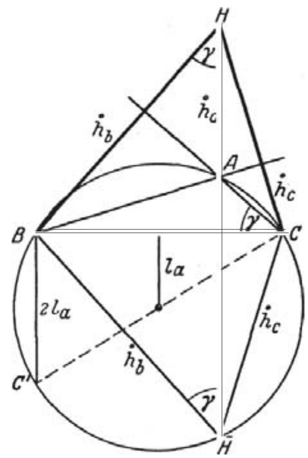
and 52, then the start and end points of the chord are diametrically opposite each other. The chord *OIIBO'* therefore fulfills a semicircle.

Conversely, if such a semicircle move is given in a circle
If b" é" é, is the upper height abachniHe, then a triangle can be specified on the basis of these two images.

The wood conatruction task is therefore comparable to that according to DÖRRIF- ßO mentioned above are identical. For given F" kp, *h*



Flg. 81



Plg. 82

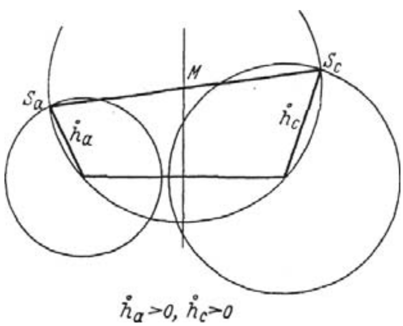
a semicircle, i.e. a radius, is to be found in which the J" é" A, form a semicircle chord. The task can be solved as follows using a transparent sheet and a compass: Since at least one of the is positive, we ateta choose the designation so that *kp* > 0 iat. Aladann we draw on a sheet a line of length and draw circles around its end points with the radii é and F, . At the center of the line I we set up a perpendicular and draw the center ff of the circumcirclea of the semicircular line on it. To do this, we draw a straight line on a transparent sheet and try to place ao over the drawing sheet so that its intersection ff with the center perpendicular of ä is as far away from the end points of Ii as it is from two matching intersections 'S and 6 with the two circles around the end points of é . To do this, the transparent sheet is moved over the drawing sheet until the test with the compass shows that thea distance condition is fulfilled (Figs. 83 and 84).

8-

I also derive the equation of the third degree, which connects the radius of the circumcircle with the three upper height sections. lat r theaer radii, ao one has

That means

CO8ß 2r , COßβ - $2r$, $\cos(\alpha+\beta) = -\frac{h_c}{2r} = \frac{h_a h_b}{4r^2} - \sqrt{1-\frac{h_a^2}{4r^2}}\sqrt{1-\frac{h_b}{4r^2}}$.



I'tg. 93

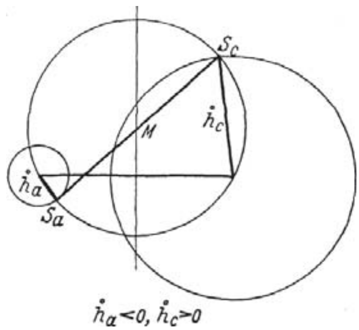


Fig. 84

This is obtained by squaring

4r° - *r (qls* -1- *hp* -l- *h* j - h kp h.* -- 0.

From this equation, the conditions for the solvability of the problem and for the number of its solutions are taken according to the rules of algebra. This may be l e f t to the liver.

I will give *another example*. It aei from a *point P* eiti *fetch* to a *parabola* given by the focal point and the directrix. If p" p are the coordinates of the point *P* and is

y°-Sgm-0 1)

is the slope of the parabola, ao the insertion of the coordinates z, y of the parabola point at which the normal meets the parabola leads, in addition to (1) nor the equation

$$V - pt + \frac{y}{p}(x - p_1) = 0 \qquad (2)$$

to pose. The task therefore requires you to find the **intersection of** the parabola (1) with the isopod (2). **Plan can** of course immediately reduce this to an equation third degree. However, instead of solving this according to the procedure described at the beginning of these paragraphs, it is more convenient to approach the problem directly with the help of a transparent sheet of paper, a straight line drawn on it and the diverging circlea , i.e. to find the point of intersection of the two curves immediately. The remark that the focal point of the parabola is equidistant from the two points at which the perpendicular of point T meets the parabola and the parabola axis leads to this. In Fig. 86 i is the focal point, i is the directrix, *A* is the point at which the perpendicular of *P meets* the parabola, and *B* is the point at which the perpendicular meets the axis of the parabola. According to (2), the equation of the normal at the parabola point (z, y) in current coordinates }, p

$$y - \eta + \frac{y}{p}(x - \xi) = 0 . \qquad (3)$$

The intersection point *B* of the normal (3) with p = 0 is at = z -l- p. ξTherefore *AB* -- z -J- p/2. On the other hand, the distance of point *A* from the directrix is also fid -- z -t- p/2 and therefore *II'A* -- z -}- p/2 according to the fundamental axis of the parabola.

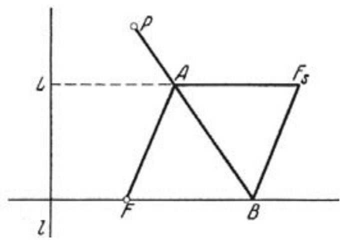


Fig. 85

requirements, point *A* can be constructed by placing the axis, guideline and *P* on the drawing sheet and drawing a straight line on a transparent sheet.
- the normals are drawn aoll -. Then place thea sheet over the drawing sheet so that the straight line *passes* through *P*. If *B ib:r* intersects the axis, ao use the compass to *trace* the line *Ii'B*, hold one point in feat, place the other on the line *PB, Maa* may give a point *A*, then aetse the other point of the compassa at this point *A* and check with the opening *'B* of the compassg whether a circle placed with thea radius around *A* touches the guiding line i. The position of the transparent sheet must be changed until this effect occurs.

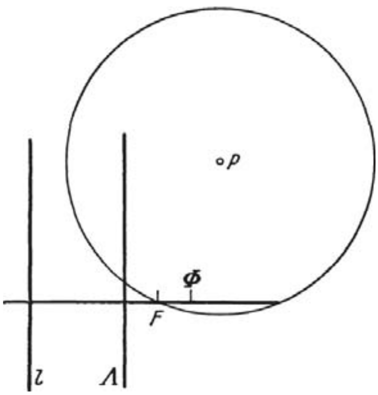
Fig. 85 shows another convenient construction of the parabolic normal through the point *P*. In Fig. 85 you can also see the imirror image J, off in relation to the parabolic normal of the point *P*. *Il'$* lies on the circle of the radius ET around the point *P from the* center. Further, however, the following statement results for the geometric location of the mirror points of F with respect to all parabolic normals: Since the quadrilateral visible in Fig. 85 is a rhombua and since *MA -- AB - c + p|2 has* already been established, it follows that the coordinates n, z of J are these:

$$\sigma = 2x + \frac{p}{2}, \quad \tau = y . \qquad (4)$$

Where z, y are the coordinates of the parabola pointsa at deaaen normals was mirrored in J, . Since equation (l) therefore applies to z, y, J, lies on the parabola

$$\tau^2 = p\left(\sigma - \frac{p}{2}\right) \tag{5}$$

with half the **parameter** pJ2 and the vertex J. Its focal point & and **their guiding line** 2 can be seen in **Fig.** 86. The construction of J, - and

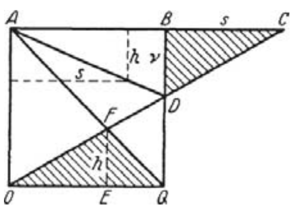


Flg. 8fi

so that the **parabola normal ala the perpendicular** from *P* to the straight line NJ, - therefore amounts to the intersection of the circle *P* (NN) with the parabola (5), and this intersection can be accomplished with the dividers, as described on p. 98/99.

Ea is obvious, as is the construction of the common tangents of two conic sections discussed in § 20 using the method dieaea
§ 25 can be amended.

Another example. **Mr.** A. SPEISER has me occasionally to one from Arabic sources known fiia- *echiebunqoau|gabe by* AnGHTHEDES, which is closely related to the construction of the regular Siebenecka. As in sig. 87, there is a



Flg. 87

Square of the edge lines I and a of a diagonal are drawn. A straight line ao is to be laid through aa corner 0 so that the two triangles hatched in Fig. 87 have the same content. This problem can be solved using the method described in this paragraph. Place a straight line through corner *0* using a transparent
Blattea has a trial straight marked on it.

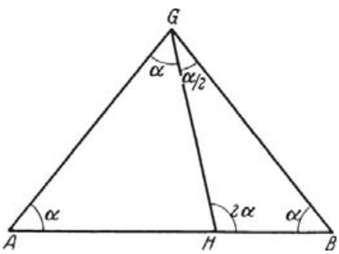
The condition of the task requires that the relationship Ur = *s*|*is* exists for the pieces marked in Fig. 87. To check whether it is fulfilled for the test line, lay another test line through the points *A, D* of Fig. 87 with the aid of a second transparent sheet, determine the point *P* on it with the compass, which has the distance *h* from *AB,* and see whether this point *P* has the distance s from *OR*. To do this, take the line Ä on p. **117** and find the point on *AD* that is at a distance fi from *AB*, i.e. choose it so that a circle drawn around it with radius *h* touches the line *AB*. Then hold the point of the compass on *AD* firmly and grasp it with the compass.

circle the distance die8ea points from *OA*. See if this distance is equal to s and change the position of the test line until this effect is achieved.

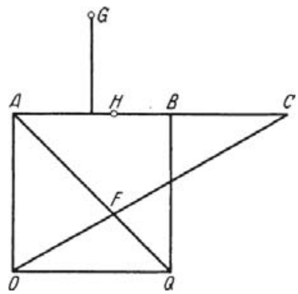
The connection between the task and the regular heptagon is shown by the following consideration: The test line drawn through *0 in* Fig. 87 has the equation y = y z. It intersects y = - z -l- 1 at

$$y = h = \frac{\mu}{1+\mu}, \quad x = \frac{1}{1+\mu} \tag{6}$$

and meets y = I at z = Ug = I -t- s. Therefore iat s = (1 - du)/y. Therefore



Flg. 84



E'lg. 89

the task

1 -|- y *(1 - /z) .

Daa iat converted to s

$$\frac{1+s}{2+s} = s^2 \text{ wegen } \frac{1}{1+\mu} = \frac{1+s}{2+s} \tag{7}$$

or, which is the same thing,

If you set s = Um, you get H -{- 'r° - 2s - I = 0. According to p. 59, this is the equation for the number 2 cot (2 7ZJ7) associated with the regular heptagon. Therefore iat e = lJ(2 coa 2 uJ7).

To a *noois better* - held entirely in the spirit of HJsLoscsv -
The following remark, also based on Anc2tIbtEDss, leads to the *construction* of " with the compass. JYfan construct over the base line *AB* of length 1 an equilateral triangle A *BU* with the initially arbitrary isides *AG* -- *BO* -- *e* (Sig. 88). Its base angle is at e. Then ' x = s/2. **If** the distance *AH* -- e° is further subtracted from *AB,* then

a triangle *AIIG* is obtained. According to the cosine theorem, the side *äd* -- s° is in it. It is therefore also equiaxed. Now remember that according to (6) and (7) the distance *OE* and thus the distance of the pointF from the squared side *OA* is just e[t] . Therefore, the following construction of a i s obtained with the dividers, which is illustrated in Fig. 89: Using a transparent sheeta , lay a test straight line through the corner 0 of aa drawn squarea with an edge length of 1 and bring theae with the diagonal *A Q* theaes squarea in one point to the intersection. to the intersection. Plan a perpendicular in the little of the square side *AB*. Then use the dividers to measure the distance of the points *f r o m* the square side *AB*. Now insert one point of the dividersa into *A* and move to Zf so that *All* is equal to the distance of the points from the side *AB*. Then hold the point of the dividerg at H and, holding the divider in place, bring the other point to the center line of *AB*. This gives you a point d on this. Now use the dividers to measure the distance *OB* and see if theae *is* equal to *BC!*

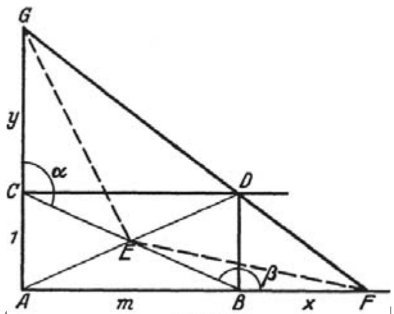


Fig. 90

kes for this effect to occur.

It is not known whether AnouiuzDze thought o f solving the problem in the way described here. However, the fact that the Hjelmalevache method of **construction** with the compass and the test line (without a transparent cover sheet) was not far removed from antiquity may be shown by a final example going back to Arozconms. It is a 2photographic representation of *the third root.* If the third root is to be taken from the number m, assume a rectangle *ABC!D* with sides 1 and m, as shown in Fig. 90. Determine aa center point A and place a straight line p ao through a corner *D* so that its intersection points and O with the extended sides *AB* and *A€l* of *E* have the same distance. This can, of course, be done immediately using a compass and a transparent cover sheet (on which the straight line g is marked). Then iat the distance *BH'* marked z in Fig. 90

even]/m . **Alan** denotes CO with y. Then it follows from similar triangles Um = (1 -t- y)J(z + m) and l/z = Um. Stan denotes the diagonal of the rectangle by d. Then the application of Coainugaatzea to two triangles with the angles et and Q teaches

From this follows

$$\frac{x}{u} = \frac{y - d\cos\alpha}{x - d\cos\beta} = \frac{y + 1}{x + m} = \frac{1}{x}.$$

So in total And

from this follows

z and y are the two middle proportions between 1 and m.

§ 56. sewing ionization conditions

The belief that one must be able to solve any construction problem, e.g. also the trisection of any angle, with compass and ruler, cannot be eradicated, although the impossibility of constructing a circle with a ruler is a quite obvious example of the fact that there are limits to the scope of any construction aid, and although the proof given in § 13, e.g. that the angle of 80 degrees cannot be trisected with compass and ruler alone, is sufficiently simple to be made accessible to anyone who feels called upon to deal with mathematics. For example, the proof given in § 13 that an angle of 80 degrees cannot be divided into three with a compass and ruler alone is sufficiently simple to be made accessible to anyone who feels called upon to deal with mathematical problems. Nevertheless, it is not only outsiders, but also academics, even qualified mathematics teachers at higher education institutions, who repeatedly offer compass and ruler constrictions for the trisection of any angle. The constructions, which the originators consider to be exact, naturally turn out to be approximate constructions whose result is more or less close to the desired goal. It cannot be the task here to enter into an enumeration of such approximations. But the goal of fundamental completeness that the book has set itself makes it seem appropriate not to completely ignore these procedures.

We call approximation aconatructions constructions with compass and ruler in the sense of § 6, which, for example, do not exactly divide the angle into three equal parts, but which nevertheless provide values that differ so little from the true value that they arouse our interest, especially if the difference in drawings of normal size proves to be imperceptible. The construction given by the great painter MaRECBT DÖRER, fully aware of its approximate character, is of the highest dignity and rarely surpassed accuracy. Instead of dividing the arc corresponding to the angle 'p into thirds, DüRsR divides the chord of the arc into thirds and b u i l d s the perpendiculars in these partial points up to the circle arc. These peripheral points are connected to each other and to the beginning and end of the arc by a chord.

connected. The arithmetic mean of these three chords is then taken as the chord of the approximate angular third. The recalculation not to be presented here shows that the difference against the third part of Q for
0 Q uJ2 increases with 'p and reaches its maximum for Q = u/2 with 18". Fig. 91 shows the construction given by DöRsR. The chord *AB* is divided into three parts in *O* and *D* and perpendiculars are constructed which meet the circle in A and . Make *AG* -- *AUH* and *BH* -- *BA,* thirds UCr in and *BH* in A with JR = 1/3 f7R and *KH* *1 J3 DH* and finally makes *AG* -- *AJ* and *B* f -- *BK.* Then fi and ff are the approximate third points of the arc of a circle *AB* according to DöBER.

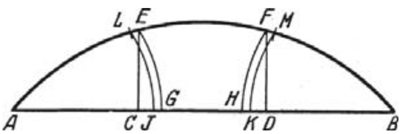


Fig. 91

Among the many other possible approximations for the trisection, two more may be mentioned, which can be found on p. 78 and p. 79.

The exact "constructions by means of insertion are given if the conchoidal or Pascal's scroll playing in there is replaced by paagendeKreig arcs. First the insertion between two straight lines (Fig. 45, p. 78). The distance of length 2 is drawn between the straight lines z - cos g and y = sin g on a straight line through the vertex *0* - which is also the origin of the coordinates. In other words: The conchoid with pole *0,* base z = cos g and interval 2 is intersected by the straight line y = a g. This conchoid meets y = 0, the one leg of the

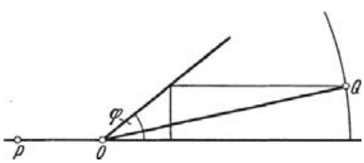


Fig. 92

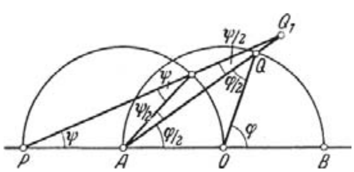
angle to be thirded **from** the **first** quadrant, in z - 2 -l- cos g. Now bring a circle **instead of** the conchoids

$$(x + \alpha \cos \varphi)^2 + y^2 = [2 + (\alpha + 1) \cos \varphi]^2,$$

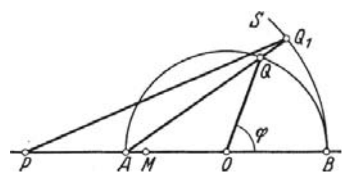
which also meets y = 0 in z = 2 -}- cot g, with y - a 'p to the cut and also et to be chosen in such a way that **the** maximum error **occurring** for 0 Q **u/2** is as small as possible. Numerical calculation shows that e = 1.27 is almost correct. The **circle used** is the **circle of curvature** in z = 2 -l- cos Q, y = 0 of the conchoidal curve belonging to an angle of approximately 57° 18' 59". It is easy to construct as e defines its center. With g = 0 and
'p = uJ2 the construction is accurate. The maximum error is about 40". Apply alao to y = 0 to the left of the origin I,27 cos g to obtain **the** center T* of a circle, which is to be placed **through the point** z = 2 -l- cot 'p, y = 0 and intersected by y = a 'p. The resulting point Q of

Sig. 92 is connected to 0. OQ approximately includes the angle 'p/3 with y = 0. The quality of the approximation can also be judged by the difference between the points Q and Q where the creia and conchoidal line
;y = sin 'p meet. This difference is smaller than8 2- 10 *. For example, if the unit of length chosen for the drawing is 10 cm, ao is 'Jp mm. For about
'p 72° 30' the approximation angle is too large, for 'p > 72° 30' the approximation angle is too small. The fact that e is chosen as favorably as possible can be seen from the fact that the absolute values of the maximum deviations upwards and downwards are equal. For each angle 'p, the angular error changes monotonically with e.

Another approximate construction can be obtained by using a Modification of Archimedes' construction Yon p. 79, Fi{;. 46, the Pascal



Flg. 03



T "lg. 04

scroll replaced by a circle. This modification can be seen in Fig. 93. The two circles have the same radius *r*. The centers are points *A* and *0*. *BOQ* is to be divided into thirds. The line *r* is inserted on a straight line through *P* between the circle around *A* and the straight line *AQ*. It forms the angle QJ3 with *PA* in *P*. According to Fig. 93, 'p/2 = y -{- y/2,
i.e. 'p = 3 y. The construction is based on the intersection of the Pascal's helix of the circle *A(r)* with pole *P* and distance *r* with the straight line *A Q*. For approximation, take a suitable circle with the center on *AB* through the point *B* where it touches the snail. For example, the circle of curvature of the worm at *B* will give a good approximation for small angles. Its radius is (9/5) *r*. As FmsLEa has calculated, for 0 N 'p 22° 30' you get a three-part approximation with a maximum error of 0"- 074 at g = 22° 30'. In Fig. 94, first draw the circle of radius *r* around *0* as the center and extend its diameter *AB* by *r* to *P*. Then determine the point ff as the center of curvature of the scroll in *B,* i.*e*. choose ff ao that f& - (9/6) *r*, and draw the circle of curvature with *MB* as the radius. *BiS* is an arc of the same. Then enter the angle 'p = @ *QOB* in *0* and intersect the straight line A Q with the circle of curvature in Q_i . The angle Q_k *PB* is then almost g/3.

The approximation of the screw by an arc of a circle is also the basis of the approximation construction of the master tailor Korz in Ludwigghafen, which became famous through a work by Mr. PzRRon. The approximating circle has to be chosen so that it provides an exact construction for 'p = 90º. In Fig. 95, the tangent to the circle *O(r j* is laid through T*, the distance r is traced on this from the point of contact, i.e. the point of intersection of *O(r)* with *A(r),* to the outside of *A[r)* and through the point 'S' obtained øo and through & a point in & the screw is laid.

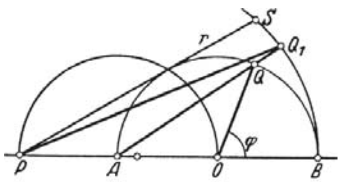
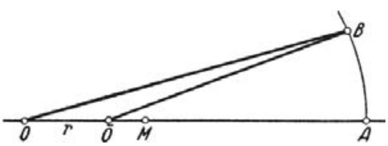


Fig. 95

touching circle. It - which which has point 6 in common with the worm – is cut, instead of the worm, with the straight line *AQ* in a point Q. Then iøt @ Q_A *PA* almost ø/3. The maximum error is 14" - 867 at g = 69º 56' 2" 447. When they became known through Herm Pzenon because of their high ge-

This surprising construction is, as we know today, only one of a whole chain of constructions of even greater accuracy based on the same principle; for example, we can note with P. Fusses that every angle in the form 'p = n - 45º -I- y, 0 y 22º 30', n whole, can be written. Then ø/3 = I-t 15° -}- y/3. However, since 16º all difference between 60º and 45º can easily be con- øtructed, øo the construction mentioned above, which is based on the approximation of the screw by its circle of curvature in *B*, provides an approximation con- atruction for any angle with a maximum error of only 0" - 074.



Fîø. 88

The following final construction, which approximates another curve by a circular arc **instead of** the scroll, is even more accurate. In Fig. 96 Rei *OO -- r, OA --* 3 *r, ôfA --* (2 -I- 4/7) *r.* A circle is drawn around if as the center point with *MA at* radius. g aei the angle to be thirded. It aei @ *BOA -- 'p|4.* Then @ *BÕA* is almost øJ3. For 0 ø 22º 30' the maximum error is 0" - 016 for 'p = 2 2 º 30'. Thus, according to the above, if suitable multiples of 45º are subtracted for all angles, we have an approximation with this maximum error. The congruence uses

in A the curvature circleø of the curve e - - $\frac{}{ßin \frac{\varphi}{3} \quad \frac{\varphi}{4}}$. You can aie according to **FINSLER** to the n-division.

Starting from the construction with the carpenter's hook shown in Figs. 65 and 66, it is also possible to approximate the trisectrix with a circular arc, following n'Oczonz.

§ S7. Regular polygons

As an example of the scope of the cubic constructions, the regular polygons congruent with the insertion ruler should be determined. These are the ones for which the equation of the three-division can be traced back to a chain of equations of the second and third degree. öflecting the Cardan formula, it can also be said that these are the ones for which the equation of the three-division can be solved by repeated extraction of square and cube roots. This **leads** to a consideration similar to that made in § 14 when dealing with the question of regular polygons that can be constructed with a compass and ruler. Again we start from an irreducible equation of the first degree /(z) = 0 in the body *K* of rational numbers. By adjuncting } to R. we obtain a body *K(5 j* which has the relative degree tt with respect to *K.* (See § 14.)

By multiple adjoint of square roots and cube roots we obtain an upper body *Kg* over A, in which Zf(}) is contained as a lower body. The relative degree n of A(}) with respect to R is therefore given by a
p. 60ff is a divisor of the relative degree of *Kg* with respect to *K.* The relative degree, however, is of the form 2° 3 since square and cube roots are added one after the other. Therefore n is also such a product n - 2' 3 .

The condition found is, as p. 64 in the case of constructions with compass and ruler, a necessary, not a sufficient condition. A necessary and sufficient condition is provided by Galoigache's theory of slopes in conjunction with group theory. It can also be expressed here in the following form: Under }, }1.-
f" - i we understand **all-**
roots of an irreducible equation /(z) = 0 in *K.* For the fact that aich
even one of these roots can be represented by a square-cubic root expression, it is necessary and sufficient that the relative degree of the body *K [$,* d" ... , }q) with respect to *K* is of the form 2' 3*.

Let us apply this result to the circle division equation. If it is to be solvable by square and cube roots, its degree g(zt) must be of the given form. (Cf. §14.) Thus

$$\varphi(n) = p_1^{\mu_1 - 1}(p_1 - 1)\, p_2^{\mu_2 - 1}(p_2 - 1) \cdots p_\nu^{\mu_\nu - 1}(p_\nu - 1) = 2^\alpha 3^\beta.$$

it follows that the prime factors pt are either 2 and 3 or only
Daraus

inof the eratepower in tt and are of the form 2' 3* + 1. The result is therefore: the *regular* tt-I *attn can only be congruent* with the Aimcfiiebe- *linml if its corner number n is of the form 2'* 3* *p* *p,* *whereby the p lazier different primes are of the form 2^e^ 3° + I.* That this condition is also sufficient can be seen from the algebraic theory of circle division. This should not be explained here. Possible prime factors of a are therefore 3, 5, 7, 13, 17, 19 ... . In contrast, for example, the regular pentagon cannot be configured with the single-slide ruler.

§ S8. The quadrature and rectification of the circle. Qnadjustable circular arc branches

The task of constructing a number that is not the root of an algebraic equation with rational coefficients is called non-algebraic or tranacendent. This includes the task of determining a content from the radius of a circle (quadrature) and finding a circumference (rectification). These tasks cannot be solved with compasses and rulers in the sense of § 6 and cannot be solved with a single-axis ruler. This is because the number a, on which ea depends, is a tremmnent *SaM, i.e.* it does not satisfy an algebraic equation with rational coefficients. This was proved in 1882 by **FanoTNND LlND£mnn** according to a basic idea stated by C aau.ss **HERMIzz** when proving the transcendence of the number e. Since then, many mathematicians, including some of the greatest such as KAnc WEZER- sein andDamn Hnazez, have developed new proof variants following the basic idea. The proof has been included in so many textbooks and can be found in so many places in the literature that it is not necessary to reproduce it here. Rather, I prefer to follow
C. L. Sizosz, who uses a method derived from A. 0. G£LFOirD. Although the Beweia approach may have some inconveniences in magician implementation, ao it is based on a very simple memorable basic idea and was applicable to other tranazendence questions, as shown by A. 0. **GELFOND, C. L. SIEGEL** and also Ta. SGHNEZDzR have shown. While one is at a loss to characterize the basic idea of Hermite-Lindemann's proof in a few words, the proof to be given here is based on the assumption that exp(zz) with all aone derivatives at the points z = 0, 1, 2, ... is not compatible with the growth properties of thea function for z -+ m. To carry out the proofs, some tools are necessary from the theory of algebraic numbers and from the theory of functions, which are familiar to every trained mathematician. These will now be explained.

1. A number d is called algebraic if it satisfies an equation (1) with rational coefficients. Sum and product of algebraic numbers
are algebraic numbers again. For ea aeien }1 llnd p, the two algebraic numbers and d2 , ... , d", the roots of an algebraic equation

with rational coefficients , ... , ng and furthermore p" . . . , pp are the roots of a second algebraic equation

$$x^m + b_1 x^{m-1} + \cdots + b_m = 0 \tag{2}$$

with rational coefficients. Then the two products have

$$\underset{\alpha,\beta}{\Pi}(x - \xi_\alpha - \eta_\beta), \quad \underset{\alpha,\beta}{\Pi}(x - \xi_\alpha \ \eta_\beta)$$

according to the main theorem on symmetric functions, are themselves rational coefficients if these products are extended over aße z from 1 e ii and all Q from 1 Q m . Therefore }t -J- pt aand } p are also algebraic numbers.

2. The }; and the p$ are called *algebraic* weapons if the coefficients , ... , ap and t'2 , ... , b", in (I) and (2) are integer rational numbers 8. Then the coefficients of the two specified products are also integer rational numbers (and again the coefficients of the highest powers of z are exactly 1). Therefore, the sum and product of integer algebraic numbers are again integer algebraic numbers.

The equation of lowest degree (1) with rational coefficients, which is satisfied by a given integer algebraic number }, in other words, the irreducible equation in the body of rational numbers, which is satisfied by the number, has in turn integer rational coefficients, if the coefficient of the highest power of this equation is assumed to be 12). The absolute value of this equation is different from zero for n > 1.

3. An algebraic integer z that is also rational is a rational integer. For then the degree of the irreducible equation in fi, which satisfies e, is one, i.e. the equation z - z = 0, and its coefficients are integer rational numbers.

4. If } is any algebraic number, ao always gives ea rational integers p -t- 0 such that p} is an algebraic integer. I f eq z' -l- - - -l- ag = 0, e0 -|- 0, any equation with rational integers

') Cf. Biznzaeace-Bznzs, \"orlesungen über Algebra, 6th ed., p. 2d7: Gxoss' theorem: If a polynomial /{z) with integer ra4ional coeoisients can be decomposed into two factors with rational coefficients, it **can** also **be decomposed** into two factors with integer **rational ifoefficients.**

coefficients, one of whose roots is }, then o is an algebraic integer. By multiplying the equation for } by ab we see that y = o } of the equation

$$y^n + a_1 y^{n-1} + a_2 a_0 y^{n-2} + \cdots + a_n a_0^{n-1} = 0$$

is sufficient.

5. Finally, remember the remarks in § 14 about algebraic number fields. If an algebraic number } is added to the body fi of rational numbers, which satisfies an equation of degree a that is irreducible in this body fi, ao a body fi(}) of algebraic numbers is created, which has the relative degree li with respect to fi. Each number e of this body then satisfies an equation of at most nth degree irreducible in fi. If }, = }, }" ... , }q are the roots of the irreducible equation for }, then according to § 14 daa e has a representation

$$\alpha_1 = \alpha = A_0 + A_1 \xi_1 + \cdots + A_{n-1} \xi_1^{n-1}$$

with rational coefficients, and are called

$$\alpha_k = A_0 + A_1 \xi_k + \cdots + A_{n-1} \xi_k^{n-1} \qquad (k = 2, \ldots, n)$$

the numbers conjugated to z. Then Zf (z - e) = G(z) = 0 is an equation with
i
coefficients on fi of degree n, which satisfies e. The equation q (z) = 0 of the lowest degreeB in fi, which satisfies z, is a factor of the same. Its degree is therefore at most a. More precisely, the degree of *g* is a divisor of n. According to the proof in § 14 Satz about the relative degree, the relative degree n of fi(}) over St is the product of the relative degreesg m of St(e) over fi and dea relative degree of fi (}) over St (e).

If we again adjoin to fi(}) an algebraic number p which satisfies an equation (2) of degree si with rational coefficients which is irreducible in A, then according to § 14 we obtain a body fi(}, ty) of algebraic numbers whose relative degree with respect to A according to § 14 is both a multiple of tt and a multipleB of m iat. Such bodies fi(}, p) are, however, completely identical with the iforbs fi(ß), i.e. they can also be obtained by adjunction of a single algebraic number ß to fi. If } = }, }" ..., }q are all the roots of (1) and p = p, p" ..., p", are all the roots of (2), then choose the rational numbers e and ö so that the a to numbers

$$\vartheta_{\nu\mu} = a\,\xi_\nu + b\,\eta_\mu\,, \quad \vartheta = a\,\xi + b\,\eta\,, \qquad (\nu = 1, \ldots, n, \mu = 1, \ldots, m)$$

are all different from each other. All you have to do is assume e -1- 0 and then weigh *b* so that there is no difference between two of the n m numbers #,p disappears. Then

$$\prod_{\nu, \mu} (x - \vartheta_{\nu\mu}) = \Theta(x)$$

a polynomial (ti m)-th degree with rational coefficients, deasen Nussatellen jene #, p aind. If then &(}, p) is any rational function of } and p with coefficients from fi and if &, p = 'f'(}" p) are the values obtained by replacing the }, p in &(}, p) by any }" p, ao iat also

is a polynomial with rational coefficients. From this it follows for z = ß that

$$\Phi(\xi, \eta) = \frac{'(4)}{0'(6)}$$

is. This proves that fi({, p) is - fi(ß).

6. The absolute implied of U(z) is the product (- I)' Zf e$ over z= 'zt and
t
its all conjugates and is called the norm of et: *N('x* jand also the norm of the conjugates with et. *The norm iet OI8O defines ats product of all K -im!-<--*

•• i• *lu'i algebraic number rone of N2tH "different from zero".*
rational number. The norm of a *9integer algebraic number other than zero* is a qarise *rational number other than zero.* The absolute value of the norm of a non-zero algebraic integer is therefore at least one. By definition, the norm *N("j* depends on the body fi(}).

7. One can avoid the somewhat subtle discussions in 5. and 6. in a manner sufficient for the purposes of these paragraphs if one understands by *N(5 j,* i.e. the norm of the algebraic number }, the absolute term multiplied by (- 1)" of that in fi irreducible equation (1) with the highest coefficient 1, which satisfies }. If the su } conjugate numbers are then understood to be the remaining roots of this equation, then *N( )* is equal to the product of d and its conjugates except fora sign. For every non-zero algebraic number }, *N(j* is therefore a non-zero rational number. Furthermore, the norm of an algebraic integer is a rational i n t e g e r . The degree of an algebraic number is then called the degree of the irreducible equation in fi that the number satisfies. If }, p are then two algebraic numbers of **degree** n and m which may satisfy equations (I) and (2) respectively and if ß - *r(, p j* is a rational function of } and p with rational coefficients, ao satisfies # the equation ff [z - r(}" p)] = 0 of degree In n if the product over }, p and all is extended to both conjugates. The equation has rational coefficients. The in fi irreducible equation satisfying ß corresponds to a divisor of its left-hand side, and therefore ß has degree a m at most.

Now the general statement about algebraic numbers should be applied to n. The assertion that neither e nor u are algebraic numbers is contained in the *more general assertion* that doc e andexp (e) *for a.-f-* 0 cannot be qfeic£- temporiy *algebraic numbers*. For for e = I this is the assertion that e is transcendent, ufid for z = 2ai it is the assertion that 2ai and thus u is transcendent. (For if a were algebraic, ao would also be algebraic as a product of algebraic numbers).

The proof that u is trangcendent is now given by the proof of the more general theorem: e and e' are never simultaneously algebraic for z -{- 0.

To prove this, we assume on the contrary that cc aeien et -|- 0 otid e° *alqebraioch.* Then we adjoin aie to the body fi of rational numbers and obtain a body fi(e, e') of algebraic numbers to which the numbers ot' and n'e' also belong for each integer rational s. I8t is then the whole r a t i o n a l ZaM *g* 0 is such that *qa* and pe' are algebraic integers'), ao q'".' and *g'+' a.'e'* are also algebraic. *h* is the degree of the body fi(e, e'),
i.e. a relative degree in relation to fi.

The function-theoretical tools required for the proof are Cauchy's integral theorem, Cauchy's integral formula and the concept of the residue, which the reader will find inevery textbook on function theory.

The proof is based on an *interyolator representation of the* joa£iion /(z) - e''. Here z -J- 0 is a number, z is the complex variable.
Stan divides the identity

by (} - z) (} - z). Then it becomes

$$\frac{\ }{\zeta - z} = \frac{\ }{\zeta - z_k} + \frac{\ }{(\zeta - z_k)(\zeta - z)} \cdot \qquad (3)$$

Bier Bind } and z complex variable and rd with £ = 0, 1, ... , n dio places where interpolation is to be performed. Stan writes down (3) for £= 0, I, ... , n and insert the identity of the number £ -t- 1 into that of the number £. Daa

') As stated above, ea gives an integer rational number g, -[- 0 such that 9,a is algebraically integer, and an integer rational number -[- 0 such that g,e' is algebraically integer. Then for g = g, q is obviously both qm and ge' completely algebraic, because the product of whole algebraic numbers is again completely algebraic. (The whole rational numbers obviously belong to the whole algebraic numbers).

•) Cf. E.g. L. BIBBERBzcn, Lehrbuch der Funktionentheorie Bd. I, 4th ed. 1834, American reprint 194ö or I . Bizazaazcu, Introduction to Function Theory, p. ed., Bielefeld 1951.

leads to the idontity

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)(\zeta - z_1)} \qquad (4)$$
$$+ \frac{(z - z_0)(z - z_1)}{(\zeta - z_0)(\zeta - z_1)(\zeta - z_2)} + \cdots + \frac{(z - z_0)\cdots(z - z_n)}{(\zeta - z_0)\cdots(\zeta - z_n)(\zeta - z)}\,.$$

Now multiply theae by /({) and integrate over a **circle** r in the **positive sense,** which contains all the variables z_0 , q, ... , zg, z **l i n** **the** **interior.** Then, according to N. E. Nösnrir, miin has the following interpolatory representation of the t'nnction {(z):

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)(z - z_1) + \cdots + a_n(z - z_0)\cdots(z - z_{n-1}) \qquad \mathbf{(5)}$$
$$+ r_n(z)(z - z_0)\cdots(z - z_n)$$

$$a_k = \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)\cdots(\zeta - z_k)}\,, \qquad £ = 0, 1, \ldots, n \qquad ()$$

$$r_n(z) = \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)\cdots(\zeta - z_n)(\zeta - z)}\,. \qquad (7)$$

If the points z$ are assumed to be different from each other, the result is a representation of the Newtonian interpolation formula with remainder. The interpolation points should be chosen differently for the purpose of the proof of sanscendence. To do this, take an integer rational number m > 1, which is then suitably disposed of, and generally use

z f for n 0, I, 2, .. . , (8)

n = m s -1- i, 0 ts m - 1; in other words, set

$$z_m = 0,\quad z_{m+1} = 1, \ldots, z_{2m-1} = m - 1, \qquad (8')$$
$$z_{ms} = 0,\ z_{ms+1} = 1, \ldots, z_{ms+t} = t\,.$$

Because (Ii) becomes

f()=•,+q•+az(•-1)+---

$$+ a_{ms} z^s (z - 1)^s \cdots (z - m + 1)^s$$
$$+ a_{ms+1} z^{s+1}(z - 1)^s \cdots (z - m + 1)^s \qquad \mathbf{(5')}$$

+"m-+'*"(-)+'"'h-'+')+ih-)" h-' + f

$$+ r^{(z)}_{ms+t} z^{s+1}(z - 1)^{s+1} \cdots (z - t)^{s+1}(z - t - 1)^s \cdots (z - m + 1)^s,$$

and it is

$$a_{ms+t} = \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{\zeta^{s+1}\cdots(\zeta-t)^{s+1}(\zeta-t-1)^s\cdots(\zeta-m+1)^s}$$

$$r_{ms+t} = \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{\zeta^{s+1}\cdots(\zeta-t)^{s+1}(\zeta-t-1)^s\cdots(\zeta-m+1)^s(\zeta-z)}. \qquad (9)$$

Now the proof of the following assertion is started: If the function l(-) -- -" øamŁ all derivatives at the points z = 0, a = 1, ... , z = øi - 1 with a paøaender choice of m has algebraic numerical values, then /(z) must b e a polynomial. This nonsensical 1 'olgerung proves that œ -}- 0 and e' cannot be algebraİBch at the same time. This is because the stated values of the function and its derivatives aare

e°J'# o,i-0,1,2,...,

and they would all have to be algebraic if œ and e° were algebraic.

The proof, however, that {(z) must be a polycom is based on the fact that the oq for sufficiently large numbers a and that the remainder of (5') for n -- n disappears. The eg and daø Regtg1ied must be estimated for these digressions.

I first estimate the coefficients e,ø,q t upwards. Letu be = s* - 2 (øi - 1) mil any fixed s from 0 < e 1 the Into- grationøweg. Then for all I from 0 tñ m - 1

$$|a_{ms+t}| \leq \frac{\exp(|\alpha|\, s^{1-\varepsilon})}{s^{(1-\varepsilon)s}(s^{1-\varepsilon}-1)^s\cdots(s^{1-\varepsilon}-m+1)^s}$$

-expose'-'-m -e)#log4- log 1- $\frac{k}{1-\varepsilon}$ (10)

< ezp (ø| a* -- + m (m - 1) a' - m (1 - c) a log a) .

Because it is available for 1£ ü m - 1

$$-s\log\left(1-\frac{k}{s^{1-\varepsilon}}\right) = s\log\frac{s^{1-\varepsilon}}{s^{1-\varepsilon}-k} =$$

$$= s\log\left(1+\frac{k}{s^{1-\varepsilon}-k}\right) < s\,\frac{k}{s^{1-\varepsilon}-k} = \frac{s^\varepsilon k}{1-\frac{k}{\cdot}} < 2ks^\varepsilon$$

and therefore

$$-s\sum_{k=1}^{m-1}\log\left(1-\frac{k}{s^{1-\varepsilon}}\right) < 2s^\varepsilon\sum_{k=1}^{m-1}k = s^\varepsilon m(m-1).$$

Further, for - e '- and | s1 according to (9) for iille Ot N
m - 1

$$\left| r_{ms+t}\, z^{s+1} (z-1)^{s+1} \cdots (z-t)^{s+1} (z-t-1)^{s} \cdots (z-m+1)^{s} \right| \tag{II}$$

ezp (| o | a* "') (m!)*

___" 8' " "* (8' "* - 1)' - - (8 "' - Nt -|- 1)*

< ezp() o a' ' -|- (a -I- I) log(m!) -I- m (m - 1) a' - m (1 - c) *e* log8*).*

(1I) shows that for feates or and featea e in z 1 of the

lizn r"" + z^" - - (z - m -|- 1)* 0 (12)

iat. e° is therefore represented for z_1 I is represented by the heg level unend- interpolation series resulting from (5').

(10) also shows for feate m and c that lim ap, + t = 0. Hence

could only be derived from a"" t t = G if the ag,+ t were rational integers. But this is not the case. However, we will soon establish that the a""+ t are algebraic numbers from the body A('x, e'). We will also determine the factors that must be applied in order to convert the ep,+ t into whole algebraic numbers. The conclusion that the a""+ t we are large s zero, however, must then be drawn by estimating the norm. And for this purpose, the numbers conjugated to the *ag,+* t must also be estimated. This is alla accomplished by the residuals theorem of function theory, which should now be applied to the representation (9) of the coefficients *a*", +. According to this, a""+ t is equal to the sum of the Reaiduen at the singular points of the integrand enclosed by the integration path. However, these are the places 0, 1, 2, ..., m - 1. The Reaiduum at each of these places is the coefficient of the - I- th power in the Laurent expansion of the Irite- grand at the relevant place. Since, for example, at = 0 the integrand has a (s + 1)-fold polar cell, the Taylor expansion of the integrand freed from the factor I J{- + ', i.e . from

$$\overline{(\zeta-1)^{s+1} \cdots (\zeta-t)^{s+1}(\zeta-t-1)^{s} \cdots (\zeta-m+1)^{s}}$$

at } = 0 and, in particular, use the coefficients of the members }' of the Taylor expansion. However, this is known to be the s- th derivative of the function just written at } = 0, divided by e! Similarly for the other zeros of the denominator of the lntegrandon. This gives us the following representation (13) of a", + t:

$$a_{ms+t} = \frac{1}{s!}\sum_{\mu=0}^{\mu=t}\frac{d^s}{d\zeta^s}\left(\frac{f(\zeta)(\zeta-\mu)^{s+1}}{\zeta^{s+1}\cdots(\zeta-t)^{s+1}(\zeta-t-1)^s\cdots(\zeta-m+1)^s}\right)\Big/_{\zeta=\mu}$$

$$+\frac{1}{(s-1)!}\sum_{\mu=t+1}^{\mu=m-1}\frac{d^{s-1}}{d\zeta^{s-1}}\left(\frac{f(\zeta)(\zeta-\mu)^s}{\zeta^{s+1}\cdots(\zeta-t)^{s+1}(\zeta-t-1)^s\cdots(\zeta-m+1)^s}\right)\Big/_{\zeta}$$

$$=\frac{1}{s!}\sum_{\mu=0}^{\mu=t}\sum_{k=0}^{s}\binom{s}{k}f^{(k)}_{(\mu)}\frac{d^{s-k}}{d\zeta^{s-k}} \Big/_{\zeta} \tag{13}$$

$$+\frac{1}{(s-1)!}\sum_{\mu=t+1}\sum_{k=0}\binom{s-1}{k}f^{(k)}_{(\mu)}\frac{d^{s-1-k}}{d\zeta^{s-1-k}}$$

$$\times\left(\frac{(\zeta-\mu)^s}{\zeta^{s+1}\cdots(\zeta-t)^{s+1}(\zeta-t-1)^s\cdots(\zeta-m+1)^s}\right)\Big/_{\zeta=\mu}.$$

In (13), /*$ = 'x* e°*. Therefore, a s already noted above, +"/(*) and thus also q'+ '^ /J*) for 0 k *(e ,* 0 y to - 1 is quite olgebraic if q x and pe' are gonz a1gebraic and p -1- 0 i s a pitaaing rational integer. When the differentiations occurring in (13) are carried out, the denominators are again integer rational functions, but they a r e all divisors of

gQ)= Q}-1):(§-mdl "+1/Q-p@ּ*1

are. The fact is t h a t every denominator *N[(j* has whole rational coefficients and that every *N[(j* has a whole rational Q({) with likewise whole rational coefficients such t h a t ip(d) = *N[(j* Q({). Therefore, for every integer rational number ,tt every denominator *N(g j* is a divisor of g(/i) = [p! (m - I - y) ! (- 1)"' -j" + ', and according to the binomial theorem, this number iis a divisor of (m - 1)! " + '. Because according to theaem

sentence is ') p (pJ an integer rational number. One has

Therefore, as a result of the consideration that all denominators occurring after the differentiations in (13) are divisors of [(m - 1)! + ' are. All in all, therefore, the

" ;#+1 !@"+-1)]"'1 (14}

integer algebraisoho numbers.

The conjugates 't ,+ , of op + t are obtained from (13), if in //*$ = ot'e°* which replaces et and e° with their conjugates. Ea may have z the degree p, e° the degree n. Then according to 7. of p. 129 e", ,+ t has a degree A n p. It aei e = , e° = q, and ot" ... , e" *e , ... , e* aeien the other conjugated. Ea aei

$$A = \textbf{Max} \quad (|\alpha_j|,\ |e_k|,\ 1). \qquad (é)$$

Then /* < A' +'^ for all conjugates. We use this to estimate the 'i ,+ u p w a r d s in (13). In order to estimate the upward derivatives of the rational functions there, remember the formula for the differentiation of a product of m - 1 factors, which is proved by independent induction and which was already used for m = 3 in the derivation of (13). It reads')

$$\frac{d^k}{d\zeta^k}(f_1,\ldots,f_{m-1}) = \sum_{\varrho_1,\varrho_2,\ldots,\varrho_{m-1}} \frac{k!}{\varrho_1!\cdots\varrho_{m-1}!}\frac{d^{\varrho_1}f_1}{d\zeta^{\varrho_1}}\cdots\frac{d^{\varrho_{m-1}}f_{m-1}}{d\zeta^{\varrho_{m-1}}}. \quad (16)$$

As usual, 0 ! = 1. p" p" . . . . pp_t means non-negative **whole** numbers.

') See e.g. 0. Römer and G. Amen, Differential- und Integralrechnuog, vol. 2, Bsrlin 1838, p. 20. There seem to be only a few modern books in which the formula is given. Therefore, the proof is given below:

(18) is obviously correct for £ = 1. Because then

$$\frac{d}{d\zeta}(f_1\cdots f_{m-1}) = f_1' f_2\cdots f_{m-1} + f_1 f_2'\cdots f_{m-1} + \cdots + f_1\cdots f_{m-1}'$$
$$= \sum_{\sigma_1+\cdots+\sigma_{m-1}=1} \frac{1!}{\sigma_1!\cdots\sigma_{m-1}!} f_1^{(\varrho_1)}\cdots f_{m-1}^{(\varrho_{m-1})}.$$

Assume that (18) is correct for *k - I.* Let it therefore be known

$$\frac{d^{(k-1)}}{d\zeta^{k-1}}(f_1\cdots f_{m-1}) = \sum_{\sigma_1+\cdots+\sigma_{m-1}=k-1} \frac{(k-1)!}{\sigma_1!\cdots\sigma_{m-1}!} f_1^{(\sigma_1)}\cdots f_{m-1}^{(\sigma_{m-1})}.$$

$$\frac{d^k}{d\zeta^k}(f_1\cdots f_{m-1}) = \sum_{\sigma_1+\cdots+\sigma_{m-1}=k-1} \frac{(k-1)!}{\sigma_1!\cdots\sigma_{m-1}!}\frac{d}{d\zeta}\left(f_1^{(\sigma_1)}\cdots f_{m-1}^{(\sigma_{m-1})}\right)$$

Here the et, . . . tp _ coefficients still to be determined, and the sum is again over all p" p"- - . es - t with the 8sum £. Each item of the last sum is created by a single differentiation from an item of the (k-1)th derivative. Therefore *I,* depending on which of the factors is differentiated again,

$$a_{\varrho_1\cdots\varrho_{m-1}} = \frac{(k-1)!}{(\varrho_1-1)!\,\varrho_2!\cdots\varrho_{m-1}!} + \frac{(k-1)!}{\varrho_1!\,(\varrho_2-1)!\cdots\varrho_{m-1}!} + \cdots$$
$$\frac{(k-1)!}{\qquad}$$

Only for eacha p 0 a summond has to be added, since, as mentioned above, each poat is a unique differentiation of one of the (£-l)th derivatives.

$$a_{\varrho_1\cdots\varrho_{m-1}} = \frac{(k-1)!\,\varrho_1}{\varrho_1!\cdots\varrho_{m-1}!} + \frac{(k-1)!\,\varrho_2}{\varrho_1!\cdots\varrho_{m-1}!} + \cdots + \frac{(k-1)!\,\varrho_{m-1}}{\varrho_1!\cdots\varrho_{m-1}!} = \frac{k!}{\varrho_1!\cdots\varrho_{m-1}!}.$$

The $\frac{}{!'}$ p_are the polynomial coefficients'), i.e. whole rational numbers. len. Therefore iat

$$\sum_{\varrho_1+\cdots+\varrho_{m-1}=k} \frac{£!}{\varrho_1!\cdots\varrho_{m-1}!} = (m-1)^k\,,\ \text{wie auch}\ \sum_{k=0}^{s}\binom{s}{k} = 2^s$$

is. Formula (18) can be applied to (13). Because there the. products of m - I factors are to be differentiated. The item ({ - y)- + * or

') The polynomial theorem - a generalization of the binomial mismatch to more than two summands under the 6th power - is

$$(x_1+\cdots+x_{m-1})^k = \sum_{\varrho_1+\cdots+\varrho_{m-1}=k} \frac{2!}{\varrho_1!\cdots\varrho_{m-1}!}\, x_1^{\varrho_1}\cdots x^{\varrho_{m-1}}.$$

It is proven by complete induction. Obviously, for £ = 1 is correct

$$x_1+\cdots+x_{m-1} = \sum_{\varrho_1+\cdots+\varrho_{m-1}=1} \frac{1!}{\varrho_1!\cdots\varrho_{m-1}!}\, x_1^{\varrho_1}\cdots x_{m-1}^{\varrho_{m-1}},$$

because there is always only one pi -|- 0, namely = I. Assuming cc Aoi

$$(x_1+\cdots+x_{m-1})^{k-1} = \sum_{\sigma_1+\cdots+\sigma_{m-1}=k-1} \frac{(k-1)!}{\sigma_1!\cdots\sigma_{m-1}!}\, x_1^{\sigma_1}\cdots x_{m-1}^{\sigma_{m-1}}$$

richtig. Dann ist

$$(x_1+\cdots+x_{m-1})^k = \Big(\sum_{\sigma_1+\cdots+\sigma_{m-1}=k-1} \frac{(k-1)!}{\sigma_1!\cdots\sigma_{m-1}!}\, x_1^{\sigma_1}\cdots x_{m-1}^{\sigma_{m-1}}\Big)(x_1+\cdots+x_{m-1})$$

$$= \sum_{\varrho_1+\cdots+\varrho_{m-1}=k} a_{\varrho_1\cdots\varrho_{m-1}}\, x_1^{\varrho_1}\cdots x_{m-1}^{\varrho_{m-1}}.$$

Here the et, ... t", $_{-1}$ coefficients still to be determined, and ea is the sum again to be extended over all non-negative p" p" ....e - i with the sum £. Each pole of the £th power arises aue aom the (le-1)-tsn by blulbiplication with a zj. Therefore, depending on which zj al8 factor was added,

$$\frac{(k-1)!}{(\varrho_1-1)!\cdots\varrho_{m-1}!} + \frac{(ß-1)!}{\varrho_1!(\varrho_2-1)!\cdots\varrho_{m-1}!} + \cdots + \frac{(k-1)!}{\varrho_1!\varrho_2!\cdots(\varrho_{m-1}-1)!}$$

Only one summand has to be written down for each pt -[- 0, since, as already mentioned, each Pmtsn arises from one of the (£- 1)th power by multiplication with an sj. Therefore, again

$$a_{\varrho_1\cdots\varrho_{m-1}} = \frac{(k-1)!\,\varrho_1}{\varrho_1!\cdots\varrho_{m-1}!} + \frac{(k-1)!\,\varrho_2}{\varrho_1!\cdots\varrho_{m-1}!} + \cdots + \frac{(ß-1)!\,\varrho_{e-1}}{\varrho_1!\cdots\varrho_{m-1}!}$$

$$\overline{\text{pi I - - - p - 11'}}$$

The analogy with the £th derivation is striking. This is why it is also customary to write

$$\frac{d^k(f_1\cdots f_{m-1})}{d\zeta^k} = (f_1+\cdots+f_{m-1})^{(k)}.$$

If you set s,-s - - - = em i - 1, ao you get the formula mentioned in the **text** concerning the sum of all polynomial coefficients.

() - y)' in the numerator only means that the corresponding item of the denominator is to be omitted before differentiation. Nowfor 0 N t

$$\frac{d^\lambda}{} \frac{(\zeta-\mu)^{s+1}}{} \qquad -$$

it' t-+'- "-"-+'(t -'- i)-- " -- + i)'

$$\sum_{\varrho_1+\cdots+\varrho_{m-1}=\lambda} \frac{Z!}{\varrho_1!\cdots\varrho_{m-1}!} \frac{(-1)^{\varrho_1+\cdots+\varrho_{m-1}}(s+1)\cdots(s+\varrho_1)\cdots s\cdots(s+\varrho_{m-1}-1)}{\zeta^{s+\varrho_1+1}\cdots(\zeta-m+1)^{s+\varrho_{m-1}}}$$

Therefore, for 0 2 ü s

$$\frac{d^\lambda}{d\zeta^\lambda} \frac{(\zeta-\mu)^{s+1}}{\zeta^{s+1}\cdots(\zeta-m+1)^s}\Big/_{\zeta=\mu} < (m-1)^\lambda (s+\lambda)^\lambda \leqq (m-1)^s (2s)^s$$

This Abachiitzung') was so rough that, as can be seen at a glance, it also remains correct for the other derivatives of rational functions occurring in (13). Therefore, according to (13) we have the valid estimate for all conjugates

., / A'+"' 2(m-1)(2) (m-l)<C' a. (17)

Here U means a number independent of a. To estimate the norm of e""+t, we have to use (10), (14) and (17). Thus, we find the following estimate (18) for the norm of an integer a1gebraized by (14) by using (10) for op + t itself and (17) for the *h* - l conjugates:

') It is also possible to avoid the formulas used in differential calculus and still arrive at a useful conclusion. You can then use the Oauchyache integral formula

$$\varphi^{(\lambda)}(\mu) = \frac{\lambda!}{2\pi i}\int_k \frac{}{(\zeta-\mu)^{\lambda+1}}, \quad k: |\zeta-\mu| = \frac{\cdot}{2}$$

$$\varphi(\zeta) = \frac{(\zeta-\mu)^{s+1}}{\zeta^{s+1}\cdots(\zeta-m+1)^s}, \quad 0 \leqq \lambda \leqq s.$$

Plan then takes place

$$\varphi^{(\lambda)}(\mu) | \leqq \frac{1}{2}\lambda! \max_{|\zeta-\mu|=\frac{1}{2}} |\zeta-\mu|^{s-\lambda} \Big/ \prod^{£ - m - 1} |\mu-k+e^{i\varphi}/2| \leqq \lambda!\, 2^m.$$

That means

with a number independent of o . for which you can take 2(m - 1), as in the text.

N (o ,+t g""- s! [(m - 1)!/*°"')

$$< C^{(h-1)s}\, s^{(h-1)s}\, g^{h(s+m-1)}\left[(m-1)!\right]^{h(2s+1)}$$

Xe ()t1- -+ y(y) - ()) | g) (18)

< C'j s" ')* e ((") 8' -|- ex (m - 1) a' - m (1 - c) slogs)

< e () " a! ' -|- m (m - E) a* -j- s logC/ -|- th - 1 - m (1 - c)/ aloga).

Here U means another number independent of e. If you therefore choose

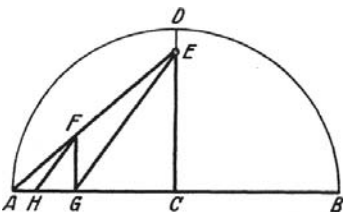


Fig. 97

$$m > \frac{}{1-\varepsilon}$$

and if such a si is fixed, the following applies

lizaN(*" ,+,* '+*-'s!(("r - 1)!)*+')- o.

For sufficiently large s, this norm therefore has an absolute value less than I and is therefore zero as a rational integer. Hence the o""+ t for large s are also large nuts. If we now look at (12), we see that (5') with the coefficients (9) merges into an infinite series which, however, breaks off because its coefficients are all zero from a certain number onwards. The series arepresents the function e°' in z 1. Theae is therefore a polynomial for z 1 and therefore, according to the principle of analytic continuation, for all z. This obvious nonsense teaches us that e and e° for 'x -J- 0 can never be algebraic numbers at the same time.

Approximations for n can be made to the chain break development

$$\pi = 3 + \frac{1}{7+\frac{1}{15+\cdots}}$$ –- s, 7, 1s, i, **292,** i, 1, ...]

extract. Approximation breaks aare

$$3+\frac{1}{7} = 3{,}142\ldots \text{ (Archimedes)},\quad [3,\,7,\,15] = 3+\frac{15}{106} = 3{,}14150\ldots,$$

p3, 7, 15, 1] = 3.14159292 ... = 3 -l- $\frac{4*}{7*-F8"}$

The latter can be constructed like this: Fig. 97. Here *CAD -- I, C!lil --* 7J8, *AG --* **1/2, JG** *OD, EH EG, AH --* **4°/(7°** -I- 8-). The subsequent proximity fractions for a are:

[3, 7, 15, I, 292] - 3 -l- $\frac{4687}{33102}$

[3, 7, 16, 1, 292, 1] = 3 -l- ~~703$~~,

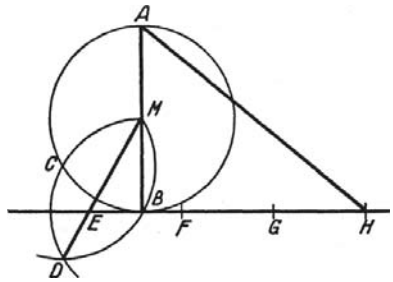
[3, 7, 15, 1, 292, 1, 1] ^ 66317 '

According to the general **theory** of **continued fractions,** the **approximationB-** break 3 -|-$_0$ ~~g um~~ hÖChatens $\frac{1}{106\cdot 113} <$ 10* too small. The approximation is alao only slightly beaaer than the general theory suggests. However, the general theory further teaches that the next approximation truncation

3 -{- $^6{}_3$ with the only slightly larger denominator 113must give a much better approximation. For it can

highest data at .3. 33102< - ' o*

be large.

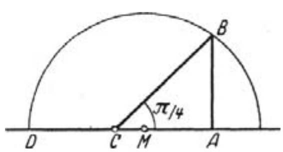


Ftg. 88

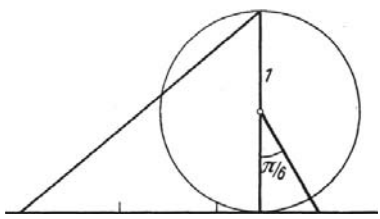
The proximity breaks described above can be seen one at, that

$3 + \frac{4703}{33215}$ is expected to be particularly good again.

The old approximation construction of Koc xnsxY is particularly elegant



Flg. ß8



Flg. 100

from 1685, which uses only one compass opening. In Fig. 98 *MA* -- *MB* -- *BO* -- *BD* -- C f -- *ID* -- fiJ-- J9-- Call and *AH* – $\sqrt{2^2+\left(3-\frac{1}{\sqrt{3}}\right)^2} =$

0 3 i - 3 , 141533 ... a good approximation for a.

Particularly simple and sufficiently accurate for all practical purposes did 'lie the following recently published approximation aconatruction by HEnt-itICß STILLER. It is shown in Fig. 99. The circle has a radius of 1 and *AB* is perpendicular to the marked diameter. Ea is G f = 1J4

and *DA* -- US (7 -{- J3J = 1, 5710 ... a good approximation to u/2.

Incidentally, the construction which L. Cesuona described in a graphical calculation of 1875 to the engineer CERADrxi

ascribes. It iat according to Fig. 100)/ 2- + (3 - tg a/6)- = 3.14153 . . .. It should be noted, however, that this approximation according to Fig. 98 agrees with the much older one by Kocaansxr, with the only difference that Koemnsxv has given the construction a particularly elegant fit by the mere use of a single circular opening.

We understand a *triangle of circular arcs* to be an area bounded by two circular arcs. If *A andö B are the* two corners, ao the two arcs have the chord *AB* in common. The area of the triangle is then equal to the sum or the difference of two circle segments with this common chord. Let *r* and *r$* be the radii, 2 Q and 2 gt be the angles of the two circular sectors to which these chords belong. Then iat

$$r_1^2 \varphi_1 - \frac{r_1^2 \sin 2\varphi_1}{2} \pm \left(r_2^2 \varphi_2 - \frac{r_2^2 \sin 2\varphi_2}{2}\right)$$

the content of the Kreiabogenzweieck. Due to the common chord, there is still the condition

$$r_1 \sin \varphi_1 = r_2 \sin \varphi_2 .$$

If we take *r$* -- 1, ao then becomes *r* - $\frac{\sin \varphi_2}{\sin \varphi_1}$ and the content of the Circular arc branches

$$\frac{\sin^2 \varphi_2}{\sin^2 \varphi_1} \varphi_1 - \frac{\sin^2 \varphi_2}{\sin^2 \varphi} \cdot \frac{\sin 2\varphi_1}{2} \pm \left(\varphi_2 - \frac{\sin 2\varphi_2}{2}\right).$$

The following is the *problem of the two arcs of a circle.* It is assumed that both the arc bisector and its content are congruent with compass and ruler for a given distance 1 (Tt adius of one circle). In addition, it should be assumed that g, = m #, g = n # with whole rational m and n and fitting angle ß. Then the in- stop of the circular arc bisector

$$\frac{}{\sin^2 m\vartheta} m\vartheta - \frac{\sin^2 n\vartheta}{\sin^2 m\vartheta} \cdot \frac{\sin 2m\vartheta}{2} \pm \left(n\vartheta - \frac{\sin 2n\vartheta}{2}\right)$$

as a square root augdruok an a1gebraic number. Since ginm ß, sinn # are also aIge- braigohe numbers due to the assumed congruence of the arc of the circle, then

ßin-m0 ' " "

must be an a1gebraic number. If we divide by ml, we see that even aJtn ß must be an algebraic number. lat here a -}- 0, it follows that ml must also be an algebraic **number** a. Therefore ml and ainni ß and therefore also i m ß and e"" would be algebraic numbers at the same time. But it was proved earlier that this is impossible. Therefore iat o - 0.

The problem of squaring arc bisectors therefore boils down to the question for which positive integer rational numbers si and n

ßin-m0 m '

becomes an algebraic equation for a # that is solvable by a square root expression. Since this is not possible with plus signs, there are no configurable bisectors of arcs of circles, and the question for the others is for which integers m and n an equation solvable by a square-root expression is possible.

---

can exist for ain ß. Up to now, five such triangular bifurcations are known;

m - 2, n1 ; m3, n - 1 ; m - 3, n - 2 ; m - 5, n1 ; m - 5, n - 3.

If m is prime and a = 1, then L*nce has also shown that for qua drier- ble arc bisectors this prime must be a Gaussian, i.e. one of the form 2° -{- 1 a. Tgcmxacorr again showed that m = 1 does not lead to squareable monads.

In 1771 **EULER** gave four monads whose quadrature is a third degree problem. Ea are theae:

m = 4, ti = 1 ; m = 4, " = 3 ; m = 5, Ii = 2; m = 5, ii = 4.

§ S9. Ifonstructions with compass and ruler on the ifngel surface

When constructing on the surface of a sphere, a ruler is an instrument for drawing great circles through two given, non-diametrically located points. As geodesic lines on the sphere, the great circles represent the role of straight lines in the geometry of the plane. By vortex we mean an instrument for drawing circles around a given point as the center with a given compass span. We call the circular *span* the length of the chord between two points on the surface of a sphere. The compass span of a circle is therefore the length of the chord between a)Y center point and a peripheral point. We distinguish the 2tadiu" of the circle from the circular span. This is the distance of the center pointsa from the periphery measured on the surface of the sphere.

Of course, you can also use the compass to draw great circles as soon as you know the corresponding compass span, the great circle compass aperture. If *r is* the radius of the sphere, ao iat *r is the great* circle aperture. With the Constructing on the surface of the sphere, new points are created from given points. points as intersections of circles (small circles and large circles). My first aim was to *gain* an *overview of all the points that can be constructed from given points using a compass and ruler.*

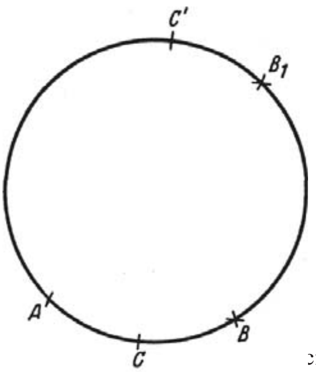


Fig. 101

If only one point or two diametrically opposed points are given, you cannot construct another point using a compass and ruler. Let us assume that there are two points *A* and *B* that are not diametrically opposed to each other. Then, as is initially clear, if *A* and *B* are two corners of a simultaneous triangle inscribed in a great circle, you can only construct the third corner and no further point. If this is not the case, ao aei the apherical distance of *A* and *B*. Then the great circle apanne is given. Place around *A and B* as circle span great ciriae. They are perpendicular to each other and intersect at the)¥center points of the two great ciriae passing through *A* and *B* and perpendicular to the first two. Let us now assume that the spherical distance between the two points is not (aJ2) r. Then let the closest sun be less than (2s/3) r. Then the two circles placed around *A* and *B* ala center points with the circle span *AB* intersect at two non-diametrical points U and Ut. A great circle is drawn through both with the ruler. It meets the great circle through *A* and *B* at two diametrical points C and U', which bisect the two great circle arcs defined by *A* and *B*, and is perpendicular to the great circle through *A* and *B*. If you can now also bisect the great circle arc 6'f7', ao you have a third great circle perpendicular to the first two and thus the great circle span again. This does not work according to the construction described for arc *AB,* because UC' is half a great circle. If, however, the arc *C!B* > (a/6) *r, co* is traced from N' as in Fig. 101 as arc *C'B, and* then you have arc *BB* < (2s/3) *r.*

)Yfan can therefore bisect the arc *BB* according to the construction described for *AB* and arrive ao at a point bisecting the semicircle UN'. If the arc *AB* does not satisfy the specified condition, note that by continuing to bisect and join all arcs with of an apherical length of the form n $\frac{OB}{2^k}$ can be constructed. With *CB* iat denotes, for the sake of brevity, the aphaerial length of the arc *OB*, and n and ä are integers. Among these arcs, however, there are certainly those that $r > a \frac{OB}{2^k} > g\, r$ used earlier and which can be

can then be subtracted from N and U' to obtain a suitable arc *BB,* the center of which is also the center of the arc GC/'. Finally, if the spherical length of the given arca *AB is* between (2u/3) r and a *r*, double this arc and consider the rainbow arc of the great circle passing through *A* and *B*. *This arc will then be shorter ala (*2u/3) r. This is then shorter ala (2s/3) *r,* so that the described construction can b e applied to it. The case in which the arc *AB* has the spherical length (2a/3) is the exceptional case just mentioned, in which the great circle span cannot b e found from the two given points. In a.Hen other cases, three pairwise perpendicular great circles have now been found. Their six intersection points are *N, N'* ,-0, 0' -, *P, P'* and two points labeled with the same letters a r e diametrically opposed. Then, for example, *N, O, P* are the three corners of a spherical octant, i.e. a spherical triangle with three right angles. We now want to introduce the planes of these three **pairs** of perpendicular great circles as the planes } = 0, p = 0, d = 0 of a right-angled Cartesian coordinate system. The designations are chosen so that *N* has *the* coordinates (r, 0, 0), 0 the coordinates (0, *r,* 0) and T* the coordinates (0, 0, *r j*. We now project the sphere from *P* onto the equatorial plane = 0. In this plane we choose *N* and *0 from* the r-point of the coordinates and denote by z, y the coordinates of the stereographic projection of the pointsa }, p, {. Then, as is well known

$$\frac{r}{r-\zeta}, \quad y = \frac{g\, r}{r-\zeta} \qquad (*)$$

$$\xi = \frac{\sim\sim}{x^2+y^2+r^2}, \quad \eta = \frac{2\, y\, r^2}{*\wedge + y^\circ + r^-}, \quad (- r\, \frac{*' + ¥' -}{s- + y- + r-} \qquad '''$$

The double spatial coordinates 2}, 2 p, 2 and r a r e known as circular margins. For example, 2} is the circular gap between the spherical point *K,* one coordinate of which is }, and its mirror image at the great circle } = 0. However, *K* is also used to construct thea mirror image. For if, for example, *K lies* in the eucloctant with the corners *N, 0, P,* ao place around 0 and *P* as the center the two ciriae through *K,* which are in the mirror image of *K* at } = 0 again.

Conversely, it is easy to see that a point with the double spatial coordinates 2 }, 2 p, 2 can be found as the intersection of three circles if the circle spans 2 }, 2 p, 2 § are given and the sum of squares 4 r-have. The sign of the }, p, indicates the octant in which the point is located. If, for example, it is the octant with the corners *NOP* and all three coordinates are positive, then first obtain two points *z4 B* on the great circle *N P* with the circle span 2 d = *AG*. Bisect the great circle arc *AB* at a point M. (fr/2) *r* reduced by the spherical distance between 0 and (u/2) *r* of the two points 1 and ff is then the spherical radius of a circle around *P* as the center on which all points with twice the spatial coordinate 2 d of the spherical surface lie. The same applies to the other two spatial coordinates.

A construction with compass and ruler on the fuyefober/föcbe *becomes a construction with compass and ruler in the aegualo plane and vice versa through this stereoqraphic projection*.

As is well known, the circles on the surface of the sphere merge into the circles and straight lines of the equatorial plane in a reversible and unambiguous manner. With each circle on the sphere you know three of its points and thus their stereographic projection. However, a **circle** (or straight line) through three points can be constructed in the plane using a compass and ruler. Conversely, if a circle or a straight line is g i v e n in the equatorial plane, you know three of its points and thus their projection onto the sphere. However, you can construct a circle through three given points on the sphere again using a compass and ruler (according to the definition of these instruments given at the beginning of this paragraph). This can be done in exactly the same way as finding the center of a circle through three points in plane geometry. This consideration now gives an *overview of the collectivity of the points, which can be constructed on the given sphere using a compass and ruler. Determine* the *coordinates of their* sfereoyrapbic *projection* according to (1)*, form from them* iryetidwefcfie quodrofwurzefamdrüc£e *and enter in* (2) *any of these* qundraftrurzefousdrüc£e. *iso one obtains the (double) spatial coordi- nates of all points that can be conslrued from given points with compass and* direct *on the sphere.*

The result can also be expressed as follows: Plan Wilde from the spatial *coordinates of the given points* irpendtrefcÄe QandraftrurzektusdrticLe. *Any three* eofcÄe *outdriicke, whose sum of squares is r*, are then the spatial coordinates of constructible points, and one thus obtains the* spatial *coordinates of all points constructible with compass and* direct *from given points on the sphere.*

Finally, the result can also be expressed (independently of a right-angled coordinate system) 8o : *Starting from given points with given mutual compass spans, one can construct all and only those compass spans on the sphere with compass and ruler which can* be *represented from the given compass spans by* Q,uadratttnirze/ausdriic£e.

I8t *ar* the great circle arc corresponding to the compass span *o r*, then s = 2 i s an et/2. Therefore, the sats can also be expressed in this way: *Starting from* a *given orbital arc, you can construct all and any great circle arcs on the sphere using a compass and a ruler,* the *trigonometric* functions of which *can be represented by square-inexpressions from the trigonometric II'unclions of the given orbital arcs.* The only exceptions to this rule are the following two cases:

1. There are only two points with the aperture distance (2 7r/3) *r*.
2. There are only two diametrically opposed points.

§ 30. Ifonstrnctions with the compass alone on the upper edge of the sphere By **compass** we mean the instrument explained at the beginning of § 29,

init in which circles are created around existing points as an open center point. New points are created as intersections of such circles. Again, we call a point existing if it is either given or already constructed. Constructing with compass and ruler amounts to constructing with the compass alone ifthe great circle span occurs among the existing pieces. It is therefore obvious to *ask* how Irina can emRmmi with the *compass alone* if *the oropcircle compass* is one *of the given pieces.* The first question is on which given pieces you can or cannot construct the great circle span with the compass,

First of all, I assume that there are two *points on the surface of the sphere* and ask which points can be constructed on them with the compass. I maintain that the *oropcircle span cannot always be constructed from this using* the *compass* alone. If, for example, two **corners** *A* and *B* of a regular tetrahedrona are given, then the other two corners of the tetrahedrong can be constructed with the compass alone, but no other points. This is because the circles around *A* and *B* with the circle apex *AB* as the radius intersect in the other two corners, and this statement applies to every pair of corners of the regular tetrahedron. If, on the other hand, the two end points *A* and *B* of an edge of the regular icogahedron are given, the remaining corners of the regular &oaasdera and no other points can be constructed with the compass alone. Among the constructible circular spans is only the circular apex of the spherical dihedral, which connects two diametrical points, but not the great circle circular span.

In these two examples, the set of constructible points was finite. But even if this was not the case, the great circle span does not always belong to the constructible circle spans. I *r* the spherical radius, *AB* the given pair of points whose radii are at the center of the sphere include the angle œ, then œr is the spherical distance and *2r* em '*x*|*2* is the circular span of the pair of points. The fiÆz then applies: Zøł a *point f'aor GB ąegeben, fiir dae* øincæ/2 a *transcendent ZaM iet, øo iet the* Oro#£reitzir£ef- *8panne r pŹ mil dem Zirket allein aue B mold: konetruierbar.* To prove this, I first note: If *2 r is* an e,/. 2 are the six circular spans of the rantøn of a tetrahedronø, øo each can be represented by a square root expression from the five others according to § 29. òfan can easily write the algebraic equation that exists between the six circular apertures explicitly. If , ..., (t are the vectors from the center of the sphere to the corners of the tetrahedronø, øo there is a relation between them

Œ;G; -0 vomRsngEMs 1)

since four Yektorøn of the three-dimensional spacea find linearly dependent. Multi-plizing (1) with the vectors Rt økalar, øo one obtains four Rslations

$$\sum a_i \mathfrak{C}_i \mathfrak{C}_k = 0. \tag{2}$$

Since the a; oicbt all XčI1 siztd, dafter is the detemú0aztte

|| E; IEt| 0. (3)

This is the relationship we are looking for, which we will now transform a little. If "t$ is the angle of the vectors Ri and Rt at the center of the sphere, then

$$\mathfrak{C}_i \mathfrak{C}_k = r^2 \cos\alpha_{ik} = r^2\left(1 - 2\sin^2\frac{\alpha_{ik}}{2}\right).$$

It therefore follows from (3) that

$$\left\|1 - 2\sin^2\frac{\alpha_{ik}}{2}\right\| = 0. \tag{4}$$

Theø **is** for eachøs **ain° œ;ş/2 a square sliding.** If, for example, in (4) five of the ainp$/2 are equal to each other and equal to a transcendental number øincæ/2, øo the øøohsts øin is also a transcendentalø number. Otherwise it would be **(4) an algøbraic** equation for sins/2 **with** algebraic number coefficients tsn, and therefore**,** according to a known, easily') mi provable **theorem**

') Is n8mIioh

"- + œ ' -l- ---- -l- - = o Isl

an algebraiøohe Gleiohung mi4 **algøbraiøohen Koeffiziøntøn, which satisfies sin** œ/ß, **øo bøo trachtø** man **gømåß Ziffer** 6 von § **28 einen K.örpsr 2t(}), dem øåmtliohe koøffizientøn ay** øugehörøn. Theøø øare **then ra4ionaIe functions o¡** = r;( ) of **with rational coeffi-** nientøn. It øøien - Ş" " **...,** **",the to conjugatedøn numbers. If we then denote** the polynomial in (5) **by p(s,**) in order to make the representation of the **coefficients** without coefficients, then p(z, 1$), £ = I, **...,** m are the polynomials with the **coefficients conjugated** to the **coefficients** in **(6). This is naoh the foundationøløøtz of the** øjrmmetriøohen **functions**

/ff(*, fi) = -m -I- $i - m-i -I- - - - -ł- I" "

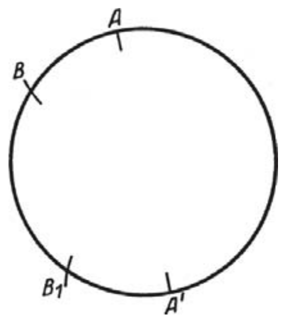
a polynomial **with rational coøffizientøn,** and **sin œ/ß** iø4 a nullstølle deøøelben **and dømnaoh ølø algebraiøohe Zøhl kenntlioh.**

of algebra, sin z/2 is also an algebraic number. This remark can be generalized as follows: If in (4) five of the sines can be represented by square root expressions from a transcendental ain e/2, then the sixth sine is also a transcendental number. Otherwise, form from (4) all the conjugate expressions obtained by any changes of sign in the square roots occurring therein, and multiply all these expressions together. Then, according to the fundamental theorem of symmetric functions in algebra, we obtain an algebraic equation for sin z/2 with algebraic coefficients, and again sin e/2 would be algebraic.

From this, however, follows the proof of the **theorem** stated above. For a construction with the compass alone produces new points from existing points as intersections of two circles around existing points with existing compass spans. Thus, at each construction step, a new circle span appears as the sixth edge of a tetrahedron. So if the circular spans used in the construction of the tetrahedron are

- between the two centers around which we placed circles, and the two radii, resulting in

together provides five tetrahedral edges - from one transcendental sin z/2 by square root-



F1g• 102

expressions, the new compass span, which appears as the sixth edge of the tetrahedron, must also have a center angle with a transcendental sine. This therefore teaches us that all circular spans *2 r* sin Q/2 that can be constructed from a circular span 2 *r an* e/2 with transcendent an eJ2 with the compass alone also have a transcendent ain Q/2. Therefore, the large

circular span *r*)"2 and also the diameter circular span *2 r* cannot be constructed. This proves the theorem.

Let's go one step further and prove the following theorem: Zs/ *auper any two PvnWn with the circle span AB still the circle span of the cuqel diameter is* given*, int A' the point diametrical to A and int finally on the half circle ABA' still a point B 8o is given, Since the circle span A'B* is *equal to the circle span AB, it is not possible to construct all the points from the pair of points AB using the circle alone, which can be constructed from it using the circle and* Direct. *However,* in *addition to the use of the compass as described in § 29*, it is *also necessary to allow the* vertebra *alone to be used to determine whether two designated circles intersect, overlap or* are separated.

In Fig. 102 you can see a large circle that carries the given points.

We beat around *B* and Pf$_{l}$ with the circle span BB_G Kr ise. They intersect i0-

in two points P, *P* of the great circle whose two equidistant points are *A* and *A′*. Therefore, *A P* -- *A'P*
-- *A P* -- *A'P* the great circle span.

There are several comments to be made on the construction described. If, for example, *B* and *B* coincide, *AB* -- *A'B* already did the great circle apan and the construction does not apply. So that in the case that *B* and B_1 are different points, the two intersection points *P* and *P* exist, the spherical distance of *BB* must be
(2a/3) must be *r*. If the
sign, the two points *P* and f* are combined. If, however, the distance *B B* > (2 a/3) r, we first note that we may assume that the points *A, B, B , A′* have the arrangement on the semicircle indicated in Fig. 102 ; for this can be achieved by changing the designation *(B* atatt *B)* if necessary. The *fact* that the distance of *BB* > (2 aJ3) *r* means that the spherical distance of *AB* < (a/6) *r*. In this case, the point *B* diametrical to *B* is congruent with the help of the spherical girth circle *and the* construction is repeated with the girth circle belonging to the arc *BB* *I* staH *AB* twice as long. The pair of points *B B* takes the place of the pair of points *AA′*. To carry out the construction, a point is then required on half of the great circle *B*$_I$ *BB*.
B$ in such a way *that* &$_1$ B = *B B*. It is obtained as a8 diametric point of aa points *B′* , which itself is created by mirroring *B* at the great circle omega B_Z *B′*. This reflection is achieved by making circles around *BI* with the circular aperture *BQB* and around *B* with the circular aperture *BGB*, which meet in *B* as well as in *B*. If the distance *BB* < *(*u/6) *r*
aein, double again8 and so on until you have a sufficiently large arc.

The example of the edge of the icogahedron mentioned above shows that it is not sufficient to take the circular apex of the spherical ameaaerg as given in addition *to* the pair of points *AB*. The example of the edge of the regular tetrahedron shows that it is not sufficient to take only a pair of points *A, B as* given. Instead of assuming the point *B,* one *can* also assume that another point *B* is given on the great circle *AB* in such a way that *A* is the center of the arc *BB*. *B$* is diametrically opposed to *B* .

The description of the construction now assumes that you can use the compass alone can decide whether a spherical distance $\lesseqgtr$ (2 a/3) *r* iat, i.e. that the compass can be used to decide whether two circles intersect, touch or do not intersect. Otherwise, the compass is used in the manner defined in § 29. All compass spans between existing points are permitted.

There is *another* atiff0R8unq *of the 1t'raq position.* Only small circles should be drawn with the compass and only those should be cut.

It is also possible to use a suitable additional specification, such as naoh D. Foo of the added bisection of any great arcs with small circles around existing center points with YOrhandenen compass aßes, aßes on a given pair of points, waa you can construct from it with compass and ruler.

I also prove the following theorem: If a non-diametral pair of points is given and, in addition, an instrument with which one cannot draw great circles, but can determine the center of each great circle arc, then one can construct all the points on *AB* with the compass alone, which one can construct with compass and ruler. This simply follows from the fact that the centers of the two great arcs of a great circle, defined by *A* and *B*, form a diametrical pair of points. Then the correctness of the theorem8 just stated follows from the previous one. You only have to give the center of the arc *AB* the role that point *A* had in the previous theorem.

Notes and additions

The following notes also provide references. They are not intended to be exhaustive. Some are references, some are intended to provide the reader with suggestions, and some are references that are not found in older reference works and textbooks due to their novelty. These include in particular *Enz yklopädie* der mathematischen Wissenschaften ; Tsorzxz, Geschichte der Elementarmathematik ; **EB£R- WELLSZEIN,** Enzyklopädie der Elementarmathematik ; ENRIQTfES, Fragen der Elementarmathematik ; ENRIQozS, Enciclopedia delle matematiche elemen- tare ; Tu. Vmzzu, Constructions and approximations; AnLER, Theory of geometric constructions. In some respects my account is in agreement with the subsequent work of H. LEBESOoz, Lepons zur lea conatructions géométriques, Pari8 l9Ii0. - It was only after the corrections had been made that I became aware of them: G. von Sz. Nzor, Geometrai azerkesztések elmélete, Kolozgvär 1943.

§1 For the questions just touched on here, please refer to: **J. HJELAIS-LEV**, The geometry of reality. Acta math. Vol. 40, pp. 35-66 (1916). - J. JELiisczY, The natural geometry. Hamb. math. Einselachr. 1928, Reft I.

§§ 2, 3, 4, 5 J. STEINz's is fundamental for the questions dealt with here: **J. ÜTEINER,** Die geometrischen Konstruktionen, ausgeführt mittels der geraden Linie und eines featen Krei8ea. Berlin 1833 (Gea. t'erke vol. 1, pp. 461 bia 529, 1882 ; Ogtwalds Klassiker vol. 60, 1895.) - Many of Steiner's theorems are of older origin. Many can be found, for example, in T BEsz, and the theorem of the fine circle with center **comes** from PoiroELET: Traité der propriétés projectivea des figurea, p. 187 (1822). The constructions in limited plane have found a monographic representation : P. Züaixs, Konstruktionen in begrenster Ebene. Math.-phys. Bib1. Vol. 11. 2nd ed. 1930.

The i$teiners' ifonatructions are treated succesfully in Gzoes
äfoait, Euclides curioaua. Amsterdam 1672. the contents of this lost book are known from the correspondence between GRsooRv and GOLLINS. See J.E. Horror, Die Entwicklungsgeschichte der Leibnizachen Mathe- matik, p. 107, Munich 1949.

§ H. TisTzz has devoted three works to points that can be constructed with the right-angled ruler: **H. TIETZE,** Über die Konstruierbarkeit mit Lineal und Zirkel. Sitzgsber. kaiserl. Akad. d. Wis8. Vienna, math.-nat. Kl. 118, Abt. II a, pp. 735-757 (1909). - H. Trszzz, über die mit Lineal und Zirkel und die mit dem rechten Zeichenwinkel löabbaren Konstruktionaaufgaben. I. Math. Z. vol. 46, pp. 191-203 (1940). - **H. TiETZE,** Zur Analyse der Lineal- und Zirkelkongtruktionen. I. Sitzgsber. bayr. Akad. d. Wiaa., math. nat. Abt. 1944, pp. 209-231. - The main result of this work is: If in the plane n -t- 2 points are given whose coordinates in a suitable rectangular coordinate system are (0, 0), (1, 0), (, *b), ...,* (ag, bq), then aße and only those points are conatruable with the right-angle ruler whose coordinates (z, y) are rational functions of , ..., *b* with rational number coefficients, in such a way that z is an even, y an odd function of be-. - ., *bq*.

§ F. **SEVERI,** Sui problemi risolubili colla riga e col compasao. Pal. Rend. Vol. 18, pp. 256-259 (1904). - F. IlözTzmnu, A contribution to the Steiner constructions. Tber. German. Math. vol. 43, pp. 184 to 185 (1933).
- D. Moeoc i-BocTovsxoi, Sur les congtructions au moyen de la régle et d'un arc de cercle fixe, dont le center ert connu. Periodico di mat.
(4) 14, PP. 101-11I (1934). - D. Corso, On the construction of the center of a circle with the ruler alone. Math. ann. Vol. 73, pp. 90-94 (1913). - D. CaU£R, On the construction of the center of a circle with the ruler alone (correction). Math. ann. Vol. 74, pp. 462-464 (1 9 1 3). --
E.A. WEisg, Ifonstruktionen mit hängenden Linealen. German. Mathematik vol. 6, pp. 3-15 (1941).

Perhaps it is also of interest to note that instead of a *ge*-
circle with center can also take any arc of another cone with center and focal points, or in the parabola case the focal point and a diameter, in order to be sure that one can then carry out all quadratic constructions with the ruler alone. - lt. J. S. SbftTTt, Mémoire aur quelques problemes cubiqueg et bi- quadratiqueg. Annali di mat. Ser. 2, Vol. 3 (1869) ; Collected math. Paperg vol. 2, pp. 1-66 (1894).

§ H. TIETZE (1. c. § 4) has made the following remark on the result of this paragraph: When carrying out the construction, several points are obtained at the intersection of a circle with a straight line or another line. When continuing the construction, it is then necessary to specify which of the two points obtained in the nth step should be used when continuing the construction. Ofen can ask for which congruction tasks it is always indifferent which of the points generated in the nth step is used in the (tt -l- 1)-th construction step. H. TIzzzE has found that these are exactly the problems that can be solved with the right-angle equation in the sense of § 4. (These constructions are not to be confused with the constructions with the right angle £a£en to be considered in § 9).

From this point of view, the result of the considerations in §§ 5 and 6 and later § 10 can only be understood to mean that the booked points are to be found and selected among those whose coordinates can be represented from the coordinates of the given points by the four basic arithmetic operations and square root operations. This finding or selecting means using the arrangement axioms of geometry or, in other words, deciding on the signs of the square roots that occur in each case. This can be done with the aid of the *circle* if it is determined that it is to be used not only for drawing circles but also for comparing sizes, i.e. for the *pris/tittq* of *relationships*. If, for example, the two points (0, 0), (1, 0) are given and the

If you are given the task of constructing the point (1/2, + 1/2 3), this task is not determined by the given pieces because it is not specified on which side the points with positive second coordinates should lie. The pair of points (1/2, -}- 1/2 J3) can be found using a compass and ruler, and the

point to be found or searched for. If, however, by specifying a further point with a positive second coordinate, this additional

If the third point is given, the point (1/2, -{- l/2 J3) is the one of the two constructed points that is closer to the third given point than the other point obtained during the construction. The plan shows which of the two constructed points this condition applies to by using the compass. The example is not kinetic. Because if you think of the elementary geometric theorem that the side dea regeimö§ipea Schreck is the larger section dea circle radius divided according to the golden ratio,
i.e. the mean proportional between the whole radius and the smaller section, the correct one must likewise be sought among two points (even if this is somewhat cached in the usual description of the construction ibid). TIEzzz's remark can also be understood to mean that between

A distinction must be made between conatructions at the oriented and non-oriented level.

The considerations of R. C. ESPzNSEn, Nogle geometriake forsog. bfat. Tidsskr. A, PP. 55-61, 94-96 (1937). Certain given pieces may only be **used** to a limited extent. For example, no straight line and no circle through one of the given points may be used in the congruence. The a u t h o r only knows this work from the papers in the Tb. Fortschr. Math, where no results are given.

§ 7. the congructions with the compass alone were made known by the Italian M SGTtERONI in **1797.** They are **usually** named after him. òfan has known since 1928 through **J. JELMSLßV** that they were first mentioned **by** GEOso MoøR İn his Euclideg Danicus, Amsterdam **1672. -** J. HJELbfS- **LEV,** Om en af den danske Matematiker GEOBO MoiiR udgivet Skrift "Euclides Danicus" i Amsterdam 1672. mat. Tidsskr. B, p. l -7 (1928). - **J. HJSLAISLEV,** Beiträge zur Lebensbeschreibullp" **vOH** GFORO MoilR **(1640-** 1697). Det kgl. Videnskaberneg Selskab Math. fys. Medd. Vol. 11, 4 (1931). - H. **GEPPERT,** CEOso Moøn e la geometric del compasso. Periodico di mat. Ser. 4, Vol. 9,
S. 149-160 (1929). - ĞEORG MOTlR, Euclides Danicus, Amsterdam 1672, with a preface by Joømnzs HJELMSLEV and a German translation by JimiUS Pin, Copenhagen 1928.

H. C. **ESPENSEN** makes similar considerations to those mentioned in § 6 for constructions with the compass alone. H. C. Esr£1NSEN, Geometriøche Untersuohungen mit dem Zirkel allein. (Danish.) Mat. Tidsskr. A, PP. 11-23 (1941).

§ 8. **F. ÜEVERI** (1. C. § 5). F. SEYERi, Complementi di geometria projettiva. Bologna 1906, p. **303.** Hiør **shows** how to get by with a limited parallel ruler, while the representation of the text assumes an unlimitedø parallel ruler.

§ 9. H. Fume, Congtructions with the drawing angle. Z. math.-nat. Unter- riclit, vol. 65, pp. 279-287 (1934). - F. BAGBuANN, Geometries with Euclidean ăfetric, in which for every straight line through a point not lying on it there a r e several non-intersecting lines. I. Math. Z. Vol. 51, pp. 752-768 (1949).

. § 10. H. GErPERz, Sulle costruzioni geometriche che øi eseguigcono colla riga ed un compasso ad aperture fissa. Periodico di mat. Ser. 4, Vol. 9, pp. 292 to 319 (1929). - J. JELosczv, Konstruktion ved Passer med fest Indstelling uden Brug af Lineal. Mat. Tidsskr. A, PP. 77- 85 (1938).

§ 11. J. HJ Et*MSLEv, Konstruktioner med normeret Lineal. flat. Tidsskr. B, PP. 21-26 (1943). Here the proofs are given without the parallel axiom.

§ D. HILBERT, Grundlagen der Geometrie, 7th ed. 1930 - E. MNDAU, Über die Darstellung definiter binärer Formen durch Quadrate. Math. Ann.

57, P. 6M64 (1903). - **E. ABTIN,** On the decomposition of definite functions into squares. n "b. Abh""dl. Vol. 6, pp. 100- 115 (1926).

F. SCBUR, Fundamentals of Geometry (1909). - F. BAcBatanx, Geometries with Euclidean metric, in which for every straight line through a point not lying on it there are several non-intersecting lines. II Math. vol. 51, pp. 769-779 (1949). - F. Bzcnzer, Über die Konatruierbarkeit mit Lineal, Rechtwinkel- Maß und Eichmaß. Math.-phys. Semeaterber. 1, S. 77-88 (1 9 4 9).

§ 13 The version of the text, which is often found in the literature with various modifications, goes back to a discovery by E. Lernen. Among the papers left behind by H. A. Sciiz-znz was a sheet with the heading "EDblirND Lwnzn, atud. math, Berlin, June 28, 1897" and with the following text, undoubtedly written by UND&If"s hand, like the heading: "In order to prove that the trigection of an arbitrary angle by a finite number of constructions with ruler and compass is not possible, it must be shown that there is no expression formed by a finite number of rational operations and square root extractions aua e which satisfies the equation /(ti) = 4tt° - 3 u -{- a - 0. For z - ain 'p/3 is a g - 3 z - 4 z°, i.e. 4 z° - 3 z -}- sin 'p = 0. Let us assume that such an expression exists. If, to form it, sizes y" ... , y, are adjoint w, where generally y* belongs to *Pra, y ,* ... , y; _), whereas y; does not, and if z is a quantity of *P(a,* y" ... , y,), z
the form *z* -- 'x -l- Q y" where e, Q belong to P r a , y" .r. —) , Q 0 could be assumed, since otherwise z would already belong to *Pra, y , ... , y).*
It must also be z - Q y, a root of /(ti) = 0 ; because if 0 = /(z -l- Q y,)
-- *A* -{- *B y* is, where *A* and *B* belong to *P(a, y , ... , y), B* -- 0 must abe, since otherwise *y - A |B* would be, i.e. would belong to *Pra,* y" ... , y _); therefore /(et - Q y,) = *A - B y* -- 0. The two roots z -{- Q y" ot - Q y, aare different, since their difference is not 0; since the member with u° is missing in /(u), the third root = - (z -}- Q y, -}- et - Q y) - - 2 z, al8o belonging to *P{a,* yt, ... , y, _ t). If it did not belong to *Pra),* repeating the same swallowing would result in a fourth root for the cubic equation, i.e. a contradiction. Thus - 2 z would have to belong to *P(a)* ; now it can be shown in a known way that no quantity of *P(a j* can satisfy the equation; thus the assertion is proved." So much for the transcript of denen's proof from the age of twenty-one. The proof also offers a variant of the argument that does not seem to be known in the literature, namely to trace the contradiction back to a fourth solution of the cubic equation. On this page we have the first independent achievement of the great mathematician. A slightly different version of the proof first appeared in print in 1903 in : WzBER-WEccszsiN, Ensyklopädie der Elemen- tarmathematik, vol. 1. **(H. WEBER)** p. 320. Leipzig 1903.

H. Bzimxs and H. HERbIES have examined the proofs for the impossibility of the trisection of angles with compass and ruler from the point of view o f basic formalistic research and have given a conclusion which can also be applied to ala proof from this point of view (HILBERT, GENzzEx). H. BElimE and H. HERblzs, Über die Unmöglichkeit der Drei- teilung dea Winkeln. Sem.-Ber., Math. Sem. d. Universität Münster 1I, p. 23 bia 47 (1938). - B. L. vNnre WAERDEn, Moderne Algebra I, p. 1811 and 183 and p. 49, Berlin 1930, second, verb. Ed. p. 194. 1937. °l950. Now t h e explanation of the indeterminate mentioned in the text is omitted. Instead, the8e is introduced on p. 50 as an element of an infinite cyclic group. -

W. GEBER, On the uniformity of constructions with compass and ruler. Dtgch. Mathematics Vol. I, pp. 635-665 (1936). - W. WEBEB, On constructions with compass and ruler in favorable cases. Dtgch. Mathematics Vol. I, p p . 782-802 (1 9 3 6). - WEBER'S results: Let Zt, ..., Zq be inter- vaße ; a conatructiona task with the auagangaat distances z" ..., zq be solvable with compass and ruler in every single case in which eacha z lies in Z. Ea will prove that there is a uniform solution if the solution path p is an algebraic function of *z , ..., xq iBL*. However, if the algebraic function q erat is made whole by multiplication with a polynomial *N(x , ...,* zq), the uniformity of the construction erat can be achieved if those systems zt, ..., zq are excluded for which *N(x , ..., xq) vanishes*, although even then 9 still lies in a square root body over the rational functions of y, ..., zq. The possibility of a uniform construction nevertheless fails due to the occurrence of denominators. - The question of construction possibilities in favorable cases depends on whether or not the system of the Augganggat segments can be constructed from a single one using a compass and ruler. If not, t h e r e are always such favorable cases for continuous problems with at least two starting lines. However, if the starting line is to b e drawn with a compass and ruler, as in the case of the division of an angle of 90°, f o r example, this is a special feature which, given our limited knowledge of square root numbers, we do not yet know whether it has an analog in every other problem. - L. E. DIc&SON, New first courae in the theory of equationg. New York 1939. 8th ed. 1949.

§ 14. **J. R6EMELz,** The irreducibility of the Kreiateilung8 equation. Publicationa mathématiqueg de l'univeraitß de Belgrade. Vol. 2 (1933).

S. I6MI65. J. E. Honer, Natur und Kultur vol. 29, p. 477 (1932).

§ 15. monographs : A. HS'.TtRbIANN, Das Deliache Problem (Die Verdopp- lung der Würfeln). Math.-phyg. Bibl. vol. G8 (1927). - W. BSEIDENBAGH, The tripartition of the angle. Math.-phya. Bibl. vol. 78 (1933).

§§ 16, 17, 18. W. BsEIDENBAoH, Der rechte Winkel und das Einschiebe-lineal. Z. math.-nat. Unterricht vol. so, p. 4- ls (1925). - J. E. HOPMANN, A contribution to the theory of insertion. Z. math.-nat. Unterricht v o l . 57, pp. 433 to 442 (1926). - J. E. HormNN, Graphical solution of cubic equations by inserting a line between a circle and a straight line. Lessons. Math. Naturv-igg. Vol. 40, pp. 64-67 (1934). - L. BIEBERnaca, Zur Lehre Von den kubigchen Konstruktionen. J. reine u. angew. Math. vol. 167, pp. 142- 146 (1931). - R. lozisca, Über die Dreiteilung der Winkeln und die Verdoppelung dea Cube unter Benutzung von Zirkel und rechtwinkligem Dreieck. Z. math.- nat. Unterricht vol. 64, pp. 207-210 (1933). - R. GABVER, A nOte on BiEBEB- *aaeii'n* trigection method. J. reine u. angew. Math. vol. 173, pp. 243 and 244 (1935).
- R. **GARVER,** BiEBERBAca's trisection method. Scripts math. 3, pp. 251-255 (1935). - E. L. Gonr+tEY, A IlOte on BIEBERBAoH's trigection method. Scripta math. 3, p. 326 (1 9 3 5).

§ 17. M. D'OCAONE, Etude rationnelle du probléme de la trisection de l'angle. L'enseignement math. Vol. 33, pp. 49-63 (1934). - J. JELusczv, Geometrigke Experimenten. Copenhagen 1913, °l9I9. German edition: Beih. z. math.- nat. Unterricht H. 5 (1915). - P. BOGKNER, Eine Aufgabe, die mit Zirkel und Lineal nicht lösbar ist. Elemente d. Math. vol. 2, pp. 14- 16 (1947).

§ 19. I. Nzwzoir, Arithmetica universalia. Cited in I. Nzwzo", Universal arithmetic or a treatise of arithnietical composition and resolution, p. 246. London 1725. - R. Oaz4m, Zur Théorie der Konstruktionen dritten Grades. Töhoku math. J. vol. 39, pp. 1-15 (1934). - R. Oacäzii, Ida théorie des cons- tructions cubiques. Paris G. R. vol. 197, pp. 1383- 1385 (1933). - R. G. **NATES,** The angle ruler, the marked ruler, and the Carpenter Square. Nat. Math. Mag. Louisiana vol. 15, pp. 61-73 (1940).

§§ 21, 22 The theorem of the solvability of arbitrary problems of the third and fourth degree with compass and ruler with a firmly drawn conic section is due to
for the parabolic case from DssCARTzs and is found for ellipse and hyperbola in FnN€'OIS RznE SLtfSE, Meaolabum. Liège 1659, s1G68 and then
carried out at NEwzo" 1. c. § 19. The proof given in § 22 goes back to Tu. VATILEn, who included it in his still classic book on constructions and approximations. The proof given in § 21 was carried out by G. v. Sz. Nzor carried it out: G. von Sz. Nenn, Ein Beweis des Satzes von H. J. S. SCH und H. Koitzt u. Elemente der Math. vol. 3,
S. 95-97. (1948). The naming of the theorem after Seit andKORText has become customary because both mathematicians independently solved the task set for the Steinerpreig of obtaining this theorem, first referred to by NEwzon, in the style of the time using the means of geometric geometry. The following is said about the implementation of the approach of § 21. If 2 is chosen according to the given conic section, then

the problem presented does entail that the curve leaves (8). Then it consists of two straight lines, and this means that the problem presented is a quadratic one which can be reduced to the intersection of circles and straight lines. If the given second-order **curve is** an ellipse, this decay does not occur in problems with real solutions, and there is nothing to add to the information given in the text. In the hyperbolic case indegsen there are indeed two hyperbolas Ht for 1 -}- Z - o°/b° and H for 1 -]- 2 = $b*|a*$ in the bundle (8), whose asymptotes enclose the same angle as those of the drawn conic section. However, it may be the case that öeide are not similar to *H,* but to the hyperbola Zf' conjugated to ff and located in the other angle space of the asymptotes. In this case, the reasoning of the ellipse can not be transferred without further ado, but it is still necessary to consider how the intersection of *H'* and any **circle** *K'* **can** be **constructed with the** help of H and compass and ruler. **If** then

z- + y° + *A x* -t- *B y* -l- *C!* -- 0 or q, -z- y- 6 , - 1 - 0

the equation of *K'* or ff', ao draw the circle *K, which* is slightly congruent to *K',* with the equation

$$x^2 + y^2 + \frac{b}{a} Bx + \frac{a}{b} Ay + C + a^2 - b^2 = 0$$

approach. If *P(5, p) is* an intersection of ff and *K, then P'* [(a/ö) p, *(b|a j* $) is an intersection of Zf' and N'. Because the equation of *H* is

Therefore, the parameter representation

$$x = a \operatorname{tg} t,\ y = \frac{b}{\text{CO8}\,¢} \text{ for Zfand} \qquad x = \frac{a}{\cos t},\ y = \text{tgt for Zf'.}$$

The intersection points of ff and *K* and of ff' and *K'* are therefore determined by the same equation

(o* -|- ö*) tg*f -}- *bB* tgf -|- $\frac{aA}{\text{cosl}}$ -l-C-t-o'-0.

If t = z is one of its roots, then 'i tgz, *b|ooe c* are the coordinates of an intersection point T of Zf and *K* and o/cos z, h tg z are the coordinates of an intersection point *P'* of H' and A'.

§ W.K.B. HOLZ, Das ebene obere Dreieck. Hagen i. Tt'. 1944. -
H. Düwiz, Mathematical Miniatures. Breslau **1943.** The context of the task of constructing a triangle from its upper height sections,

W. K. B. HoLz discovered the Newtongo semi-circular task. (The task o f constructing a triangle on its heights is a circle-and-linear task, since the triangle itself has heights on the three heights that f o r m a triangle similar to the triangle being sought). Due to the realism of the problem, its execution presents particular difficulties, i.e. particular beauties for a good mathematician. P. **BtfGrESER,** Die Benutzung der Imagi- nären bei Konstruktionen. Z. math.-nat. Unterricht vol. 61, pp. 338-343 (1931). The construction of a triangle from the three upper sections of the angle bisectors can also be traced back to Newton's semicircular task, since, according to Dö sia, the equation of the third degree for this reciprocal radius, which dominates the semicircular task, exists between the reciprocal values of these sections and the reciprocal value of the radius of the inscribed circle. The construction of a triangle from its three angle bisectors leads to an equation of the tenth degree with a symmetrical group. A. koa- szcz, Z. Math. Phys. 42, p. 30M3I2 (1897). F. lEISs, On the impossibility of constructing a triangle on its three angle bisectors.

J. reine u. angew. Math. vol. 177, pp. 129-133 (1937). II. WOLPP, On the determination of a plane triangle from its angle bisectors. J. reine u. angew. Math . vol. 1 7 7 , pp. 134=-151 (1 9 3 7). B. L. YAfl DER AnRDEN, On the determination of a triangle from its angle bisector. J. pure u. angew. Math. vol. 179, pp. 65-68 (1938).

The heptagonal treatise of the AecBIMEDES was published by C. Senor on corrupted copies of the Arabic translation by TaBI'r Izo Qenea.

G. Senor, The trigonometric teachings of the Persian astronomer Anne-Rio Mna. IBN AahJYtED Ac-BIRtfNi, illustrated after M-JinAx- Ws'iini. Published after the death of the author by J. Rasse and **H.WIELEI3'NEl:t.** Hanover 1927. - J. TitorraE, Geschichte der Elementar-mathematik, vol. 3, p. 127. 3rd ed. 1937. - C. DER, Die Würfelverdopplung des ArocLONIuS. Dtgch. Mathematik vol. 5, pp. 241-243 (1940). - J. LEME£*J has shown how the construction of the regular Siebenecka with compass and ruler can be traced back more simply to the trisection of an angle. (J. RSEMELJ, Die Siebenteilung der Kreises. Monatshefte f. Math. u. Phyg. Bd. 23 (1912), pp. 309-311.) His method is theaeg: The equation (1) H -}- z° - 2 z - 1 = 0, which appears several times in the text, satisfies, among other things, z = p° -|- p °

with p = - exp - . The iside of the heptagon in the unit circle is s =

= 2 $\sin_7$ = i (p - p). It is " - 2 - z. Thus, from (1) we find the equation for the side of the heptagon in the unit circle (2) e® - 7 8* -l- 14 s° - 7 = 0. It decays into two equations of the third degree s° -j- (8* - I) - 0 or

1\3 1 / 3 3 }/7 Alao is =-{ 2 / j3 - cot - , ¢f cos / 3 $\alpha = \frac{3\sqrt{3}}{2\sqrt{7}}, \sin\alpha = \frac{1}{2\sqrt{7}}$

is. From this, the side of the Siebenecka $s_7 = \frac{\sqrt{3}}{2} : \cos\frac{\alpha}{3}$ with tg z, 1 3 3

or approached *s* . The latter is the rule of Auto West MomuEn, also called the Indian rule.

§ 26 H. SzaIO6tÜLLER, DiIRER alB Mathematician. Stuttgart 1891. F. VOOEL, On the approximation approximations for the trisection of an angle. Z. math.-nat. Unterricht vol. 62, pp. 145-155 (1931). - 0. PERRON, Über eine Winkeldreiteilung dea Schneidermeisters Korn. Bavarian Academy of Sciences, math and science. Abt. 1933, p. 439-445. - 0. NEBRTNG, On the trisection of the angle according to EnoEir **Kors.** Seatgaber. bayr. Akad. d. WiBg., math.-nat. Abt. 1936, pp. 77-79. - P. **FINSLER,** Einige elementargeometrißche Näherunggkonstruktionen. Comm. math. llelv. Vol. 10, pp. 243-262 (1938). Approximations of high accuracy for the Deliache problem, regular polygons, etc. are also given here. - W. TbfANN, Some groups of angle-divisions and the numerical size of their errors. With an appendix: On a curious property of the Pascal snail. A circle-curve problem. Dtgch. Mathematics vol. 3, pp. 556-597 (1938). - B. Doss, Approximation conatructions for the side of the regular heptagon. Tber. Dtgch. Math. vol. 44, pp. 291-292 (1934). - H. J. FisCriEe and **K. SCBMEI8Efl,** Studies on approximate circle division. SiJzgsber. Akad. d. Wi8s. Heidelberg, math.-nat. Cl. no. 18 (1934).

§ 27 T. Kuzozz, Mathematical Notes. T8hoku Math. J. Vol. 16, pp. 92 to 98 (1919). - T. KuaoTz, Histories of geometrical constructions. Annals. DMV. Vol. 37, pp. 71-74 (1927).

§ A detailed analysis of Hermite-Lindemann's proof and its connection with Lambert's proof for the irrationality of a is recently given by LEBEsonE in his subsequent book: Le§on8 zur les constructions géométriqueg. Paris 1950 - See also G. **L. SIEOEL,** Über einige Anwendungen Diophantischer Approximationen. Abh. Preuß. Akad. d. Wiaa., phya.-math. Kl. 1929 (Paggim.). - A. GEzsoun, Sur len nombreB trangcendanto. G. R. Paris vol. 189, pp. 1224-1226 (1929). - G. L. SIEOEL, On the periods of elliptic functions. Grellen J. vol. 167, pp. 62-69 (1931). SiEoEL remarks here p. 63/64 following a sketch of the proof presented in § 28 of this book: "This is probably the most natural proof of Hermite-Lindemann's theorem, and HEitiimE, as can be seen from several passages of his treatise, came very close to this proof; but he obviously lacked the simple idea of forming the norm from the number (4)." Recently, Tx. SGIfNEIDE+t has attempted to present the train of thought outlined by SisoEc in detail. TUT. **ScHnEIDER,** On the proof of the trance x-on e

') Daa corresponds to (14) in the presentation of § 28.

and u. Math.-phys. Semesterber. Vol. I, pp. 299-303 (1950). The starting point for the new method of proving transcendence inaugurated by A. 0. CiELPonn and used in the text is a groundbreaking function-theoretical theorem **by ĞEORO PÓLYA.** Gzonc Pócra, On integer integer functions. Rend. del circolo mat. di Palermo vol. 40 (1915) and Nachr. v. d. kgl. Ges. d. Wiss. Göttingen, math.-phys. This SaŁz says: Let q(z) be a gauze function of the complex variable z, which at the points z = 0, I, ... assumes gauze rational numerical values. Then either p(z) is a polynomial, or else *g(z)* for ø m is at least as fast as 2'. Sharper: If one sets 3f(r) = Max p(z) , then lim sup *M(r)* 2* " 1. a similar new functio-

nentheoretical SaŁz follows from the method of proof described in § 28. It reads: Let *g(z)* be an integer function which at the points z= 0, 1, 2, ..., m together with all its derivatives assumes gauze rational numerical values ; then either *g!z)* is a polynomial or eø grows q(z) at least as fast as eg (- (z - I) ... (z - m)). Schiirfer: It is then lim sup *M(r)* exp (- *r'^+ j* *I.* The proof of transcendence presented here is a generalization for the case that p(z) assumes algebraic numerical values of a certain kind at the m -}- 1 points mentioned. Cf. also Tø. SoREIDER, A theorem on integer functions as a principle for proofs of transcendence. Math. Ann. Vol. 121,
Here a more general approach is presented. The author did not have access to G. **L. SIEOEc,** Transcendental numbers. Princeton 1949.

-. N. E. **NÖRLÓND,** Differences calculus. Berlin 1924, p. 199 - E. Mnnzu, Über quadrierbare Kreisbogenzweiecke. Sitzgsber. Berl. math. Ges. 2, pp. 1-6 (1903). -L . TgcmmLOFz, Contribution to the problem of squareable b i s e c t o r s . Math. Z. vol. 30, pp. 552-559 (1929). - L. Tscœxmcozr, Application of the theory of algebraic numbers and ideals to the p r o b l e m of squareable circle-arc b i s e c t o r s . C. r. du premier congrès des math. des pays slaves. Warachau 1930. - H. WIELEITnzR and J.□ OFMANir, Zur Geschichte der quadrierbaren Kreiamonde. Wiss. Beilage zum Jahresber. d. neuen Realgymnasiums München für das Schuljahr 1933/34. IIETNRIGH MÜLLEB, Eine einfache Näherungakonstruktion für u. ZaãfM. Vol. 29, S. 254 (1949).

§§ 29, 30. T. BONNEsEn, Geometriske KonsŁruktionerpå kuglefladen. Nyt Tidsskr. for Max. Vol. 10, pp. 1-13, 25-35 (I599). - D. Foo, Om kon- struktioner med paaseren alene. MaŁ. Tidøøkr. A, PP. 16-24 (1935). - B. WIEDE- firm, Algebraic-geometric investigations on construc- tion possibilities on the Hugel. German. Mathematik vol. 2, pp. 520-544 (1937), vol. 7,
S. 178-184 (1942). The results and suggestions of P. E. Böii- ozR are utilized here.

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