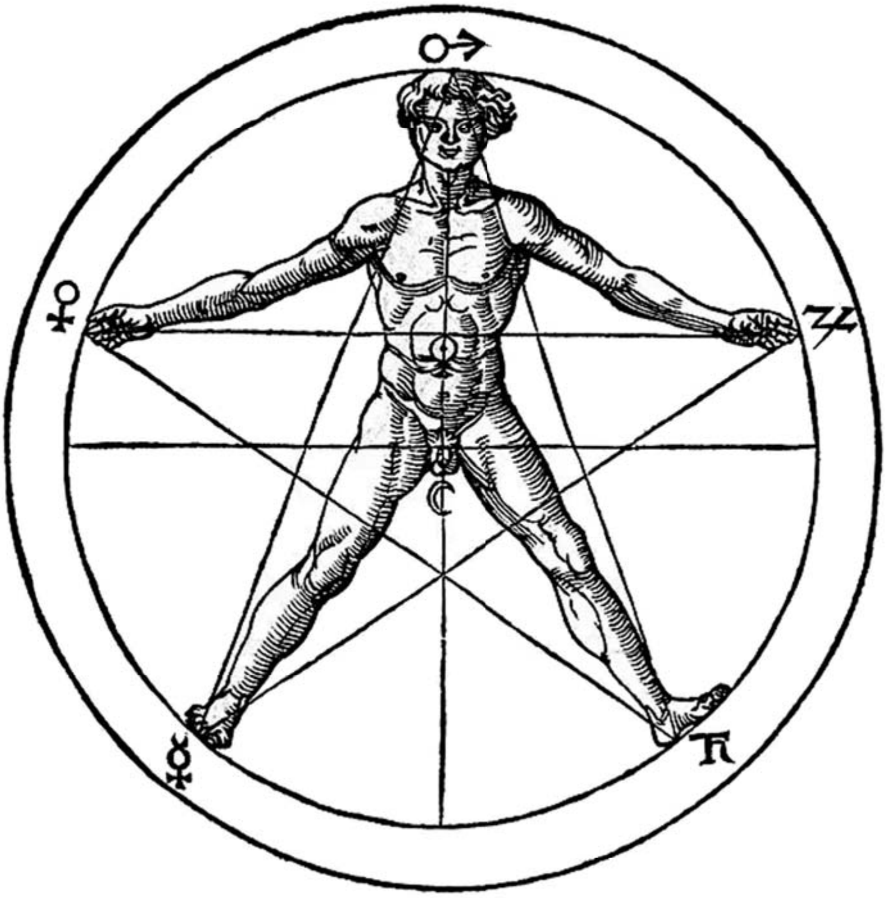
RESTITUTION OF PYTHAGOREAN GEOMETRY



ARTURO RHEGINI

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BOOKS

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ARTURO REGHINI

The Restitution of Pythagorean Geometry

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FOREWORD

1. Proclus, head of the School of Athens (5th century A.D.), left us a valuable commentary on the *First Book of* Euclid, from which commentary we derive the most precise and important information that moderns possess on the achievements and discoveries made in geometry by Proclus and his school.

According to Proclus, 'Pythagoras transformed this study and made it a liberal teaching; for he went back to the higher principles and researched theorems abstractly and with pure intelligence; it is to him that we owe the discovery of the irrationals and the construction of the figures of the cosmos (regular po- lihedra)'.

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1 PROCLO, *Com. in Euclidem*, ediz. Teubner, 65, 15-21: above translation is that Tannery (PAUL TANNERY, *La Géo- métrie grecque; comment son histoire nous est parvenue et ce que nous en savons*, Gauthier-Villars, Paris, 1877, p. 57). It is not a literal translation; and not out of pedantry, but out of fidelity to Pythagorean thought, we note that the Greek text does not say that Pythagoras reverted to the higher principles of geometry, but ἄνωθεν τὰς ἀρχὰς αὐτῆς ἐπισλοπούμενος, which means: considering from above the principles of geometry. Loria (GINO LORIA, *Le scienze esatte nell'antica Grecia*, 1914, p. 9), also quotes the passage with a translation similar to that of Tannery.

Proclus further states[2] that:

a) Eudèmus, the peripatetic[3], attributes to the Pythagoreans the discovery of the two-right theorem (in any triangle, the sum of the angles is equal to two right angles), and asserts that they demonstrated it (fig. 1) in taking the parallel to the opposite side through one of the vertices A and observing that, since the alternate internal angles formed by a transversal with two parallel lines are equal, the sum of the three angles of the triangle is equal to that of three consecutive angles forming a flat angle. This, says Proclus, is the proof of the Pythagoreans.

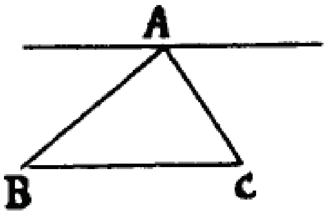

Fig. 1

b) "Six equilateral triangles joined by the vertex fill exactly the four right angles, the same three hexagons and four squares. Any other polygon whose angle is multiplied will give more or less than four right angles; this sum is given exactly only by the

2 See P. TANNERY, *Le Géométrie Grecque*, p. 102. PROCLO, Teubner edition, p. 379. ALDO MIELI reports the passage in the Greek text on p. 273 of his work: *Le scuole ionica, pythagorica ed eleatica*, Florence 1916.

3 Eudemus of Rhodes, the eminent disciple of Aristotle. Aristotle died in 322 BC; Euclid flourished around 300 BC.

polygons assembled according to the given numbers. It is a Pythagorean theorem'.[4]

c) Pythagoras discovered the theorem on the square of the hypothesis of a right-angled triangle: 'If one listens to those who want to tell the story of olden times, one can find them attributing this theorem to Pythagoras, and making him sacrifice an ox after the discovery'[5].

d) "According to Eudemus (οἱ περὶ τὸν Εὔδημον) the *para- bola* of areas, their *hyperbola* and *ellipse*, are discoveries due to the muse of the Pythagoreans".

With this nomenclature, classic since Euclid, and no longer used today, Proclus designates the problems of simple application, application in excess and application in defect, i.e. he attributes the geometric construction of the three unknown equations to the Pythagoreans[6]:

$$ax = b^{(2)}; \qquad x(x+a)=b^2; \qquad x(a-x)=b^2$$

e) The use of the star pentagon, or pentagram, as a sign of recognition.

f) The construction of regular polyhedra, and in particular the inscription of the (regular) dodecahedron in the sphere[7].

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4 PROCLO, ed. Teubner, p. 304.

5 PROCLO, ed. Teubner, p. 426. This theorem is also attributed to Pythagoras by DIOGENE LAERZIO, VIII, 12, by PLUTARCO, by VITRUVIO (*De Architectura*), IX, cap. II, and by ATENEO.

6 PROCLO, ed. Teubner, p. 419.

7 PROCLO, ed. Teubner, p. 65. For this last point see also JAMBLICO - *De Vita Pythagorae*, 18.

2. These, along with a few others that we will have occasion to look at later, are the little information have today about the geometric discoveries of the Pythagoreans; we owe them to Proclus, who in turn drew them from the reliable source of Eudemus. It should be noted, however, that Tannery, in the magnificent study quoted above, not only agrees with the unanimously accepted point that Proclus did not personally know any geometrical work prior to Euclid, but also supports the thesis that Proclus did not even directly use the geometrical history composed before Euclid by Eudemus, although he quotes him very often [8], and that he knows and quotes Eudemus only second-hand, namely through Geminus, an author of the 1st century BC, a Greek, pro- bably, despite his Latin name.

As for Eudemus, in order to explain the origin of the passably numerous and circumstantial indications given through him concerning the work of the Pythagorean school, Tannery argues [9] that there must have existed a relatively considerable work on geometry, which Eudemus must have had in his hands, a work composed after the death of Pythagoras, approximately in the middle of the 5th century. It is perhaps the work that Giamblicus describes as: *the tradition about Pythagoras*. Tan- nery[10] observes that, according to Proclus' historical summary, in the treatise on Greek geometry, whose existence can be suspected, there is no doubt Pythagoras was a Greek geometer.

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8 P. TANNERY, *La Géom. gr.*, p. 14 and 15.

9 P. TANNERY, *La Géom. gr.*, pp. 82 and 86.

10 P. TANNERY, *La Géom. gr.*, p. 87.

za, the framework was already the that fill the "Euclid's 'Elements', from Book I (Theorem of the Two Right Handed), to Book 10 (Discovery of the Incommensurables), to Book 13 (Construction of Regular Polyhedra). This is the crowning of the one and the other; that is, of Proclus' summary and Euclid's Elements. "*Toute la Géométrie élémen- taire nous apparait ici, comme sortie brusquement de la tête de Pythagore, de même que Minerve du cerveau de Jupiter*"[11].

However, we know nothing about the proofs of the theorems, the solutions to the problems and in general the treatment of the issues reported by Proclus - Geminus - Eu- demos; nothing except the demonstration of the two-rectified theorem, which at first glance lacks nothing.

The demonstration given above, and attributed by Eude- mo to the Pythagoreans, does not coincide with that found in Euclid's text (prop. 32) but differs slightly. Euclid first proves that an exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles, based on proposition 29, itself based on the fifth postulate, or postulate of the parallels or Euclid's postulate. The transition to the theorem on the sum of the three angles of a triangle is immediate and is effected by Euclid in the proposition itself.

Theorem and demonstration, however, predate Euclid, as Vacca[12] observes; because, as has been observed

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11 P. TANNERY, *La Géom. gr.*, p. 88.

12 VACCA GIOVANNI, *Euclid - The First Book of the Elements*, Greek Themes, Italian version and notes, Florence, 1916, p. 78.

vided by Heiberg, Aristotle in a passage in the *Metaphysics* (*Metaph*., 1051 a 24) refers not only to this theore- but to Eudemus' own demonstration.

At this point we must raise an important question from both a historical and theoretical point of view. The demonstration to which Aristotle refers, and which is the same demonstration that Eudemus attributes to the Pythagoreans, was it also based, like that of Euclid, on a postulate equivalent to that later admitted and formulated by Euclid? Proclus uses in the passage he reports from Eudemus the term parallel, indeed he says: παράλληλος ἡ, the parallel; did Eudemus do the same, and did the Pythagoreans mentioned by Eudemus do the same? And if so, what was the meaning and definition, for them, of the word: parallel? And in connection with this historical question, another theoretical one arises: Is it necessary to rely on Euclid's famous postulate to prove the two-right theorem, or only on an equivalent postulate?

We can answer that Euclid's postulate is not necessary in order to prove the two-rectified theorem; not only that, but the demonstration referred to by Ariel, which according to Eudemus is the same as that of the Pythagoreans, can be done without admitting or assuming the fifth , or, equivalently, without admitting or assuming the uniqueness of the non-secant of a given line passing through an assigned point.

If in fact one admits, for example as Severi does[13], the postulate that: in a plane the locus of the points situated on one side of a straight line and having a given distance from it, is still a straight line, one can observe 1º - that this line is unique[14]; 2º - that in order to demonstrate that this line, that is, the only one equidistant from the given line passing through the assigned point, is also the only one not secant from the given line, Severi has recourse to the postulate of Archime- de[15], which proves that the postulate admitted by Severi is not equivalent to the postulate of Euclid; 3º - that Severi's explanation of the external angle theorem and of the theorem of the sum of the angles of a triangle[16] (which is Euclid's theorem) is based in reality only on the properties of the equidistant (Severi's parallel), and, although it is preceded by it in the text, it is not based on the property formulated by Euclid's postulate. It is sufficient to conduct the equidistant from the op- posed side by the vertex and apply the property of alternate angles inter- ni[17], i.e. it is sufficient to rely on Severi's postulate and not Euclid's.

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13 FRANCESCO SEVERI, *Elements of Geometry*, Florence, 1926: vol. I, p. 113. This is the unabridged edition.

14 F. SEVERI, *Elem. di Geom*., I, 114.

15 F. SEVERI, *Elem. di Geom*., I, 119-20. We will see later how this can be dispensed with, but one must always resort to a postulate.

16 F. SEVERI, *Elem. di Geom*., I, p. 123.

17 F. SEVERI, *Elem. di Geom*., I, p. 117.

It follows that the demonstration referred to by Aristotle can very well exist on basis of a postulate such as Severi's or equivalent postulate, and that it is legitimate to raise the historical question raised above. But we will leave it aside for the moment, because as far as the ancient Pythagoreans are concerned, it appears somewhat idle. In fact, even this single piece of information that seemed to have been acquired about the Pythagoreans' demonstrations is missing, since it is certain that the ancient Pythagoreans did not prove the two-rectified theorem in this way, but in another, quite different and, moreover, completely unknown way.

In fact Loria rightly warns[18]: "Only one thing must be noted in this regard, and that is that the Pythagoreans to whom we owe the discovery of this theorem are not the same ones who invented this reasoning, because otherwise we would not understand how Eutocius, in a passage of the commentary to the 1st book of Apollonius' *Co-Niches* (Apollonius - ed, Leipzig, 1893, p. 170) says: "Similarly, the ancients demonstrated the theorem of the two right angles separately for each species of triangle, first for the equilateral, then for the isosceles and finally for the scalene, while those who came afterwards demonstrated the theorem in general: the three angles

18 GINO LORIA, *Le scienze esatte nell'antica Grecia,* 2nd edizio- ne, Hoepli, 1914, p. 47.

interior of a triangle are equal to two right angles". "And" Eutocius con- tinues, "he who says this is Geminus"[19].

In conclusion, this information is also missing, and all we know is that the property over the sum of the interior angles of a triangle was not admitted, but rather demonstrated by the ancients, and that this demonstration was divided into three parts.

"The Pythagoreans", writes Loria again[20], "knew the value of the sum of the angles of any right-angled triangle and were able to demonstrate [how?] the relevant theory; they are universally credited with the discovery and demonstration [which?] of the characteristic property of the right-angled triangle".

We are therefore forced to speculate for both theorems, bearing in mind that for the former we must exclude the theory of paraleles, and for the latter we must exclude the demonstration contained in Euclid's text (which is also dependent Euclid's postulate), because Proclus formally attests that this demonstration of the Pythagorean theorem is not Pythagoras' but Euclid's, saying: "on behalf of

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19 Cf. ALDO MIELI, *Le scuole jonica, pythagorica ed eleatica*, Florence, 1916, p. 273; there the Greek text of Eutocius is given. LORIA reports the whole passage on page 154 of '*Exact Sciences...*'.

20 GINO LORIA, *Storia delle matematiche*, Turin, 1929-33, vol. 1, p. 67.

I admire those who first investigated the truth of this theorem; but I admire the author of the *Elements* even more, because not only did he secure it with a clear statement, but because he reduced it to a much more general theorem in his sixth book with strict reasoning·

3. We do not know what Pythagoras' proof of his theorem was, but we can state, it seems to us, that Pythagoras did not make use of the properties of the parallel lines postulate this purpose. Otherwise the ancient Pythagoreans, who were older than Pythagoras, would have already made use of it, and also for the theorem of the two straight lines, whereas we know from Euto- cio-Gemino, that only those who came afterwards gave such a hasty demonstration.

Allman has indicated how the ancients may have arrived at the two-right theorem, which he inclines to attribute to Thales. Allman observes[22] that in the case of six equilateral triangles congruent around a common vertex, the sum of the six angles being equal to four right angles, each is equal to one-third of two right angles, and therefore the three angles of a triangle have a sum of two right angles. This explanation, however ingenious, cannot be the right one, because it presupposes the recognition of

21 Mieli on p. 266 of the cited work quotes Proclus' Greek text.

22 ALLMAN GEORGE JOHNSTON, *Greek Geometry from Thales to Euclid*, Dublin, 1889, p. 12.

necessarily empirical that six equilateral triangles (whose existence is implicitly admitted, and also that they are also equiangular) can actually be described in the manner indicated; whereas Proclus clearly states that this third point constituted a Pythagorean theorem, which, unless one sophistrates on the precise meaning of the word theorem ascribed by Proclus, indicates that this was the point of arrival and not the point of departure.

From the case of the equilateral triangle, Allman easily moves on to the case of the particular right-angled triangle, which is obtained by lowering the height. Then, in the case of any right-angled triangle (fig. 2), he completes the rectangle (whose existence is thus assumed) and says that: "*he* (Thales) *could easily* (empirically?) *see that the diagonals are equal and bisect each other*". The trian-

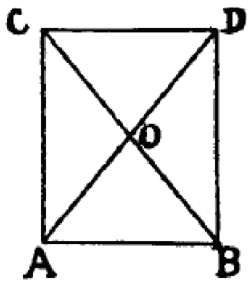

Fig. 2

rectangle is thus decomposed into two iso- chosen triangles with equal angles at the base, and since it is known that the two consecutive triangles with vertex A have a right angle as their sum, the same is true for the pair of the other two angles to

They are respectively equal, and hence it follows that the sum of the three angles of any right-angled triangle is equal to two right angles. Hence the theorem extends easily, though Allman forgets to say so, to an isosceles triangle, and from this to any triangle.

Tannery explicitly recognises that the property that the six equilateral triangles, the four squares and the three hexagons can be arranged around a common vertex *derives* logically from the theorem of the two right angles; nevertheless, he also reverses the order[23] saying: "It is also very possible that it was the empirical recognition of the property of equilateral triangles united around a common vertex, which led to the discovery of the equality to two right angles of the sum of the angles of each of these triangles; it was then, according to the testimony of Geminus, first to the isosceles triangle and finally to the scalene". We have seen that, following the path traced by Allman, we pass only to a particular case of the right-angled triangle, and that we must then make another appeal to empiricism in order to pass to the case of any right-angled triangle, only after which we finally pass to the isosceles and scalene triangles.

It does not therefore seem that the starting point indicated Tannery and Allman is the one used by the ancients. Another must be found, leading to the results in the

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23 P. TANNERY, *La Géom. gr.*, p. 104.

the order indicated by Geminus, and appeals to in- tuation in a simpler way.

4. As for the theorem on the square of the hypotenuse, "everything seems to indicate", writes Tannery[24], "that if he did not borrow it from the Egyptians, this proposition was one of the first he encountered, and by no means the crowning achievement of research", as it is in the text of Euclid's first book.

And it is precisely for this reason that the Pythagorean demonstration of the Pythagorean theorem not only cannot be the coda and consequence of other equivalence theorems, but must be independent of the theory of similarity, the theory of proportions, and the postulates of Euclid and Archimedes. On the other hand, if it is known and certain that the Egyptians knew particular right triangles with whole numbers as measures of the sides, among them the triangle known as the *Egyptian trian- gle*, it does not appear, however, that they knew the general theorem on the square of the hypotenuse, and if Pythagoras' discovery had been reduced to a simple discovery, the hosannas, the peanas and the sacrifices to the gods would be poorly explained.

Researching what the demonstration might have been, Tannery, after saying[25] that "the Greeks introduced the notion of similarity (VI of Euclid) as late as possible", states shortly afterwards that Pythagoras must have

24 P. TANNERY, *La Géom. gr.*, p. 105.
25 P. TANNERY, *La Géom. gr.*, p. 97.

The principle of similarity, the use of which had to be restricted due to the discovery of incommensurability. The principle of similarity is proved by using the postulate of parallels; "inversely [26] by admitting it a priori one could derive the postulate of parallels". Now, apart from the fact that this is a simple hypothesis, unsupported by any element, it should be noted that it is very true that by admitting this postulate of the similarity, the postulate of the parallels, the theorem of the two right angles, could be derived, the notion and properties of rectangles and squares, the theorem of proportions and the demonstration of the Pythagorean theorem by means of similar triangles, but the pre-existence of the ancient demonstration of the two-right theorem mentioned by Eutocius Geminus cannot be explained.

Also according to Loria[27] "the demonstration that has the greatest verisimilitude is the one based on the similarity of a right-angled triangle with the two that arise by lowering the perpendicular from the vertex of the right angle on the hypotenuse. With an easy metamorphosis, it becomes the same as in Euclid's *Elements*'. This possibility of reducing this demonstration to that of Euclid seems to us to prove just the opposite, namely that the demonstration mentioned by Lo- ria and Tannery, which in fact leads to the so-called first theorem of Euclid, from which the theorem of

26 P. TANNERY, *La Géom. gr.*, p. 105.

27 GINO LORIA, *Storia delle Matematiche*, vol. I, p. 67 in footnote.

Pythagoras, is not at all the original one; not to mention that, if this were the case, under the name of Pythagoras' theorem another theorem would have to be designated, namely the theorem on the square a cathetus (the first so-called theorem of Euclid). Allman much more happily observes[28] that although Pythagoras "may have discovered it as a consequence of the theorem on the pro- porality of the sides of equiangular triangles, there is no indication that he arrived at it in such a deductive way", and after having recalled that we know, thanks to Prodo, that Pythagoras took a way that is not the one taken by Euclid, he recognises that "the simplest and most natural way of arriving at the theorem is the following as suggested by Bretschneider" (fig. 3)[29].

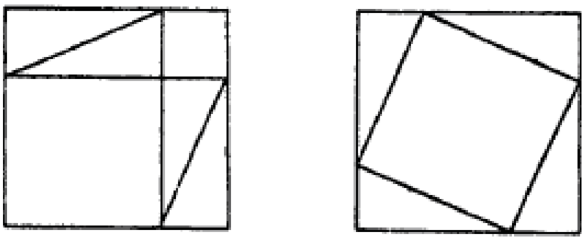

Fig. 3

This is a demonstration whose author is unknown to modern historians; however, it is known to be ancient. All that is needed for it is the notion of a right-angled triangle and a square, the properties of perpendicular lines and, as we shall see, knowledge of the theorem of the two

28 ALLMAN, *Greek Geometry*, p. 35.

29 BRETSCHNEIDER C. C., *Die Geometrie und die Geometer vor Euklides*, Leipsig, 1870.

straight; and is, as we shall see, independent of the theory of parallels.

Except that, continues Allman[30], Hankel[31] in citing this demonstration by Bretschneider says that "it can be objected that it does not present at all a specifically Greek colouring, but is reminiscent of Indian methods. This hypothesis about the eastern origin of the theorem seems to me well-founded; I would therefore attribute the discovery to the Egyptians...', from whom Pythagoras is said to have drawn it.

Indians or Egyptians seem to be the same thing, to take all credit away from Pythagoras! In any case, even if derived from India, Egypt or the Minoan civilisation, this would be, according to Allman and Bretschnei- der, the demonstration given by Pythagoras; one would at least like to admit that, even if inspired by the path suggested by the figure, the logical demonstration belongs to him; otherwise, where would be the merit that Proclus and all antiquity gave to Pythagoras in this respect? On the other hand, the assumption about the more or less Indian or Egyptian character of the demonstration does not seem to us to be sufficiently safe and impersonal, and if this criterion is applied, it is probable that other theorems, which are certainly Greek, should also be assigned an oriental origin.

We will show how a proof of the theorem based on this figure is obtained very simply-

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30 ALLMAN, *Greek Geometry*, p. 37.

31 HANKEL H., *Zur Geschichte der Mathematik in Alterthum und mittel-Alter*, Leipsig, 1874.

using the theorem of the two recti and its immediate consequences. And, in anticipation, let us note at once that in this demonstration we make use of the same criteria of composition and decomposition of figures that Plato uses in the *Timaeus* and the *Meno*[32], and that consequently this demonstration has not only the Greek colouring, but also the *Pythagorean* colouring of the demonstration of the *Meno*.

32 PLATO, *Timaeus*, XX; *Meno*, XIX.

CHAPTER I
THE TWO-RIGHTS THEOREM

1. It follows from the above that this essential and preliminary question must be resolved: To find out how the ancient Pythagoreans proved the two-rectified theorem.

We only know that they gave a proof that was not based on the postulate of parallels; and this leads with some certainty to the conclusion that they did not admit this postulate.

This indirect proof, moreover, is confirmed by the fact that not only the postulate, but the very concept of parallel lines, defined at least with Euclid as lines that never meet when extended to infinity, must have seemed particularly repugnant to the Pythagorean mentality for which the finite, the limited was the complete and perfect, and the infinite, the unlimited was the imperfect.

On the other hand, if we exclude the fifth postulate, and use only the preceding 29ª proposition of Euclid's first law, it is not possible, we believe, to achieve the goal; and we must assume, therefore, that the ancient Pythagoreans must have admitted some other simple property that made it possible to prove the theorem.

It is not strange that this should have been the case; for Tannery says that in Pythagoras' time "the number of truths admitted as primordial was undoubtedly much more considerable; and the progress... must have consisted more in the reduction of the axioms". We have seen that among these primordial truths admitted by the Pythagorean antiquarians, Tannery is inclined to believe that there was a postulate of similarity; but if this can be used to arrive at the demonstration of Pythagoras' theorem, it is of no use for that of the two recti, because it leads to the ordinary demonstration of this theorem and not to the archaic one, unknown, but whose existence and independence of Euclid's postulate we know. For the same reason, and also because of its relative complexity, we must exclude the possibility that the Pythagoreans had recourse to a postulate such as the one enunciated by Severi and which we mentioned at the beginning.

These rational considerations allow us to exclude the possibility of such postulates; but with rational considerations alone we cannot hope to grasp which postulate should be used; we can only add that it must be some property that continued to exist naturally after the adoption of the postulate of the parallels and after Euclid's arrangement of geometry, but which then disappeared from the number of the primordial properties, and probably became an obvious consequence of the new postulate. Determining which it was is a matter of inhalation rather than reasoning; let us say inhalation and not whim or fancy, and we add that we should subject it to every possible scrutiny, examine whether it arises from the Pythagorean mentality and whether it allows a development equal to the development actually achieved by the Pythagoreans and capable of leading to the results achieved by them, as Proclus has handed down to us.

It is well understood, and we say this explicitly in order to avoid misunderstandings and for the sake of precision, that by necessity and for the sake of brevity we assume and admit that the Pythagoreans accepted or showed the contents of the first 28 positions of Euclid, that is to say, what precedes the postulate of parallels and the theory of parallels, since we are interested in and must investigate how the propositions in which we know that Pythagorean geometry differed from Euclidean geometry can be . Basically, let us admit and suppose that the Pythagoreans (explicitly or not) admitted: 1º - the postulates of determination and belonging; 2º - the postulates relative to the division into parts of the line and the plane (referring, if we wish, to finite lines and finite planes); 3º - the postulates of congruence or movement.

And we believe that the Pythagoreans proved and knew the properties that can be obtained by ordinary procedures, namely:

1) the ordinary equality criteria for triangles;

2) relations between the elements of the same triangle; the theorems on isosceles, equilateral and equal-sided triangles; external angle theorem (greater than cia-

one of the non-adjacent interiors), the theorem over one side and the sum of the other two...

3) the uniqueness of the perpendicular for a point to a line, the property of perpendiculars to the same line, the properties of perpendiculars and obliques, of the axis of a segment... i.e. what is basically obtained with the ordinary postulates and procedures and without Euclid's postulate.

2. Using modern language, we have said that a new postulate must be introduced, i.e. the ancient postulate must be rediscovered, in order to be able to prove the two-right theorem. But we do not know by which term the ancients designated the primordial truths from which they logically derived the other propositions of geometry. The word *postulatum*, in which the character of logical exigency attributed to the concept thus designated is transparent, corresponds to the Greek αἴτημα and the Middle Latin *petitio*, and appears as a mathematical term in the 1619 Latin edition of Euclid's Commandino, and as a philosophical term in the Latin version of the *Reth. ad Alexan*. by Philel- phus (d. 1489). The distinction into hypotheses, axioms and postulates is Aristotle's; and Euclid, of course, makes use of the term αἴτημα.

The logical geometric edifice of the ancients necessarily contained primordial truths that were admitted without demonstration, but not necessarily out of pure logical necessity, to give reasoning the necessary starting point.

chosen and established with regard solely to ordinary intuition and sensual experience. It must be borne in mind that the geometric mentality of the Pythagoreans was quite different from the modern mentality that has as its ideal a pure, abstract geometry, existing only in the world of logic. On the contrary, observes Rostagni[33],

"Religion, morals, politics, mathematical sciences were not separate subjects for the Pythagoreans; they may have been identified over time, but they never ceased to be emanations and dependencies of cosmology... The cosmological spirit, which is inherent in Pythagorean philosophy, stands above those specifics, and dominates them all, indifferently'. Archita, the Pythagorean friend of Plato, in a fragment reported by Nicomachus and in another reported by Porphyry,[34] says that geometry, arithmetic, spherics (spherical astronomy), and music are sciences that seem to be sisters.

Geometry was not for them an exclusively logical discipline, made by man and for man, independent of cosmic reality, as the game of chess might be; it was the science that studied the *cosmos* from the aspect of position and extension. Arithmetic was the science of rhythm, ῥυθμός, ἀριθμός, of number, of time, of interval; and Archita distinguished

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33 A. ROSTAGNI, *Il verbo di Pitagora*, ed. Bocca, Turin 1924, p. 71

34 Cf. A. ED. CHAIGNET, *Pythagore et la philosophie pythago- ricienne*; Paris, 1874, vol. I p. 279.

also had a physical time and a psychic time. And the connection that the other two sciences, spherical astronomy and music, still had with these two sister sciences is evident. Moreover, it must be remembered that this synthetic vision that bound the various sciences together was presumably not based on ordinary human intuition and sensual experience alone and did not only have as its object the φύσις, nature, the world of ἄλλο, of alteration, of becoming; but also the eternal and Olympian unalterable ἐστὼ τῶν πραγμάτον, the essence things, the beyond of the περιέχον, of the cosmic *band*, which envelops the world of the four elements and the ten heavenly bodies. Ten centuries after Pythagoras, Proclus again assigned the objects of geo- metry to the intelligible and not the sensible.

Taking all this into account, the primordial truth we introduce, and which we believe is admitted by the Pythagoreans, is the following, which we shall call:

Pythagorean postulate of rotation: If a plane rotates rigidly about itself in an assigned direction about a fixed point (centre of rotation) of an assigned (convex) angle, every line in the plane also moves, and the initial and final positions of the (oriented) line, if they meet, form an angle equal to that by which the plane has rotated.

This primordial truth from the modern point of view is undeniably a simple fact of intuition, observation and experience. When a wheel spins, any segment, lying and rigidly connected with the plane of the wheel, it also moves, and it always turns in one direction if the wheel does likewise, and it turns more or less depending on whether the wheel turns more or less; and intuition and observation say that the rotation of the segment is equal to the rotation of the vector ray. On the other hand, the ability to compare angles could not have been lacking in the Pythagoreans; since, according to Eude- mus, the somewhat more difficult problem of constructing an angle equal to an assigned angle, given the vertex and a side of the angle to be constructed, is an invention rather of Oinopides of Chios than of Euclid; and Oinopides (c. 500 B.C.) is perhaps a Pythagorean.

To the adoption of this postulate, some moderns will object that it is independent of movement; but it must be observed that it is not a question here of discussing the theoretical questions of movement and congruence, but of judging whether this postulate could have been one of the primordial truths admitted by the Pythagoreans, and fact that it is based on movement, or rather on rotation, does not bear any prejudice in this respect. Movement, and in particular the movement of rotation, was a salient and characteristic aspect of cosmic life, and therefore not only could, but must, in Pythagorean terms, also have its function in geomotion. The tendency to dispense with motion as far as possible is a tendency of Euclid's, and this antipathy of his perhaps contributed to his great innovation, the theory of straight lines that prolonged to infinity never meet. These are straight lines that no one has ever po-

But Euclid was not a Pythagorean, and it was good that the definition of the parallels and the corresponding formula gave him the necessary means to proceed on his way.

3. The Pythagorean postulate of rotation does not, of course, coincide with the ordinary postulate of rotation.

The ordinary postulate of rotation tells us that when a plane rotates around a fixed point O (Fig. 4) by a certain angle α, all points on any line AB in the plane rotate around O, so that any two ray vectors OA, OB go around O respectively.

mind in OA', OB' such that $\widehat{AOA'} = \widehat{BOB'} = \alpha$, and the line AB goes into A'B' and every other point C of AB goes into a point C' of A'B' arranged with respect to points A' and B' as C is arranged with respect to A and B, and is $\widehat{COC'} = \alpha$.
Every point of AB therefore rotates by α. The pi- tagorical postulate of the rotation states that, moreover, the whole straight line AB, with this rotation, if it meets A'B', forms with it the angle α. In the case of a ray-vector OA the superposition to OA' is obtained by a simple rotation around one of its points O, in the case of any straight line AB the superposition is obtained by an equal rotation around an external point O, *or* by an *equal* rotation around the point of intersection (if any) of AB and A'B' followed by an appropriate translation. The postulate asserts the equality of these two rotations; and, if every point of AB rotates by α, it was not

natural to say that the set of such points, i.e. AB, also rotated by α?

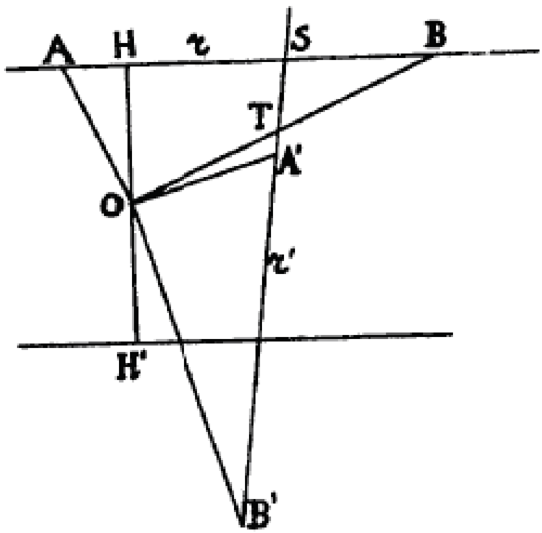

Fig. 4

From the postulate, it then immediately follows that if the retractor *r* with two consecutive rotations in the same direction is first brought into $r_{(1)}$) and then from r_1 into r_2, the angle $\hat{rr}_{(2)}$ is egua-
the sum $\hat{rr}_{(1)} + \hat{r_1 r_2}$. Therefore, the property extends
de immediately to the case of the concave angle and round angle.
In the case of the rotation of half a revolution, led from the centre of rotation by the perpendicular OH to AB, the vector ray OH is carried on the extension OH', AB is carried on the perpendicular to OH' by H', and the Pythagorean postulate tells us that if it meets AB it would form a flat angle with it. But since it is known that

two perpendicular lines at different points H, H' to the same line do not meet, we simply recognise that in this case the initial and final positions of the line do not meet. Of course, it does not follow at all that for every other rotation they must meet.

Finally, note how the postulate could also be stated in a different form. For example: If the plane rotates above itself in a certain direction around a fixed point, the angle formed by any line of the plane with its final position is constant; or: if the plane makes two consecutive rotations in the same direction with which *r* first goes into r_1 and then into r_2 then

$\widehat{rr_2} = \widehat{rr_1} + \widehat{r_1 r_2}$ (1). But it seems to us that the form that has- chosen adheres more readily to the obser-
and is therefore more likely to coexist with the primordial truth admitted by the Pythagoreans.

4. With the help of this postulate, the two-right theorem in the case of the equilateral triangle can be proved immediately.

Of course, this assumes that there are equilateral trian- gles and that one knows how to construct an equilateral triangle with an assigned side. The consideration of the equilateral trian- gle must have appeared very early in Pythagorean geometry, because of the correspondence they saw between the first four numbers, and the point, the network (identified and limited by two points), the plane and the triangle identified by three, and the space or volume identified by four points. It is perhaps no coincidence that

In Euclid, the first proposition of the first book has the equilateral triangle as its object. And since the occasion arises, we note that in it Euclid tacitly and implicitly admits the postulate that if a circle has its centre on another circle and a point inside it, it cuts it. In the same way, Euclid tacitly admits the other particular case of the continuity postulate, namely that the segment joining two points on opposite sides of a straight line is cut by it.

Having said that, to prove our theorem, it suffices to know the 1st and 2nd criteria of equality of triangles with their corollaries on the isosceles triangle and equilateral triangle, and to apply the Pythagorean postulate of the rotation.

Let us therefore demonstrate the

THEOREM: *The sum of the angles of an equilateral triangle is equal to two right angles.*

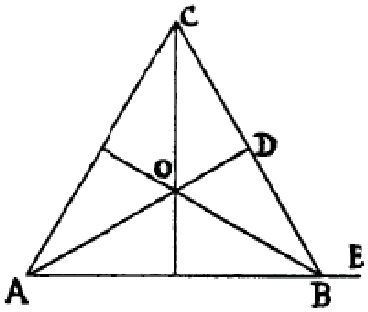

Fig. 5

Let ABC be an equilateral triangle (Fig. 5), and therefore equiangular.

The angle bisector $\widehat{CAB}$ meets the op- side at a point D inside it, and since the two points A and D are on opposite sides of the bisector of

$\widehat{ACB}$, the two bisectors cut at a point O within the given triangle. The angles $\widehat{OAC}$, OCA are egua- them because halves of equal angles, and therefore OAC is isosceles and OA = OC.

The triangles ACO, BCO are equal by the 1st criterion, and so OB= OA= OC and $\widehat{OBC} = \widehat{OAC}$; and therefore OB is bisector of the angle $\widehat{ABC}$. The three triangles isosceles OAB, OBC, OAC are therefore equal (2nd or 3rd criterion) and the vertex angles $\widehat{AOC}$, $\widehat{COB}$, BOA are equal.

By rotating the figure around O by angle $\widehat{COB}$, vertex C goes to B, B to A, and A to C, CB goes to BA and the angle formed by them, i.e. the external angle $\widehat{CBE}$ è equal for postulate to the angle $\widehat{COB}$. Continuing the rotation, with two more ro- equal, the figure overlaps itself; and the sum of the three angles of rotation, i.e. the three external angles of the given triangle, is equal to one round angle, i.e. four right angles.

On the other hand, each internal angle of ABC is supplementary to the external angle; therefore their sum will be equal to six right angles minus the sum of the external angles, i.e. six right angles minus four right angles: i.e. two right angles. c. d.

5. The truth of the theorem in the first case, according to Eutocus and Geminus, proved by the Pythagoreans is therefore an immediate consequence of the Pythagorean postulate of rotation. Having proved the theorem easily in this particular case, it was natural that the ancients wondered what happened in general, and it was natural that before the general case they studied the other special case of the isosceles triangle.

In the latter case, the demonstration is not so straightforward; several other propositions have to be taken into account, all of which can be demonstrated quite easily and without the need for Euclid's postulate, as can be found in Euclid himself and in modern texts. We will refer to these texts for the demonstrations and merely mention these properties, which are also included in those mentioned above:

a) The bisector of the angle at the vertex of such an isosceles triangle is also median and height.

b) Existence, uniqueness and determination of the midpoint of a segment.

c) Theorem of the exterior angle of a triangle.

d) The sum of two interior angles of a triangle is always less than two right angles.

e) If one angle of a triangle is greater than or equal to a right angle, the other two are acute.

f) If in a triangle one side *a* is correspondingly greater than or less than a second side *b*,

the anglê A opposite to a is correspondingly greater than, equal to or less than the angle $\hat{B}$ opposite to *b*; and vice versa.

g) If a triangle has an obtuse or right angle, the side opposite it is the greater.

h) In a triangle, one side is less than the sum of the other two sides.

i) Definition, existence, uniqueness of the perpendicular to a line for a point.

k) Inverse theorems on the median and height of an isosceles triangle.

l) Theorems on the axis of a segment and on the bisectors of angles formed by two competing lines.

Having said that, we demonstrate the

THEOREM: *The sum of the interior angles of an isosceles triangle is equal to two right angles.*

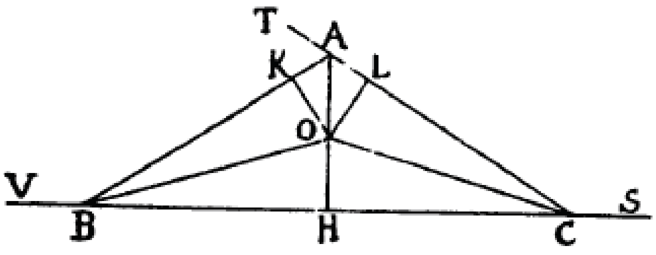

Fig. 6

Let ABC be the isosceles triangle (fig. 6) and let AB = AC and therefore $\widehat{ABC} = \widehat{ACB}$; let AH be the bisector, median and height of the isosceles triangle. It is shown as in the previous case that the bisector of the angle at the base $\widehat{ABC}$ meets AH at an interior point O, and joined O with C by the equality (1st criterion) of the triangles BAO, CAO it follows that OB= OC and therefore $\widehat{OBC} = \widehat{OCB}$, and therefore CO is the bisector of $\widehat{ACB}$.

On the other hand, since BC < AB + AC will be the half BH < AB = AC; and taken then on the sides BK = CL = BH the points K and L result internal to AB and AC respectively. If O is joined to K and L, the triangles OKB, OHB, OHC, OLC are equal by the first criterion, and therefore OH = OK = OL, and AB, AC are perpendicular to OK and OL respectively. Let us now rotate the figure around O, so that OH rotates in OK, the perpendicular BC to OH goes on the straight line BA perpendicular to OK in K, and by the postulate of the rotation, the ango-

lo exterior $\widehat{VBA}$ of the given triangle is equal to the an-

Rotation $\widehat{HOK}$. Continuing the rotation in the

the same direction OK goes to OL, AB perpendicular to OK goes to CA perpendicular to OL and the outer angle

$\widehat{BAT}$ is equal to $\widehat{KOL}$. Continuing the rotation and

By bringing OL above OH the figure returns, after one complete turn, above itself, and $\widehat{ACS}$ L= $\widehat{OH}$.

The sum of the three external angles is equal to the whole ro-
tion of four right angles; and again, since the three angles of the given triangle are respectively supplementary to the adjacent exterior angles, their sum will be equal to six right angles minus the sum of the exterior angles, i.e. six right angles minus four right angles, i.e. two right angles c. d.

6. Let us turn to the general case.

It is only necessary to state the following theorems, which are easily proved by absurdity, and which we will only state for the sake of brevity:

a) In an acutangle triangle, the feet of the three heights are inside the sides.

b) In an obtuse or right-angled triangle, the foot of the height relative to the longer side is inside the side.

This is enough to prove that:

THEOREM: *In any triangle, the sum of the three angles is equal to two right angles.*

Let A (fig. 7) be the vertex of the right or obtuse angle, if any, of the arbitrary triangle ABC. Lower the height

AH, the foot H is internal to BC and the angle $\widehat{BAC}$ is divi-

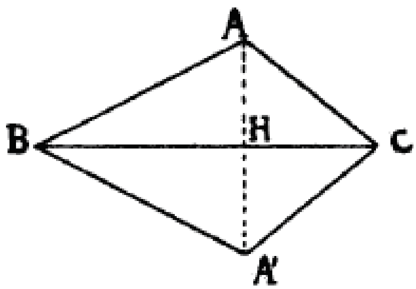

Fig. 7

so in two parts by AH. On the extension of AH we take HA' = AH and join A with B and with C. The right-angled triangles AHB, A'HB are equal for the lº

criterion, thus BA= BA' and $\widehat{BAH} = \widehat{BA'H}$; analogous-

mind C $\widehat{AH} = \widehat{CA'H}$.

By the previous theorem applied to the two isosceles triangles BAA', CAA' we have:

$$\widehat{ABA'} + \widehat{BAA'} + \widehat{BA'A} = \text{two straight}$$

and BH being the bisector of the isosceles triangle BAA', we ha:

$$\widehat{ABH} + \widehat{BAA'} = \text{a rectum}.$$

Similarly

$$\widehat{ACH} + \widehat{CAA'} = \text{a rectum},$$

and adding

$$\widehat{ABH} + \widehat{ACH} + \widehat{BA'A} + \widehat{CAA'} = \text{two straight},$$

or

$$\widehat{ABC} + \widehat{ACB} + \widehat{BAC} = \text{two straight}.$$

The theorem is thus proved in general.

7. demonstration presented itself immediately in the first case mentioned by Eutocio-Gemino, and then neatly for the other two cases they mentioned.

However, it should be noted: 1st that the demonstration of the first case is, from a modern point of view, superfluous, because the second case includes the first; 2nd that the general case can also be demonstrated directly to include the other two.

In order to obtain this general demonstration, two theorems need only be stated, which are as follows:

THEOREM: *Two right-angled triangles having an equal hypotenuse and an equal acute angle are equal.*

Let it be (fig. 8) $\hat{A} = \hat{A}' = 90°$; $a = a'$; $\hat{B} = \hat{B}'$.

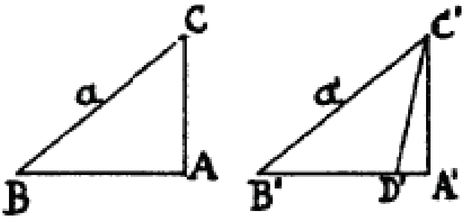

Fig. 8

If BA = B'A', the theorem is proved; but if, for example, B'A'>BA, and B'D'=BA, the triangle B'D'C' is equal to the triangle BAC by the first criterion; therefore C'D' is perpendicular to B'A', and this cannot be ac- cessed because from C only one perpendicular to B'A' can be led.

The other theorem that needs to be stated is the following.

THEOREME: *Two right-angled triangles having the hypothetical equal nuse and equal cathetus are equal.*

Let (fig. 9) BAC, B'A'C' i two triangles,

$\hat{A} = \hat{A'} = 90^\circ$, BC=B'C', CA= CA'.

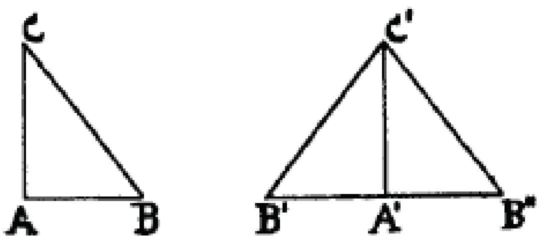

Fig. 9

Given A'B"=AB the right triangle C'A'B" is equal to CAB, C'B'=CB=CB', and in the isosceles triangle

B'C'B' the height è also median, therefore B'A'=A'B"=AB.

This leads to the following general demonstration of the fundamental theorem:

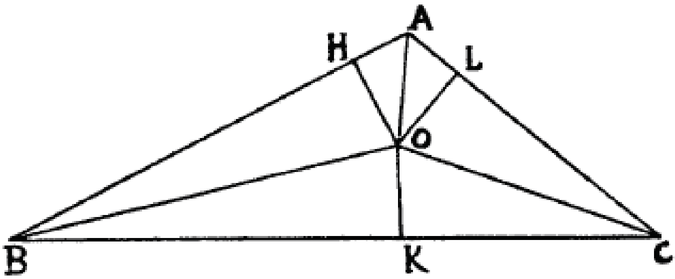

Fig. 10

Let A (fig. 10) be the vertex of the right or obtuse angle, if any, of triangle ABC; and let the bisectors de-

angles $\widehat{BAC}$, $\widehat{ABC}$. It is usually demonstrated that they meet at a point O inside triangle ABC.

Angles $\widehat{ABO}$, $B\widehat{AO}$ half convex angles are acute, so that in the triangle OAB the possible non-acute angle is that of vertex O, and therefore in all cases, by lowering from O the perpendicular OH to AB the foot H is internal to AB. When O is joined to C, the acute angle

$\widehat{ACB}$ is divided into two acute angles, so that also in triangles AOC, BOC the possible non-acute angle is that of vertex O, and also in them the feet L and K of the perpendiculars lowered from O above AC and BC are in all cases respectively internal to AC and BC.

The right triangles OBK, OBH have an equal hypotenuse and an equal acute angle; they are therefore equal, and

OK=OH. Similarly, the triangles OAH, OAL are equal and therefore OH=OL. But then the right triangles OLC, OKC have the hypotenuse in common, the cathetus OL=OK, are therefore equal and therefore OC is bisector of $\widehat{ACB}$. We therefore have that the three angle bisectors interiors of any triangle meet in a point internal to the triangle, such that, lowering the perpendiculars to the sides from it, the three feet H, L, K are internal to the three sides, and we have OH=OK=OL.

All that remains now is to rotate the figure around O, successively bringing OK to OH, OL, OK, and the line BC will subsequently go to AB, CA, BC; the external angles of triangle ABC for the Pythagorean postulate of rotation will be equal respectively at the three corners $\widehat{KOH}$, $\widehat{HOL}$, $\widehat{LOK}$; their sum will be four right angles, and that of the internal angles will be two right angles.

8. This demonstration therefore makes the previous two superfluous; and in any case, the demonstration in the case of the isosceles triangle includes that of the equilateral triangle. Must we conclude that this is not the three-stage demonstration of the ancient Pythagoreans, mentioned by Eutocius and Geminus?

To conclude in this sense be to attribute to the ancients the modern tendency and habit of generalisation, i.e. to judge according to our mentality. In order to obey our standards, they would have had to give up proving the theorebut in the first and simple case and wait until (and why would they?) they had found a way to prove it in the second and third cases. Furthermore, we must not forget that they discovered the theorem, and it is probable that the discovery was made in the case of the equilateral triangle; only afterwards, and as a consequence, did the question arise as to whether the theorem was valid in general, and only afterwards, and with much more effort, did they prove it in the other two cases.

Therefore, unless it is possible to deduce the second case from the first case in a fairly simple way, we are convinced that our demonstrations are precisely those of the ancients, and we almost believe that even in the third case they did not deduce the demonstration from the second case, but preferred to refer again to the postulate of rotation by analogy of demonstration. In any case, we must keep in mind what Tannery wrote [35]: "I believe it is useless to insist on the difficulty that the first geometers seem to have found in raising themselves to the simplest generalisations", citing for example the case of the theorem of the two right angles.

However, we have arrived at this result: We have proved the fundamental theorem over the sum of the angles of a triangle without making use of the postulate and the concept of parallel lines. It is a result of a

35 P. Tannery, *La Géom. gr.*, p. 101, footnote 2.

certain importance if the Pythagorean postulate of rotation is not equivalent to Euclid's postulate.

9. Indeed, the Pythagorean postulate of rotation is not equivalent to Euclid's postulate. And here's why.

We have seen that from the Pythagorean postulate of the rotation we deduce the two-right theorem. Conversely, assuming that the sum of the angles of a triangle is a constant, we deduce our postulate.

Let O be the centre of rotation and S the point at which the initial and final positions of the straight line *r'* meet (fig. 4). Let us take a point A on *r'* on the side O, and a point B on the opposite side; *r'* cuts the segment OB at a point T. The rotation that brings *r to r'* brings point A to a point A' and B to a point B'.

and is $\widehat{AOA'} = \widehat{BOB'}$ the angle of rotation. The triangles AOB, A'OB' are equal, so $\hat{B} = \hat{B}'$. The triangles OTB', STB have therefore the angles $\hat{B} = \hat{B}'$, $\widehat{OTB'} = \widehat{STB}$; and, if we admit that the sum of the angles of any triangle is constant, the third angle $\widehat{TSB}$ will be equal to the third angle $\widehat{TOB}$; that is, the angle $\widehat{rr'}$ equal to the angle of rotation, as had to be proved. So the Pythagorean postulate of rotation and the proposition about the constancy of the sum of the angles of a triangle are equivalent as postulates.

Admitting the constancy of the sum of the angles of a triangle, one could deduce our rotation po- stulate, and applying it to the case of the equilateral trian- gle, one would immediately find that the quantity whose constancy has been admitted is equal to two right angles.

Girolamo Saccheri proposed, as is well known, the notion that the sum of the angles of a triangle is equal to two right angles in place of Euclid's postulate, and Le- gendre demonstrated that, if Archimedes' postulate is also admitted, Saccheri's proposition is effectively equivalent to Euclid's postulate. It follows immediately that if, in addition to the Pythagorean postulate of rotation, we also admit Archimedes' postulate, it is equivalent to Euclid's postulate.

If nothing else is admitted, it is not equivalent to Euclid's postulate. In fact, Dehn (1900) has demonstrated [36] that Saccheri's hypothesis is compatible not only with elementary geometry, but also with a new geometry, necessarily non-Archimedean, where the fifth postulate does not apply, and in which non-secant infi- nites pass through a point with respect to a given line [37].

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36 *Math. Ann.*, B. 53, p. 405-439, *Die Legendre'schen Sätze über die Winkelsumme in Dreieck*; cf. ROBERTO BONOLA, *Sulla teo- ria delle parallele e sulle Geometrie non euclidee*, in ENRIQUEZ, *Questioni riguardanti le Matematiche elementari*, 3 ediz., vol. II, p. 333.

37 Dehn calls this geometry: *semieuclidean geometry*.

The same certainly applies to our postulate. Once one admits the Saccheri proposition or the equivalent Pythagorean postulate of rotation, one can:

1. Admit the postulate of Archimedes, and then postulate of Euclid is proved; and Euclidean and Archimedean geometry is obtained.

2. negate that of Euclid, and thus necessarily also that of Archimedes; and the semieuclidean geome- tria of Dehn is obtained.

3rd - to completely ignore the two postulates of Eucli- de and Archimedes and the related issues, and to develop a more general geometry, independent of their acceptance or negation (and thus valid in both cases), as a consequence of the two-right theorem now obtained.

The ancient Pythagoreans were almost certainly unaware of Archimedes' postulate[38], and had obtained the two-right theorem by a procedure independent of the theory of parallels.

By not introducing Archimedes' postulate, we find ourselves in exactly the same position. If the ancient Pythagoreans did not make use of the concept of pa-

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38 Proposition 1ª of Euclid's Book X is equivalent to Archimedes' axiom. From some passages of Archimedes, it appears that, before that, Eudoxus had made use of this 'lemma'; and Lo- ria believes that the origin of this lemma must be traced back to Hypocrates of Chios (cf. LORIA, *Le scienze esatte nell'antica Grecia*, p. 143-145 and 224). However, the ancient Pythagoreans must have been unaware of Archimedes' postulate.

rallela, it must now be possible, from the theorem of the two recti, again without resorting to the postulate of Euclid and that of Archimedes, to deduce one after the other all the discoveries attributed by Proclus to the Pythagoreans. If this happens, this more general geometry will agree or coexist with the geometry of the Italic School.

10. Before going any further, however, we would like to expound a quicker way to deduce the two-rectified theorem from the Pythagorean postulate of rotation.

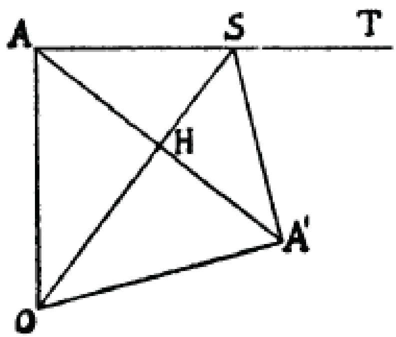

Fig. 11

From vertex A of the right angle (fig. 11) of an arbitrary right-angled triangle OAS, we take the perpendicular AH to the hypotenuse, and on the prolongation we take HA'=AH. We know that H is internal to OS; when A' is joined to O and S, the right-angled triangles OHA', SHA' are respectively equal to the two OAH, SHA; and

then OA=OA', SA=SA', $O\widehat{AH} = \widehat{OA'H}$,

$\widehat{SAH} = \widehat{SA'H}$ ed $\widehat{SA'O} = \widehat{SA'} + \widehat{OA'H} =$

$\widehat{SAH} + \widehat{OAH} =$ a right angle . Therefore, by rotating around O by the angle $\widehat{AOA}$ ', AS goes over the perpen- dicular in A' to OA', i.e. on A'S, and therefore by the po- stulate of the rotation $\widehat{AOA'} = \widehat{A'ST}$. It follows that

$\widehat{AOA'}$ and $\widehat{ASA'}$ are additional, and thus in the quadrilateral AOA'S you have:

$$\widehat{SAO} + \widehat{AOA'} + \widehat{OA'S} + \widehat{A'SA} = 4 \text{ rectification .}$$

And since the heights SH, OH of isosceles triangles SAA', OAA' bisect vertex angles the sum

$\widehat{HSA} + \widehat{SAO} + \widehat{AOH}$ is half of the previous one, i.e.

we have the theorem: In a right-angled triangle any-
that the sum of the angles is equal to two right angles.

From any right-angled triangle to an isosceles triangle (and in particular to an equilateral triangle), the bisector of the angle at the vertex being also the angle at the vertex; and since the acute angles of any right-angled triangle are now complementary, the sum of the acute angles of the two right-angles in which the isosceles triangle is decomposed is equal to two right angles. From the case of the isosceles triangle, we move on to the general case in the manner already seen.

The route taken, passing through the three stages mentioned by Geminus, is the one probably taken by the discoverers of the property; today, with the discovery made, it is more expeditious to proceed in the manner now indicated.

CHAPTER II
THE PYTHAGOREAN THEOREM

1. We needed the Pythagorean rotation postulate to prove the two-right theorem. From now on, in all that follows, we will no longer need it, because the two-rights theorem, as we know, is equivalent to it. And, since we know[39] that the Pythagoreans knew the theorem of the two right angles because they proved it, the restitution of Pythagorean geometry proceeds from now on from their certain knowledge, however obtained, but without the postulate of the parallels. Even if the way to obtain the two-right theorem had been different, but always independently of Euclid's postulate, we would still find ourselves in the same situation with the problem of restoring Pythagorean geometry as a development and consequence of the two-right theorem.

We will limit our investigation to what is needed to obtain the results attributed by Proclus to the Pythagoreans,

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39 The testimony of Eutocius, although Eutocius is also later than Proclus, is reliable. LORIA (*The Exact Sciences*, p. 721) says that Eutocius, of mediocre intellect, was, however, very diligent, accurate and conscientious; it is difficult, on the other hand, to invent such precise and circumstantial information.

often omitting demonstrations when they coincide with those known to all.

And we first see how the two-right theorem immediately allows the construction and consi- deration of the square and the rectangle and the demonstration of the Pythagorean theorem. And we note how the following consequences, among others, immediately follow from the two-right theorem:

a) The acute angles of a right-angled triangle are complementary; and in an isosceles right-angled triangle they are equal to half a right angle.

b) The angle of an equilateral triangle is equal to one-third of two right angles.

c) The exterior angle of any triangle is equal to the sum of the two non-adjacent interior angles.

2. Turning to quadrilaterals, we immediately note that Eu- clid distinguishes five in its definitions: the quadrilateral, the rectangle, the rhomboid, the rhomboid, and all the other quadrilaterals.

They are defined and distinguished by Euclid on the basis of the equality of sides and angles, and the definition of parallel lines comes *immediately afterwards*; whereas in the text, the construction of the square is based on parallels and appears at the end of the first book.

Having defined the square as a quadrilateral with all equal sides and all right angles, the construction of a square of assigned side AB, and thus its existence, They descend instead from the two-right theorem and from it alone.

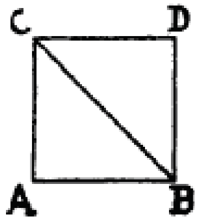

Fig. 12

Given (fig. 12) AC equal and perpendicular to AB, the two angles at the base of the right-angled iso- cle triangle ABC are equal to half a right angle. Let us lead by B the ray[40] perpendicular to AB from the side of C, and take on it BD= AB= AC; BC divides the ango-

I rectum $\widehat{ABC}$ into two equal parts; A and D are from opposite sides to CB, and thus CB divides the an-

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40 We use the term: *ray* for the sake of brevity; but the concept of lines and rays extended to infinity could not, it seems to us, be shared by the Pythagoreans. In fact, Euclid's 2ª, 3ª and 4ª definitions refer to the line and to the limited straight line, i.e. to our segment; and Euclid's se- conditional postulate only admits that the straight line, i.e. the segment, can be prolonged κατὰ τὸ συνεχές. It would therefore be necessary to say: from B let a segment perpendicular to AB be taken from the side of C with respect to D, and on it conveniently prolonged if necessary, take the segment BD= AC... Euclid'definition 23ª and postulate V *introduce* the concept of infinite lines. It is, therefore, an addition that does not conform to the spirit of ancient geometry and which does not fit well with the other definitions in the list itself that precedes Euclid's text.

triangle $\widehat{ACD}$. The triangles ABC, DBC are equal by the 1st criterion, so CD= AC, and $\widehat{DCB} = \widehat{ACB}$, $\widehat{CDB} = \widehat{CAB}$. The quadrilateral ABCD therefore has all the equal sides and all right angles; it is therefore, by definition, a a square. The diagonal BC divides it into two equal isosceles right-angled trian- gles. It is easily shown that AD = BC and that the two diagonals cut at the midpoint and are perpendicular to each other.

3. *Definition, existence, construction and properties of the rectangle.*

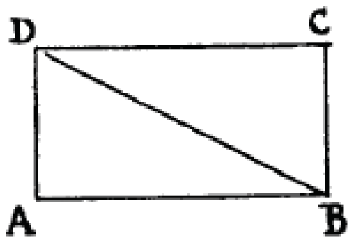

Fig. 13

Let us take the following definition: A rectangle is a quadrilateral with all right angles. Let ABD (Fig. 13) be any right-angled triangle and A the vertex of the right angle. Let B be the ray perpendicular to AB on the side of D with respect to AB, and take BC= AD, C and A remain on opposite sides with respect to BD because, being $\widehat{ABD}$ acute and $\widehat{ABC}$ straight the BD divides the right angle $\widehat{ABC}$. By joining C with D, the triangles ABD, CBD are equal by the 1st criterion, and then DC=AB, $\widehat{DCB} = \widehat{DAB} =$ a rectum,

$\widehat{CDB} = \widehat{ABD}$; and since we know that $\widehat{ABD}$ is the complement of $\widehat{ADB}$ CDB will also be the complement of $\widehat{ADB}$, i.e. the fourth angle $\widehat{ADC}$
of the quadrilateral ABCD is straight; it is therefore a rectangle.

The opposite sides are equal and their extensions cannot meet because they are perpendicular to the same line; it is easily demonstrated that the diagonal AC is equal to BD and that they cut each other in half.

On the other hand, if ABCD is a rectangle, it is first observed that the vertices C and D must be on the same side in relation to AB, because otherwise CD would be cut off from AB at a point M, and the right triangles ADM, CBM would result in that the angles
$\widehat{ADC}$, $\widehat{DCB}$ could not be rectified. Both dun-
that ABCD is a rectangle; BD determines the two right-angled triangles ABD, CBD, and the angles adjacent to the hypotenuse being acute in both, BD divides the two right angles of vertices B and D of the rectangle, and leaves A and C
from opposite sides; furthermore, $\widehat{CBD}$ is complementary to
$\widehat{ABD}$, and then $\widehat{CBD} = \widehat{ADB}$; similarly
$\widehat{CDB} = \widehat{ABD}$, and the two right-angled triangles ABD, CBD are equal, and CD = AB, BC = AD etc.

To construct the rectangle with sides equal to AB and AD,
take the segments AB, AD from vertex A of a right angle on both sides; take the perpendicular to AB through B, and on it from the D side, take the perpendicular to AB.

takes BC = AD, joins C with D and ABCD is the required rettangle.

The theorem of the two right angles and the resulting properties of the right triangle therefore immediately assured the Pythagoreans of the actual existence squares and rectangles, enabled their construction, and gave them their fundamental properties.

In order to demonstrate the property of regular polygons congruent around a common vertex, we should now move on to the consideration of quadratic polygons. However, since we need nothing else to demonstrate the Pythagorean theorem, we will now move on to the demonstration of this fundamental theorem.

4. PYTHAGORE'S THEOREM: *In any right-angled triangle, the square constructed over the hypotenuse is equal to the sum of the squares constructed over the cathexes.*

We use the ancient expression: *equal*, instead of the modern *equivalent*, also because in the demonstration we will use (as Euclid does in his) the "common notion" of *equality by difference,* and not the notion of *additive* equality, which alone leads to the concept of equivalence (Duhamel) or equicomposition (Severi).

In the particular case of the isosceles right triangle, Plato gives the following demonstration in the *Menon*[41]: pre-

41 PLATO, *Menon*, XIX - A correct and complete translation of Plato's passage can be found in LORIA's '*Exact Sciences in Ancient Greece*' on pages 115-20. Plato knew the validity

If we take a square ABCD (fig. 14) and join three other equal squares congruent at a vertex as indicated in the figure, we obtain a square that is four times as large as the given square. Dividing then each of those four squares with the diagonal, one obtains a square that is double the given square, because it is composed of four triangles equal to ABC, while the given square is equal to two.

Turning to the general case, of the seventy-plus known dimensions, the simplest are:

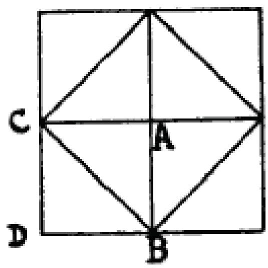

Fig. 14

1º - the one suggested by Bretschneider, whose author is unknown to the moderns, but of which we know that it is ancient; 2º - the one conceived by Abu'l Hasan Tabit (died in 901 A.D.) and of which Anarizio[42] has kept us a record; 3º - the one of Baskara, which is about three centuries later than Tabit[(43).] The first, both because it is not known to whom it should be attributed, and because of its

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of the theorem in the case of the right triangle that has the hypotenuse double the smaller cathetus; results from *Timaeus*, XX.

42 Cf. G. LORIA, *Storia delle Matematiche*, vol. I, p. 341.

43 See G. LORIA, *Storia delle Matematiche,* vol. 1, p. 315.

great simplicity, it may well be, and we are convinced, that it is Pythagoras'.

Let us see how this demonstration can be done without the postulate of parallels.

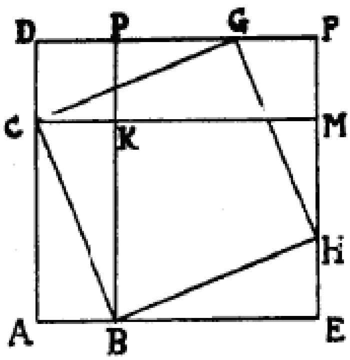

Fig. 15

Suppose that in the right-angled triangle ABC (fig. 15) Let A bê the right angle and AC the greater cathetus. At extension of cathetus AC we take CD = AB and on the extension of AB we take BE = AC. It follows that AE = AD. For C and D, we take the perpendicular bisectors of AD from the side of B to AD and take them DP = CK = AB; and we join K to P and to B. The two quadrilaterals ABKC, CKPD are by construction respectively a rectangle and a square; namely the rectangle is equal to the double of the given right-angled triangle, and the square has on its side a segment equal to the cathetus AB of the given triangle. They are separated and situated on opposite sides of the common side CK, because the three rays AB, CK, DP perpendicular to the same line AD do not meet two by two, and

Since C lies between A and D, DP and AB lie on opposite sides of CK. Since the vertex angles K of the rectangle and the square are right angles, their sum is a flat angle, and therefore the points P, K, B are collinear above a perpendicular common to the straight lines DP, CK, AB.

On the extensions of DP and CK on the opposite side of AD, we take the segments PF = KM = BE = AC, and we join M with F and with E. The quadrilateral PKMF turns out to be a rectangle by construction and it too is the double of the given triangle ABC; KMBE turns out to be a square whose side has a segment equal to the cathetus AC of the given triangle; and the three points F, M, E also turn out to be aligned on a perpendicular line common to the three straight lines AB, CK, DP. It is immediately recognised that the quadrilateral AEFD has all right angles and all sides equal and is therefore a square.

The triplet of the three straight lines AB, CK, DP and the triplet of the three straight lines AD, BP, EF are perpendicular to each other, and since K is between C and M, and between B and P, CM and BP divide the square AEFD into four parts. It is therefore equal to their sum. The square AEFD is therefore equal to the sum of the square constructed on the cathetus AB, the square constructed on the cathetus AC, and four right-angled triangles equal to the given.

Let us now take above DF and FE the segments DG= FH = AC and join C with G, G with H and H with B. The right triangles ABC, DCG, FGH, EHB are equal by the first criterion and therefore the quadrilateral CGHB has all sides equal. Furthermore, since the GC and GH rays are on the same side with respect to DF and the angles DGC, FGH are acute and complementary (because

FGH D= CG) the angle CGH which is obtained by removing-

I give from the flat angle the two angles DGC, FGH result- In the same way, the other anoles of the quadrilateral CGHB are shown to be right-angled, which is therefore the quadrilateral constructed on the hypotenuse BC of the given triangle.

Since then DCG is acute and DCM straight, the trian- CGD and the CGFM quadrilateral stand on op- placed with respect to CG. CG therefore divides the entire square into two parts, namely the triangle CDG and the polygon CGFEA. And since CGF is obtuse and CGH straight, the po- previous ligon is divided by GH into two parts, namely the triangle GFH and the polygon CGHEA; this in turn is divided by HB into two parts, namely the triangle HBE and the polygon CGHBA, which is finally divided by BC into the triangle ABC and the square CGHB.

The square CGHB is thus obtained from the square ADFE by removing four right-angled triangles equal to ABC. But removing from the square ADFE the two rectangles ABKC, KMFB, i.e. four triangles equal to the given one, we obtain the sum of the squares constructed on the cathexes AB and AC, and since the second Euclidean notion (which is already found in Aristotle) says that "taking equal from equal things we obtain equal things", so the square constructed on the hypotenuse is equal to the sum of the squares constructed on the cathexes.

5. Admitting the Pythagorean postulate of rotation and ignoring the two postulates of Euclid and Archimedes, we have thus immediately obtained the two fundamental theorems of geometry: the theorem of the two recti, and from this the Pythagorean theorem. They are both valid in the ordinary Euclidean and Archimedean geometry as well as in the more general geometry that admits the Pythagorean postulate of rotation and is independent of Euclid's and Archimedes' postulates.

The Pythagorean theorem is thus presented as the first theorem in the theory of equivalence; precisely as, according to Tannery, it was with the Pythagoreans. It lies at the base of this theory and not at the end. The demonstration we have given depends solely on the two-right theorem, known to the ancient Pythagoreans, and its immediate consequences. It is known that a demonstration based on the figure we have used existed, is ancient, and its author is not known to modern historians of mathematics. All we have done is to make it independent of Euclid's postulate, which the Pythagoreans did not use to prove the two-right theorem and which is therefore also superfluous for the Pythagorean theorem.

All things considered, it does not seem at all improbable that this is precisely the demonstration that the founder of the 'Italic School' discovered and gave twenty-five centuries ago. With it, the theorem is valid in the sense of equality by difference in a geometry that ignores or even denies the postulates of Euclid and Archimedes. It demonstrates-

Euclid' text proves the validity of the Pythagorean theorem always in the sense of equality by diffe- rence, if and even if one admits the postulate of paraleles and nothing is said about Archimedes' postulate; modern demonstrations prove its validity in the sense of additive egua- lance (Duhamel), equivalence or equicom- position (Severi), if and even if one admits Euclid's postulate together with Archimedes' postulate.

6. From the demonstration we have given of the Pythagorean theorem, the three important theorems expressed in modern notations are immediately and easily derived from the formulae:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

As for the former, simply observe the fi- gure 15 to recognise that:

THEOREM: *The square whose side is the sum of two segments* (AB and BE) *is equal to the sum of the square* (CKPD*) constructed on the first segment, the square* (BEMK) *constructed on the second segment and two rectangles whose sides are equal to the given segments.*

In the case that the two segments are equal, the theorem becomes: *the square that has twice the side of a given square is four times the side of the square*[44].

Given the following theorems:

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44 PLATO, *Menon*, XVII.

$$am + bm = (a + b)m$$
$$am - bm = (a - b)m$$

Of immediate demonstration, from Fig. 15, setting *AE=a*, AB=b we have BE=a - *b*, and $(BE)^{(2)}$ = quad. ED+ quad. DK - 2 rect. ABDP i.e. $(a - b)^{(2)} = a^2 + b^2 - 2ab$ i.e. the

THEOREM: *If a segment is equal to the difference of two segments, the square constructed it is equal to the sum of the squares constructed on the two segments minus twice the rectangle whose sides are the two segments.*

Then assuming *AE=a*, BE=b and AB=d from Fig. 15 we have: the difference of the squares constructed on AE and BE is given by the *gnomon* ADFMKB; i.e:

$$a^2 - b^2 - ad + bd = (a + b)d$$

and thus:

$$a^2 - b^2 = (a + b)(a - b)$$

i.e. the

THEOREM: *The difference of two squares is equal to the rectangle whose sides are the sum and difference of the two segments.*

This gnomon is nothing other than the *square of* the mura- tors; and in the case where a is the hypotenuse and *b* a cathetus of a right-angled triangle, the gnomon is equal to the square constructed on the other cathetus.

The three inverse theorems can be proved easily; so can the

PYTHAGORE'S REVERSE THEOREM: *If the qua-*
square constructed on one side of a triangle is equal to the sum of the squares constructed on the other two, the triangle is right-angled and the first side is the hypotenuse.

Using modern notation for the sake of brevity, let us assume that between the sides *a*, *b*, *c* of a triangle there exists the relation: $a^2 = b^2 + c^2$. Constructing the right-angled triangle of sides *b* and *c*, and calling a_1 its hypotenuse, we have by the Pythagorean theorem: $a_1^2 = b^2 + c^2$, and supposing for example $a > a_1$, we have by subtraction

$$a^2 - a_1^2 = (b^2 + c^2) - (b^2 + c^2)$$

and thus:

$$(a + a_1)(a - a_1) = 0$$

This can only be the case if $a = a_1$; but then the two triangles are equal, and thus the given triangle is rectangular, as was to be proved.

7. Two other important theorems that are deduced im- mediately are the two so-called Euclid theorems.

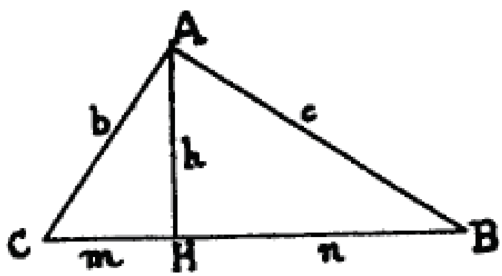

Fig. 16

THEOREM: *The square constructed above the height of a right-angled triangle is equal to the rectangle whose sides are the projections of the cathexes above the hypotenuse.*

Let AH (Fig. 16) be the height of the right-angled triangle ABC. And let *m*, *n* be the projections CH, HB of the two cathexes. Indicating for convenience, rectangles and squares with modern notions (but without introducing with this the concepts of proportion and measurement), from the right-angled triangle ABC we have:

$$m^2 + h^2 = b^{(2)}$$

and therefore:

$$m^2 + h^2 + c^2 = b^{(2} + c^2$$

On the other

$$a\, m = +\, n$$

hand then:

$$m^2 + n^{(2)} + 2\, mn = a^{(2)}$$

but

$$b^2 + c^2 = a^{(2)}$$

so also:

$$m^2 + h^{(2)} + c^{(2)} = m^{(2)} + n^{(2)} + 2\, mn$$

and for the second common notion: [α]

$$h^2 + c^{(2)} = n^{(2)} + 2\, mn$$

ma

$$c^2 = h^{(2)} + n^2$$

and thus:

$$h^2 + c^2 = 2\, h^2 + n^2$$

$$2\, h^2 + n^{(2)}) = n^2 + 2\, mn\ ; \qquad 2\, h^2 = 2\, mn$$

e

[β] $h^{(2)}\ mn=$

Having proved this theorem, we observe that the second member of [α] is the sum of two rectangles with the same height *n* and bases *n* and *2m*; it is therefore equal to the rectangle with base *n* + *2m*, and height *n,* i.e:

$$n^2 + 2\,mn = n\,(n + 2\,m) = h^2 + c^2$$

or also: and

$$n\,(n + m) + nm = h^{(2)} + c^2$$

for [β] i.e.

$$n\,(n + m) = c^2$$

$$na = c^{(2)}$$

Thus, the theorem arises:

THEOREM: *The square constructed over a cathetus of a right-angled triangle is equal to the rectangle whose sides are the hypotenuse and the projection of the cathetus over the i- potenuse.*

This is the so-called first theorem of Euclid. We can see that Proclus testifies that the theorem does not belong to Euclid and that only the demonstration found in the text of the *Elements* (Book I, 47) belongs to Euclid. In Euclid, the demonstration is based on the postulate of the parallels; from it we then obtain the theorem of Pythagoras, and from the two the other theorem, so called Euclid's theorem.

From this theorem, the following corollary immediately follows.

COROLLARY*: If two right triangles are equiangular to each other and one cathetus of one of them is equal to the i-*

potenuse of the other, the square constructed on the cathetus of the first is equal to the rectangle whose sides are the hypotenuse of the first and the homologous cathetus of the second.

Let (fig. 17) the right-angled triangles ABC, A'B'C be

ˆ∈ Ĉ and AC = B'C'= *b*.

This results in the lowering of the height AH of the first triangulum,

$$b^2 = (AC)^2 - BC - HC = ab'$$

c. d.

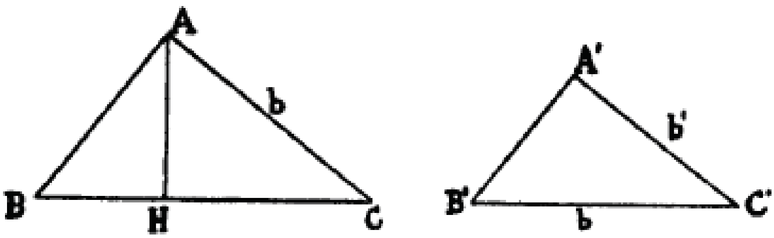

Fig. 17

We will make use of this corollary later.

Among the consequences of Pythagoras' theorem, the discovery of incommensurable quantities, which arises from the application of the theorem to an isosceles right-angled triangle, is of the greatest importance. But this is not our subject, nor are we concerned with the methods attributed to Pythagoras for the formation of right-angled triangles with whole numbers as measures of the sides [45].

8. From the study of rectangles we must now move on to that of quadrilaterals and polygons in general. From

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45 P. TANNERY, *La Géom. gr.*, p. 48.

isosceles right-angled triangle and from the isosceles right-angled triangle we obtained square and rectangle and their properties. Similarly, from the isosceles triangle and the scalene triangle, we obtain the rhombus and the rhomboid.

Rhombus, according to the definition found in Eucli- de, is the equilateral but not rectangular quadrilateral (in which case it is called a square).

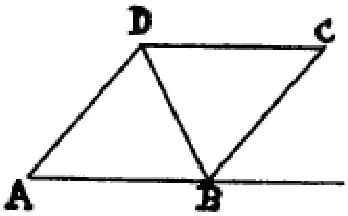

Fig. 18

Let ABD (fig. 18) be a non-rectangular isosceles triangle, and from vertex B of the base BD, let BC be the half-axis on the opposite side of A to BD, forming

with BD an angle D $\widehat{BC}$ A $=$ $\widehat{BD}$, and we take =

BA. Since $\widehat{ABD}$ is acute, it will be $\widehat{ABC}$ convex; and

so C and D are on the same side with respect to AB,

while C and A are on opposite sides with respect to BD. Let us join C with D: the two triangles ABD, CBD will be equal by the first criterion, and therefore the four sides of the quadrilateral ABCD are equal. It is therefore a rhombus.

The angles $\hat{A}$ $\hat{C}$ are equal, and are immediately

recognisable

that also $\widehat{ADC} = \widehat{ABC}$; the diagonal BD bisects the

corners of the rhombus; the axis of BD passes through A and C;

therefore the other diagonal also bisects the angles, is perpendicular to the first and their point of intersection is their midpoint.

On the other hand, if the quadrilateral ABCD is a rhombus, i.e. if AB = BC = CD = DA (assuming the vertices are ordered), we observe first of all that the vertices B and C cannot be on opposite sides of AD. In fact, if this happens, vertex C cannot be on the same side of A with respect to BD, because the two isosceles triangles ABD, CBD with the base in common and equal by the 3rd criterion would coincide and C would coincide with A. But neither can it happen that the vertex C is on the opposite side of A with respect to BD and of B with respect to AD, because the axis of the common base BD of the two isosceles triangles must pass through A, through C and through the middle point of BD, and therefore the ray AC is all with respect to AD on the side of B. Therefore, if a quadrilateral has four equal sides, two consecutive vertices are situated on the same side of the conjunction of the other two vertices. Since A and C are on opposite sides of BD, this diagonal divides the rhombus into two equal isosceles triangles.

half saw the two angles $\hat{B}$ and $\hat{D}$ of the rhombus; the other diagonal AC is merely the axis of BD; the two diagonals therefore cut internally, at their midpoint, are perpendicular to each other, and bisect the angles of the rhombus.

9. The definition of a rhomboid given in Euclid's *Elements* is as follows: Rhomboid is the quadrilateral that has

its sides and opposite angles equal to each other, but it is neither equilateral (i.e. a rhombus) nor heteromechoic (i.e. a rectangle). Euclid then calls all other quadrilaterals trapezoids.

Immediately afterwards, in Euclid, the definition of parallel lines appears, but the definition of parallelogram is completely missing, both among the definitions and in the text; this lack is also noticeable because we know from Proclus that the locution parallelogram is an invention of Euclid (46). We have already observed that the Euclidean definition of parallel lines, which is the 23ª and last, as the postulate of the parallels is the last in the list of postulates, does not agree too well with the definitions 2ª, 3ª and 4ª for which the line is always finite; now we find that the definition of quadrilaterals precedes and abstracts from the concept of parallel lines and that the definition of parallelograms is missing. One is under the impression that the list of definitions that has come down to us along with Euclid's text is the oldest or oldest, and that the classification of quadrilaterals contained therein is the oldest classification, with the 23ª and last definition attached at the end, just as the postulate of paraleleles is attached at the end of list of other postulates. This classification of quadrilaterals is more keeping with a geometry such as the one we are reconstructing than with Euclidean geometry based on the fifth postulate;

46 PROCLO, ed. Teubner, 354. II-15. Cf. ALLMAN *G*. J., *Greek Geometry*, p. 114.

and is explained by the fact that the four quadrilaterals: quadrature, rectangle, rhombus and rhomboid are obtained by operating absolutely identically over the isosceles right triangle, the any right triangle, the isosceles trian- gle and, as we shall see, the scalene (non-rectangle) triangle.

The rhomboid, in fact, is also obtained by this pro- cess. Let, in fact (fig. 19), ABD be a triangle qua- lore. Take the ray BC from the opposite side to A with respect to to BD and forming the angle $\widehat{DBC} = \widehat{ADB}$, and taken on it BC= AD, join C with D. It will be $\widehat{ABC} = \widehat{ABD} + \widehat{ADB}$ and thus less than two right angles; BC therefore lies with D on the same side with respect to AB. The triangles DBC and ABD are equal for the 1st criterion; hence CD= AB, $\widehat{CDB} = \widehat{ABD}$; e, since BD divides the angle $\widehat{ABC}$ and therefore also $\widehat{ADC}$, you also have $\widehat{ABC} = \widehat{ADC}$.

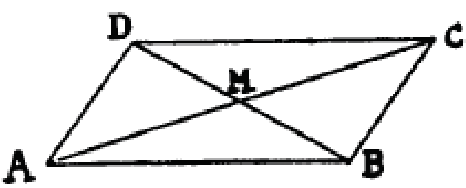

Fig. 19

We have therefore constructed a quadrilateral ABCD with equal opposite sides and equal opposite angles, i.e. a rhomboid.

Joined now the midpoint M of BD with A and with C, the triangles ADM, CBM are equal by the 1st criterion;

s $\hat{O}$ $\widehat{DMA} = \widehat{CMB}$ and therefore the three points A, M, C are aligned; MA = MC. The diagonals of the rhomboid are ta-

They are therefore one half. Each of the two diagonals dividing the rhomboid into two equal triangles, the sum of the angles of the rhomboid is consequently equal to four right angles (which is also true of the rhombus), and since the opposite angles are equal, the consecutive angles are supplementary.

Conversely, if we exclude intertwined polygons and non-convex polygons from our considerations, we show that if a quadrilateral ABCD has equal opposite sides, it is a rhomboid. With this hypothesis, the angles of the quadrilateral must all be convex; if it were in fact

$\widehat{DAB}$ a concave angle the vertex C should be

with respect to BD on the same side as A and being outside the triangle BDA and so should be A outside triangle BCD, because, if it were e.g. A inside triangle DCB, it would be, as can be shown, the sum of AD and AB less than the sum of CD and CB, whereas with the hypothesis made the two sums must be equal. But if A outside BCD, and C is outside ABD, and A and C are on the same side of BD the quadrilateral ABCD is intertwined. It follows that the quadrilateral ABCD has convex angles.

Since DAB is convex, vertex C is on the opposite side of A with respect to BD, because if it were on the same side, the quadrilateral would be intertwined or would have con-

hollow the angle $\hat{C}$. The quadrilateral ABCD, then, is diviso by the diagonal BD into two triangles that are equal by the 3rd criterion, and the opposite angles are equal; therefore, having opposite sides and equal opposite angles, it is a rhomboid.

It is also shown that if a convex quadrilateral has equal opposite angles, it is a rhomboid. Also in this case A and C cannot be on the same side

with respect to BD, because angles $\hat{A}$ and $\hat{C}$ are equal.

vertex C cannot be inside the triangle DAB, nor can the

vertex A inside triangle DCB, and because if A outside DCB and C is outside DBA, and A and C are on the same side of BD, the quadrilateral ABCD is intertwined against the hypothesis. Therefore, since A and C are on opposite sides of BD, BD divides the quadrilateral into two triangles, and therefore the sum of the four angles of the quadrilateral is equal four right angles.

Since the pairs of opposite angles are equal, there will be al-

=lora $C\widehat{DA}D + \widehat{AB} =$ two ret-

you minus the sum of $\widehat{BDA}$ and $\widehat{DAB}$. But by the theorem-

but of the two recti- this sum has as its supplement the an-

golo $\widehat{ABD}$, and therefore C $\widehat{DB} = \widehat{ADB}$. Similarly

$\widehat{DBC}$ A$= \widehat{DB}$, and thus the two triangles ABD, DBC

are equal by the second criterion, and is AB= DC and AD

= BC, and the quadrilateral ABCD is a rhomboid.

It is then easily seen, going back to the first case, that if a quadrilateral has diagonals that cut in half, it is a rhomboid[47].

10. We have thus seen, without even mentioning parallel lines, how they can be defined as square, tangle, rhombus and rhomboid, and recognise their characteristic properties.

It can be easily demonstrated that the point at which the diagonals meet in the rhomboid (and therefore also in the other three quadrilaterals) is a centre of the figure, and that the perpendiculars conducted from it to the opposite sides are perpendicular. By rotating the figure about this point, in the case of the square, one side is successively brought above the others, and each vertex is brought above the next, and the figure is superimposed on itself with each rotation of a right angle; in the case of the rhomboid, the line of one side is superimposed on the other.

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47 We are not unaware that to satisfy the modern demand for generalisation we should have treated the general case of rhomboids at once and then deduced the properties in the particular cases of the rhombus, the rectangle and the square. But our aim is not to make a new geometry, on the contrary, it is to restore the ancient Pythagorean geometry, as it really and probably was; and we believe that in order to achieve this it is necessary, if not necessary, to restore a Pythagorean, pre-Euclidean mentality, without any excessive deference to modern habits and requirements. The order to which we have adhered is that of the classification of quadrilaterals in the "Euclid definitions", and we are convinced that this order corresponds to the chronological order of discovery and to the expository order of the Pythagoreans' treatment of quadrilaterals.

placed successively to the line of the other sides, and in the case of the rectangle and the rhomboid this only happens for the half-turn.

The rhombus thus enjoys the same property as any triangle when it rotates about the meeting point of the three bisectors, and the square behaves like the equilateral triangle overlapping itself four times in a complete circle like that three times.

If we make these considerations, it is because the very name of the rhombus, and therefore also that of the rhomboid, seems to us to be linked to them. In Greek, in fact, say the dictionaries, ῥόμβος (from ῥέμβω) designates any body of circular figure or moving round. In ancient times, it was the name of the spindle, and in the operation of the spindle, the woven rows took the shape of the rhombus. The name 'rhombus' was then given to the bronze rhombus mentioned in the mysteries of Rhea, the Phrygian mother among the Greeks, and a scoliast in the Argo- nautica of Apollonius says that the rhombus is a spinning reel spun by beating it with strips of lat-th.[(48)] In one of his fragments, the Pythagorean Archita speaks of these 'magic rhombuses that are spun in the mysteries'.[49]

48 Apollonius, *Argonautics*, L. I, v. 1139. In HOMER (*Iliad*, XIV, 413) they are also called στρόμβοι. Proclus (ed. Teubner, 171. 25) also says that 'it seems that the name also came to the rhombus from movement'.

49 The Mieli reporting the Greek text of Archita translates ῥόμβοι into *drums* (MIELI - *Le scuole jonica, pythagorica*... p. 349) and the CHAIGNET (vol. I, p. 281) translates: *les toupies magi-*

Thus, the classification of quadrilaterals found in Euclid's *Elements* is not only independent of the concept of parallels, and has all the air of being pre-Euclidean, but in its terminology it seems to be related to the postulate of Pythagorean rotation and the properties triangles that refer to it.

11. The property found for the equilateral triangle and the square subsists for every convex equilateral and equiangular polygon inscribed in a circumference. Assuming the angle of the circle, or a circumference, to be divided into *n* equal parts, and taking *n* equal segments from the centre over the radii, and joining their extremities consecutively, one obtains a regular polygon, decomposed into *n* equal isosceles triangles of equal height (apo- tema of the polygon). By rotating the figure around the

centre of a $\frac{1}{n}$ of angle turn the polygon overlaps-

itself; and thus in one complete turn it overlaps *n* times on itself. For the postulate of rotation

the external angle is $\frac{1}{n}$ of four straight, and that in-

its supplement. As *n* increases, the inter angle increases and its value can be calculated for $n = 5, 6, ...$

---

ques.

We are now able to deal with the Pythagorean discovery of regular polygons congruent around a vertex that fill the plane.

The polygons must be at least three in number, and the angle of the polygon must be contained exactly in the one circle. This occurs with the equilateral triangle whose angle is the sixth part of four right angles; with the quadrangle whose angle is the fourth part of four right angles; it does not occur with the regular pentagon; it occurs with the hexagon whose angle is one-third of a revolution; and it cannot occur with other regular polygons because if the number of sides exceeds six, the interior angle exceeds one-third of a revolution.

This discovery is therefore a consequence of the two-rights theorem, i.e. it results from a demonstration, as Proclus told us, and is by no means an empirical fact that served to deduce the two-rights theorem as Tannery and Allman would have it, despite Proclus' explicit statement that he makes a pita- gorical theorem out of the property of regular polygons congruent around a vertex.

CHAPTER III
THE PENTALFA

1. The division of the circumference into 2, 3, 4, 6, 8, ... equal parts and the relative problem of inscribing in it the regular polygons of 3, 4, 6, 8, ... sides did not present any difficulty for the Pythagoreans; it is only necessary to observe that from the union of six congruent triangles around a common vertex we obtain the regular hexagon whose side is equal to the radius of the circumscribed circle.

More difficult, however, is the problem of dividing the circumference into 5, 10 equal parts and the inclusion in it of the regular pentagon and decagon; a problem that must have aroused special interest in the Pythagoreans because the arc subtended by the side of the decagon was in the whole circumference as the unit in the decade. They certainly solved this problem, because otherwise they could not have constructed the regular icosahedron and dodecahedron as we know they did. Let's see how they could have done it, again without taking into account the theory of parallels, similarity, proportions and the two postulates of Euclid and Archimedes.

mede.

2. The problem simple application, which Euclid solves after proving the theorem of complementary paralla- lograms (*parapleromes*), can also be solved in a particular case without admitting the parallel postulate. The problem can be stated as follows: To construct a rectangle with a given base and equal to an assigned rectangle or square; a problem that corresponds to the determination of the solution of the first-degree equation:

$$ax\ bc=$$

or:

$$ax= {}^{\mathrm{b}(2)}$$

If $a > b$ or $a > c$, the problem can also be solved in our geometry. Let (fig. 20), for example, $a > b$ and let HBCK be the rectangle given with HB = b and BC = c. Taking BH from B and the segment BA = a from the side of H, we complete the rectangle ABCD. Since H is between A and B, these points remain on opposite sides of HK, and so do the points C and D; therefore HK cuts the diagonal AC at an interior point P. Finally, let us con- duct for P the MN perpendicular to AD, HK, BC. Due to the equality of the pairs of triangles ABC, ADC; PNC, PKC; AHP and AMP, we subtract that the rectangle HBNP is equal (in extension) to the rectangle MPKD, and adding to both the rectangle PNCK, we have that the rectangle MNCD is equal to the given rectangle HBCK. The segment CN is therefore the inco- gnite x of the equation.

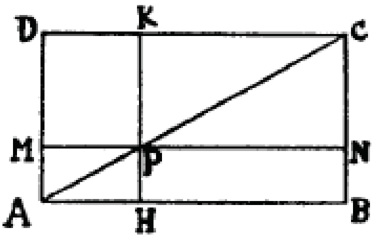

Fig. 20

If, on the other hand, *a* is smaller than both *b* and *c*, i.e. if H is outside the segment BA, we no longer have the certainty that the prolonged AC meets at a point P the prolongation of the side HK. This certainty can only be obtained by the proposition that constitutes Euclid's postulate.

Now it is worth noting here that Proclus in his commentary on Euclid I, 43 (theorem *of the gnomon*) says that the three problems of application are discoveries due to the Pythagorean muse according to οἶ περὶ τὸν Εὔδημον, and does not say as in all other cases that what he says is based on the authority of Eudemus. The testimony is not this time the personal testimony of Eudemus, and to this indeterminacy in the testimony corresponds the fact that the ancient Pythagoreans, without the theory of parallels, could only solve the problem in the case now seen.

It is, after all, what interests us, because it allows us to solve the questions that will arise later.

In order to solve the other two application problems after the simple application problem (parabola), the following theorem and its inverse must be stated:

THEOREM: *The midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices, and in- verse if in a triangle the midpoint of a side is equidistant from the three vertices, it is right-angled.*

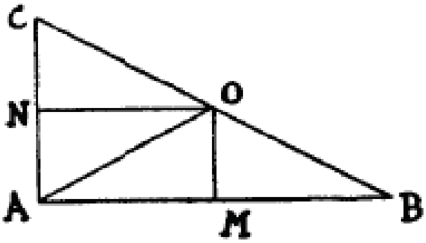

Fig. 21

Let ABC be the right-angled triangle (fig. 21), and A the vertex of the right angle. Let us draw the ray that forms an angle with AB on the side of A from C to AB.

equal to the (acute) angle $\widehat{ABC}$. It is internal to the an-

rectangle $\widehat{CAB}$, then saw the hypotenuse BC into a point.

to inner O, forming two isosceles triangles OAB, OAC (the second has the angles at the base complementary to equal angles); therefore O, the midpoint of the hypotenuse, is equidistant from the three vertices.

Conversely, if in triangle ABC is O the midpoint

of BC and is OA= OB= OC, results in $\widehat{OAC} = \widehat{OCA}$;

$\widehat{OAB}$ O= $\widehat{BA}$, , since by the two-rights theorem the sum of these four angles is equal to two right angles is

will have: $O \widehat{AC} O + \widehat{AB} = a$ rectum.

We note that the two heights of isosceles triangles divide them into equal right-angled triangles and we have:

$$OM = \frac{1}{2} AC; \quad ON = \frac{1}{2} AB$$

3. Let us move on to the other two problems of the application.

The problem of the default application (*ellipsis*) can be stated as follows: Construct a rectangle of given area b^2 such that the difference between the rectangle of equal height and assigned base and it is a square. More modern and more clearly: Construct a rectangle of given area b^2, knowing the sum of the sides a.

That is, it is a matter of solving the second-order equation:

$$x\,(a - x) = b^2$$

Let ABCD (fig. 22) be the square of side AB = b. Having taken on AB from the side of A the point O such that DO is equal to half of a, determine on AB the points E and F such that OE = OD = OF; by the preceding theorem triangle EDF is rectangular, and therefore the square constructed on the height AD is equal to the rectangle with sides AF, AE. Having constructed the rectangle EKGF, with EK = AE, if we remove from it the rectangle AHGF, i.e. the square ABCD, the difference AEKH is precisely a square. The rectangle AHGF thus solves the problem, and is EA the

x of the given equation. For the problem to admit a real solution, it must be $a>2b$.

The problem of the excess application (*hyperbola*) can be stated as follows: construct a rectangle of given area b^2 and such that the difference between it and the rectangle of equal height and assigned base a is a square. The pro- blem is equivalent to constructing a rectangle by knowing its area and the difference of its sides, i.e. it corresponds to the solution of the equation:

$$x\,(a+x)=b^2$$

and always admits a solution.

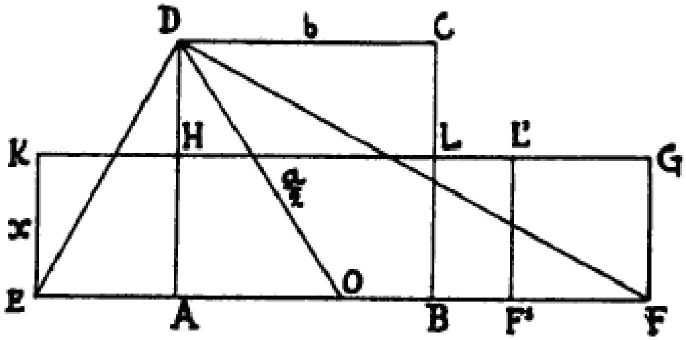

Fig. 22

Let ABCD (Fig. 22) be the square of side b, and take the segment AF'$=a$ from the side of B on AB. Let O be the midpoint of AF'; and take the segments OE= OD= OF on AB. The triangle EDF is right-angled, and the square of the height ABCD is equal to the rectangle whose sides are the projections EA = EK, and AF = EF' of the sides.

If we remove from this rectangle the rectangle AHL'F' of equal height and assigned base AF'= *a*, we obtain ap- point a square EKHA. The rectangle EKL'F' thus solves the problem, and EA is the *x* of the equation.

4. PROBLEM: *Determine the golden part of a* segment; that is, divide a segment so that the square having the largest part (golden part) on its side is equal to the rectangle having the entire segment and the remaining part on its sides.

This problem is a special case of the excess application problem; namely the case where $a = b$.

We construct (fig. 23) the square ABCD on the assigned segment AD.

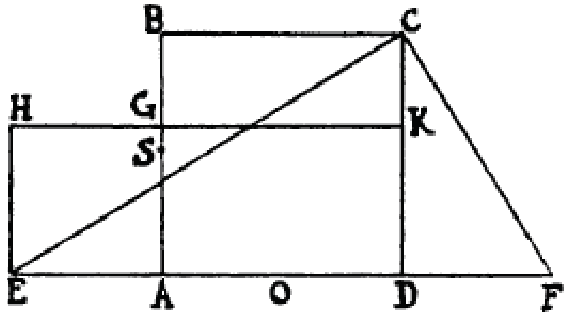

Fig. 23

Let O be the midpoint of AD, and take the segments OE = OF = OC on AD. The triangle ECF is right-angled, so the square whose side is CD is equal to the rectangle EHKD whose sides are DK = DF and ED.

Since OC and therefore OF is less than OD + DC, DF and therefore DK is less than DC; the height of the rectangle EHDK is therefore less than the side AB of the given square while the base ED is evidently greater; Therefore HK divides the square into two parts, and removing from the rectangle EHKD and the square ABCD the common part AGDK we have that the square EHGA is equal to the rectangle BGKC, which has for sides the assigned segment BC and the segment BG, which is what remains of the side AB = BC when AG is removed, that is, the side of the square EHGA. The point G therefore divides the segment AB in the required manner, i.e. AG = EA is the golden part of AB.

The figure shows that AD is the golden part of ED, while the remaining part EA is the golden part of the golden part AD; similarly BG is the golden part of AG etc.

The uniqueness of the golden part of a segment is proved by absurdity. Let AS < AG be another solution; that is, with modern notation: let it be:

$$(AS)^2 = AB \cdot BS$$

For the hypothesis made one has:

$$AG = AS + SG \qquad \text{e} \qquad BG = BS - SG$$

and

$$(AS)^2 + (SG)(2) + 2AS \cdot SG = (AG)(^2 = AB \cdot$$

therefore

$$BG = AB \cdot BS - AB \cdot SG$$

ma

and thus

$$(AS)^2 + (SG)^2 + 2AS \cdot SG = AB \cdot BS - AB \cdot SG$$

e

$$(AS)^2 + (SG)^2 + 2AS \cdot SG + AB \cdot SG = AB \cdot BS$$

from which, removing the first

$$(SG)^2 + 2AS \cdot SG + AB \cdot SG = 0$$

or

$$SG\,(SG + 2AS + AB)\ 0 =$$

This rectangle should be null; and this can only happen if SG = 0, i.e. if S coincides with G.

5 THEOREM: *The base of an isosceles triangle with the angle at the vertex equal to the fifth part two sides is the golden side.*

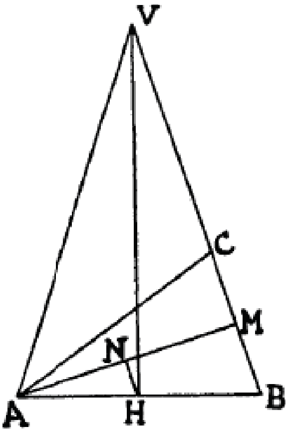

Fig. 24

An isosceles triangle VAB (fig. 24), whose angle at the vertex is 36° and whose angle at the base is 72°, is divided by the bisector of one of the angles at the base into two isosceles triangles CAV, ACB, and the three segments VC, AC, AB are equal. Triangle VAB and triangle ACB are also equiangular to each other.

By lowering the heights VH and AM, and leading from H the height HN of the isosceles triangle AHM, we have

$$NH = \frac{1}{2} BM - \frac{1}{4} BC$$

The right-angled triangles VAH, AHN have equal angles, and the cathetus AH of the first is the hypotenuse of the second; therefore a corollary of the previous chapter gives:

rect. (VA, NH)= quad. (AH)

and thus:

4 rect. (VA, NH)= 4 quad. (AH) rect.

(VA, 4 NH) = quad. (AB)

rect. (VA, BC)= quad. (VC)

Therefore VC, i.e. AB is the golden part of VB; c.d.

By absurdity, the inverse theorem is proved: *If an isosceles triangle has a base that is the golden part of the side, it has the angle at the vertex equal to the fifth part of two right angles.*

Let V'A'B' be the given triangle and the base A'B' the golden part of the side V'A'. If we construct the isosceles triangle VAB with VA = VB = V'A' and the angle at the vertex one fifth of two right angles, then by the preceding theorem AB will be the golden part of VA, i.e. of V'A'; and by the uniqueness of the golden part it will be AB = A'B' and therefore the two triangles equal c.d.[50]

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50 LORIA (*Exact sciences*, p. 41) attributes the construction of the isosceles triangle with the angle at the vertex half that of the base to Pythagoras, bringing it back to the construction of the golden part; but to demonstrate that the base is the golden part of the side, he resorts to the similarity of the triangles VAB, ABC (fig. 24), and it seems that in-

6. To construct an isosceles triangle with the angle at the vertex half that at the base, i.e. to construct an angle equal to one-fifth of two right angles or to one-tenth of the angle of the circle, it is sufficient to take any segment as the side and its base as the golden part. By making this triangle make 10 rotations around the vertex equal to the angle at the vertex, the plane around the vertex is filled and a regular decagon is obtained.

Conversely, if a circumference is divided into 10 equal parts, the side of the inscribed regular decagon is the golden part of the radius. We are therefore able to solve the

PROBLEM: *Divide a circumference into ten equal parts.*

Let us join (fig. 25) the midpoint C of the radius OA with the extremity B of the radius perpendicular to OA, and take from the side of A the segment CD on OA equal to CB; AD is the golden part of the radius. Since AD is less than OA, the circumference of centre A and radius AD bisects E, P the circumference of centre O and radius OA. This happens, of course, by tacitly admitting (as Euclid did again, two centuries after Pythagoras) the postulate of continuity in a particular case, that is, by admitting that if a circle has its centre A above a circumference of centre O and passes through a point D

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tends to mean that this path was also taken by Pythagoras. The development we have shown, on the other hand, starts from the Pythagorean theorem and only uses the consequences of this theorem, in particular the corollary on page 53, and the application problems that we know were solved by the Pythagoreans.

outside and one inside this circumference, the two circum- ferences cut each other off. This property, so axiomatic that Euclid did not feel the need postulate it, must have been a fact, a primordial truth for the Pythagoreans.

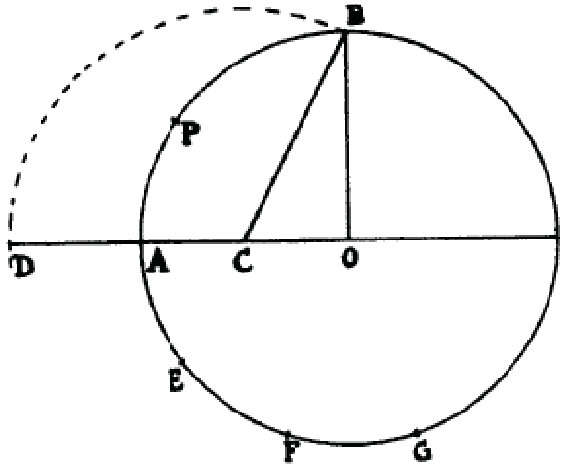

Fig. 25

The arcs AE, AP are therefore a tenth of the entire circumference. By successively centring at E and P etc. with the same radius, the other points of division are determined, two diametrically opposed, 10 being an even number. By successively bringing them together, obtain the inscribed regular decagon; by joining the first with the third, the third with the fifth, etc., we obtain the inscribed regular pentagon. It can be seen, therefore, that starting from the Pythagorean theorem and using the simple procedures outlined above, the Pythagoreans were able to divide the circumference into 5 and 10 equal parts, and to inscribe in it the regular decagon and pentagon. The starry pentagon or pentalpha (or pentagram) is also obtained im-mediately by conducting the five diagonals of the pentagon; and since the pentalpha was the symbol of the Pythagorean association, the discovery of the division of the circumference into 10 and 5 equal parts and the construction of the regular decagon, the regular pentagon and the pentalpha are undoubtedly attributed to Pythagoras.

7. The reasons why the pentalfa was chosen as a symbol by our School are not all geometric in nature. This is natural, given the connection between geometry, the other sciences and Pythagorean cosmology. But the geometric properties that link the circle, the sides of the pentagon and the inscribed regular decagon, and those of the pentalfa and the starry decagon or decalpha, are so many and so simple and beautiful that they have undoubtedly aroused the admiration of the Pythagoreans and have helped to determine or justify the choice of the pentalfa as the symbol of the School and as a sign of recognition among the Order's members.

Let us look at some of them neatly.

By successively joining (fig. 26) the points view A, B, C,... of the circumference into 10 equal parts, we have the regular decagon ABCDEFGHIL, whose side we denote by $l_{(10)}$. This is the golden part of the radius.

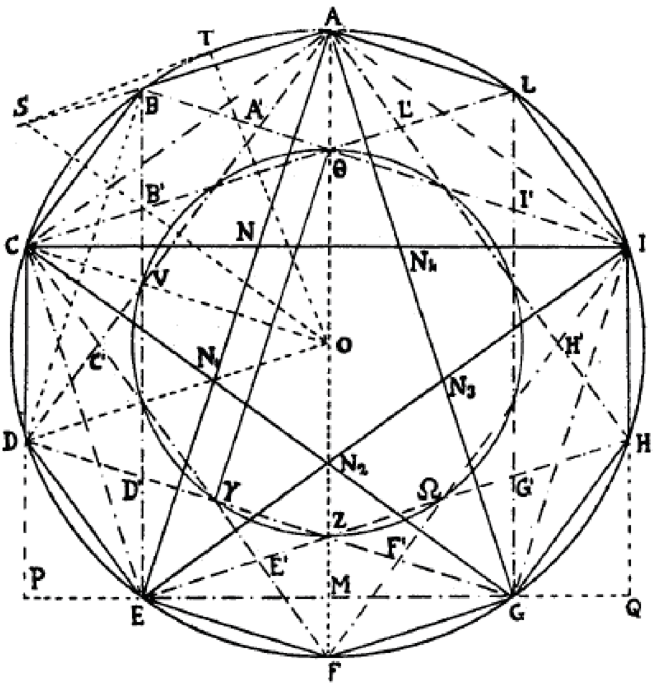

Fig. 26

If A is joined to C, C to E etc., we have the regular penta- gon ACEGI, whose side we will denote AC by l_5; if A is joined to D, D to G etc., we have the starry decagon ADGLCFIBEH or AA'BB'CC'... LL' or decalpha whose side we denote by s_{10}; joining A with E, E with I etc., we have the pentalfa AEICG or $ANCN_1EN_2GN_3IN_4$ whose side we denote by s_5.

Joining A with F gives the diameter, and drawing from A the chords AG, AH.... of the sixfold etc. arcs of arc AB, the re- gular polygons already obtained are obtained in reverse order. The inscribed regular and star polygons in the circumference, and which are obtained by dividing it into 10 equal parts, are four and only four.

The pentalpha evidently owes its name to the five α (A of the Greek alphabet) like the one formed by the strokes AE, AG, NN_4 in the figure. The name is used by P. Kircher in his *Arithmetica* (1665)[51]; but we are aware that this is the original Pythagorean name, and that similarly decalpha is the original name for the starry decagon.

We have already seen that placing the arc AB ten times successively on the circumference exhausts the circumference, just as the sum of ten units exhausts the whole decade. And just as the elements of geometry: the point, the line (straight line or segment determined by two points), the surface (plane, triangle determined by three points), the volume (tetrahedron, determined by four points) fill and exhaust the (three-dimensional) space, correspondingly, the sum of the first four whole numbers gives the decade, the fundamental Pythagorean relation that from unity through the sacred *tetractis* leads to the decade. The same, of course, happens in our figure where the arc AB added with its double BD, with the triple DG and with the quadruple GA gives the sum of the entire circumference.

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51 Cf. G. Loria, *Storia delle Matematiche*, vol. I, p. 66.

The quadrilateral ABDG, which has for sides $l_{(10)}$, $l_{(5)}$, $s_{(10)}$, s_5 and for diagonals AD = s_{10} and BG = $2r$, is divided by the diagonal BG into two right-angled triangles, and thus we have

[1] $l_{10}^2 + s_5^2 = 4r^2$

[2] $l_5^2 + s_{10}^2 = 4r^2$

from which

[3] $l_5^2 + l_{10}^2 + s_5^2 + s_{10}^2 = 8r^2$

relationship linking the radius of the circle and the sides of the four polygons, which is stated with the

THEOREM: *The sum of the squares constructed on the side of the regular decagon, regular pentagon, pentalfa and decalfa inscribed in a circle is equal to eight times the square constructed on the radius.*

It is easily recognised that the AOF diameter is perpendicular to the EG side of the pentagon and the CI side of the pentalfa, and the angle $\widehat{EOF}$ being 36° and the trian- gle EOA being isosceles, the angle $\widehat{EAF}$ is 18° and thus $\widehat{EAG}$ is 36°. This results in the

THEOREM: *The sum of the five angles of the pentagon is equal to two right angles*, which is easily proved to be true for any intertwined pentagon.

The isosceles triangles AEG, ANN_4 having the angle at the vertex of 36° have the base the golden part of the side. Thus the *side of the inscribed regular pentagon is the golden part of the side of the pentalfa*; and NN_1 is the golden part of AN.

Since $\widehat{DOF}$ is 72° $\widehat{DAO}$ is 36°; similarly, $\widehat{CAO}$ is 54° and $\widehat{BAO}$ is 72°; i.e. that the perpendicular for A to the diameter AF e the conjunctions A with the other points of division into 10 equal parts of the circumference divide the flat angle about A into 10 equal parts; and similarly for the other vertices. It follows that AN = NC = CN_1 = N_1E etc.

The triangle ECN has two equal angles at the base CN of 72° and is isosceles; therefore EN is equal to the side l_5 of the pentagon, the quadrilateral NEGI is a rhombus, and the diagonals of the regular pentagon, i.e. the sides of the pentalfa, are divided into corresponding equal parts, the largest of which is equal to the side of the pentagon. On the side AE of the pentalfa, NE = EG = l_5 is the golden part of AE, then N_1E = AN is the golden part of EN; and NN_1 the golden part of AN. Obviously $NN_1N_2N_3N_4$ is a regular pentagon.

Finally, we note that the apothem of the regular pentagon is half the side of the decalpha, as obtained from the ACF rectangular triangle. Other properties we shall have occasion to recognise later.

8. We must now establish another important relationship that arises in the construction of the icosahedron, and which the Pythagoreans must therefore have known.

Assuming that every line passing through a point on a circumference is a secant, it is shown that the perpendicular to the radius at its extremity is the perpendicular to the circumference at that point. And since we know that the geometric locus of the vertices of right-angled triangles of a given hypotenuse is the circle whose diameter is the hypotenuse, we are also able to conduct the tangents to a circumference from an assigned point. We then take (Fig. 27) from a point P outside a circumference the tangent PN, the diameter PO and any secant PCD.

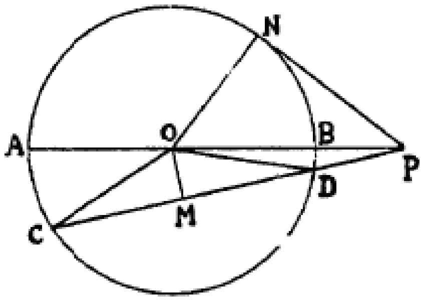

Fig. 27

The median of the isosceles triangle OCD is perpendicular to the base CD, and the rectangle with sides PD and PC, i.e. PM + CM and PM - CM is equal to the difference of the squares constructed on PM and MC. We have:

$$PC - PD = (PM + MC)\ (PM - MC) =$$
$$= (PM)^2 - (MC)^{(2} =$$
$$= (PM)^2 + (OM)^2 - [(OM)(^2 + (MC^{)(2)}] =$$
$$= (PO)^2 - (OC)^2 = (PO)(^2) - (ON)^{(2} =$$
$$= (PN)^2.$$

Let us then take in Figure 26 on AB from A the segment AS = OA: the isosceles triangles OAC, ASO, having the equal side and the equal angle at the vertex are equal, and therefore OS = AC = l_3; and since in these triangles the angle at the vertex exceeds that at the base, the base

OS is greater than the side OA and the point S is outside the circumference.

Taking the tangent ST from S, it will be by the theorem now proved:

$$(ST)^2 = SA - SB$$

and, since AB is the side of the regular decagon, it is the golden part of AS, i.e:

$$(AB)^2 = SA - SB$$

so

$$ST = AB = l_{10}$$

From the right triangle OST we then have:

$$(ST)2+ (OT)^{(2}=(OS)^2$$

i.e. the relationship:

[4] $$l_{10}^{2}+ r\ 2)= l_{5}^{2}$$

which is enunciated as follows:

THEOREM: *The side of the inscribed pentagon is the hypothetical side of a right-angled triangle, whose cathexes are the ray and the side of the inscribed regular decagon.*

9. In Figure 26, the segments OC and AD are cut at a point V, resulting in $\widehat{AVO}\ D= \widehat{CV}= 72°$.

From the isosceles triangles AVO, DCV with the angle at vertice of 36° we have VO = VD = DC = l_{10}, and AV = OA= r; thus VD is the golden part of AV i.e. of *r* and AV is the golden part AD. *The radius is therefore golden part of the side of the decalfa,* and we have the simple relationship:

[5] $$r+ l_{10}= {}_{s(10)}$$

From this relationship and the others obtained, we geometrically deduce the following, which we write for brevity with the usual notations:

$$s_{10}^2 + r^2 = s_{10}^2 + l_5^2 - l_{10}^2 = 4r^2 - l_{10}^2 = s_5^2$$

[6] $$s_{10}^2 + r^2 = s_5^2$$

and substituting in [1]

[7] $$s_{10}^2 + r^2 + l_{10}^2 = 4r^2 \quad \text{e} \quad s_{10}^2 + l_{10}^2 = 3r^2$$

and therefore from [3][52]

[8] $$s_5^2 + l_5^2 = 5r^2$$

One also has:

$$r^2 = (s_{10} - l_{10})^2 = s_{10}^2 + l_{10}^2 - 2 s_{10} l_{10}$$

so

[9] $$r^2 = 3r^2 - 2 s_{10} l_{10} \quad \text{e} \quad s_{10} l_{10} = r^2$$

$$(s_{10} + l_{10})^2 = s_{10}^2 + l_{10}^2 + 2 s_{10} l_{10} = 3r^2 + 2r^2 = 5r^2$$

and thus

[10] $$(s_{10} + l_{10})^2 = s_5^2 + l_5^2$$

Let us now take the right-angled triangle ABC (fig. 28) with cathexes AB = l_{10} and AC = r; the hypotenuse is BC = l_5, and taking the cathexes' extensions BD = r and CF = l_{10} we have AD= AF= s_{10}; CD= s_5. Taking AM= $s_{10}+ l_{10}$, and

52 The relationship $s_5^2 + l_5^2 = r^2$ is found (cf. LORIA, Exat-you, p. 271) in the 14th book of Euclid (which is by Ipsicles, 2nd century BC), and so is the other: $_5 = \frac{r + l_{10}}{2}$.

But this does not prove that they were unknown before him. In fact, Ipsicles also proves that the apothem of the equilateral triangle is half the radius, a property that was certainly known much earlier.

on the perpendicular to AM the segment ML= r also BL = s_5; and the triangle CBL is rectangular, because CL = AM = s_{10} + $_{l(10)}$.

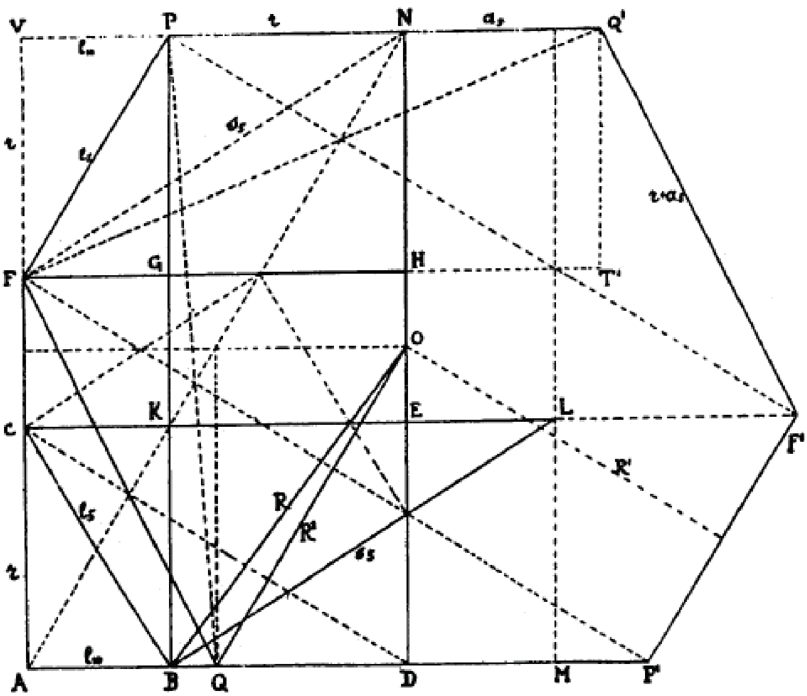

Fig.

The same five elements appear in this right-angled triangle as in formula [3]. Its cathexes are the sides of the inscribed regular pentagon and the side of the pentalpha, its height is the radius of the circumscribed circle, and the two projections of the cathexes on the hypotenuse are equal to the side of the inscribed regular decagon and the side of the decalfa, respectively.

golden projection of the greater cathetus. The magnus cathetus is the golden part of the major cathetus, and the sum of the squares constructed over the three sides is equal to *ten* times the square constructed over the height, that is, over the radius of the circumference circumscribed by those regular polygons.

Moreover, since the rectangles ABKC, BMLK are divided in half by the diagonals BC, BL, the right-angled triangle CBL is half of both the rectangle with sides CB and BL and the rectangle with sides CA and AM; there is thus a third relationship between those five elements:

[11] $l_{(5)} - s_5 = r (s_{10} + l_{10})$

denoting by a_5 the apothem of the pentagon and by $a_{(10)}$ the apothem of the decagon, we add the relations to the previous ones:

[12] $2 a_5 = s_{(10)\ r} = + l_{10}$

[13] $2 a_{10} = s_{(5)}$

We will see later the relationships that link these elements to the various elements of the regular dodecahedron.

10. The pentalfa was the symbol of the Pythagorean sodality. It was drawn, (fig. 29) with the point upwards by writing at the vertexes the letters that make up the word ὑγίεια, Latin *salus*, to be understood in the double sense that the word 'salute' has in Dante and in the 'Fedeli d'Amore', i.e. in the sense of that salvation or survival privi- legiata indicated at the end of the 'Golden Verses'.

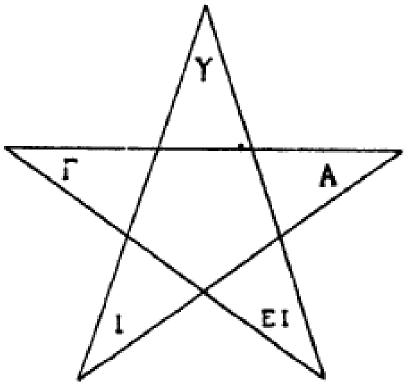

Fig. 29

This ancient Pythagorean symbol reappears here and there in the Western esoteric tradition, usually designated as 'the figure of Pythagoras'. Sometimes the letter G, the initial of Geometry, is written in the centre, as for example in the 'flaming star' of a well-known Western Order whose aim is to perfect mankind, i.e. literally, the *teleté* of the mysteries. But now is not the time to tell the story of its transmission to become the fa- tidic "flaming star" of Italy. We will only say, to close this chapter, that the pentalpha and the fascio littorio (between which there is more than one link) are the only truly important Western spiritual symbols. The rest, good or bad, comes from the East.

CHAPTER IV
REGULAR POLYHEDRA

I. To see how Pythagoras arrived at the construction of regular polyhedra and their inscription in the sphere it would be necessary to do for space what we have done, in part, for the plane. That is to say, to reconstruct the Pythagorean geometry of space without introducing the concepts of parallel lines, parallel lines and planes, and parallel planes, and to show how the results that Eudemus, through Proclus, hands down to us as having been achieved by Pythagoras can also be achieved. But in order not to lengthen our study too much, we will limit ourselves to indicating in brief the path to take, or one of the paths to follow, leaving aside the demonstrations that everyone can find for themselves.

Therefore, if we admit that a plane divides space into two , we also admit the half-space postulate: *The segment joining two points on opposite sides of a plane is cut at one point by the plane.* It may be that this special case of the continuity postulate was also tacitly admitted as a primordial truth. It is then proved in the ordinary way that:

a) A line not lying in a plane and having a point in common with it is divided by it into two semi-straight lines on opposite sides of that plane.

b) If two planes have a point in common, their intersection is a line passing through that point; any two planes are divided by the common intersection into two half-planes on opposite sides of each other.

c) If two perpendiculars *a* and *b are* drawn to a point H on a line *m* in different planes, every other line in the plane *ab* passing through H is perpendicular to *m*, and vice versa every perpendicular to *m* in the plane *ab* lies in the plane *ab*. The plane *ab* is said to be perpendicular to the line *m* in H; and the perpendicular line *m* to the plane *ab* in H.

d) For a point A, whether or not it belongs to a straight line, there is a plane and only one plane perpendicular to it.

e) *The three-normal theorem*: If a line *m* is perpendicular to a plane α and a line *a* perpendicular to a line *r* of α exits the plane from the foot H (whether or not it passes through the foot H), the third line *r* is perpendicular to the plane *am* of the first two.

f) Two intersecting planes divide space into four parts (dihedra). The definitions of convex, flat and concave dihedrons follow.

g) Let β (fig. 30) be a plane perpendicular to a line *a* and let H be its foot. Let *a* be any plane α, and let *r* be αβ; and let H be β

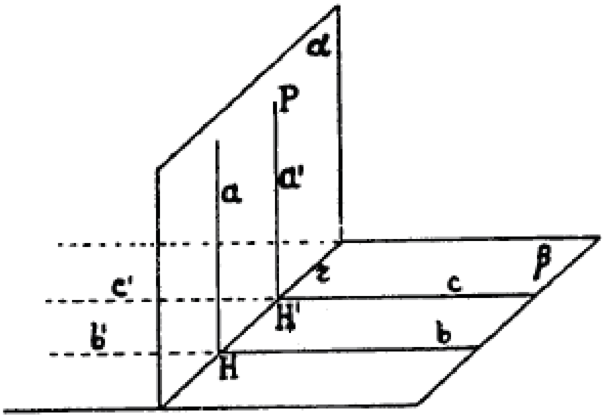

Fig. 30

bb' perpendicular to *r*. By the three-normal theorem, *b* is perpendicular to the plane α and thus to

a; the two angles $\widehat{bHa}$, $\widehat{aHb'}$ are right angles. By rotating the plane *ab* around H on itself, it remains perpendicular to *r*, and when the ray *b* goes on *a* and *a* on *b'*, the half-plane β

goes on the half-plane α and α on β'. The two dihedrâ βα and

$\hat{}$αβ' overlap, they are therefore equal; the se-

miplane α therefore bisects the flat dihedral $\widehat{\beta r \beta'}$.

Every other half-plane for *r* is internal to the one or al-

the other of the dihedralŝ αβ and $\widehat{\alpha\beta'}$; thus for a line *r*

of the β plane can be conducted one and only one α plane

bisecting the flat dihedral $\widehat{r\beta'}$. The plane α is said to be

perpendicular to the β plane; the angle $\widehat{aHb}$ is said to be normal section of $\widehat{\alpha\beta}$, and is straight.
If for a point P at α we conduct the perpendicular *a'* to *r* from the foot and the *c* at β perpendicular to *r*, the plane *a'c* is also perpendicular to *r*; by rotating around *r* the half-plane β goes to α and α to β', the half-plane *c* goes to *a'*, and *a'* to *c'*; thus $\widehat{ca} = \widehat{a'c}$ =a straight line, and therefore *a'* is also perpendicular to β and the normal section $\widehat{a'c}$ of the dihedral $\widehat{\alpha\beta}$ is equal to the other $\widehat{ab}$.

h) Perpendicular line to a plane for a point. Let H be a point in a plane β (fig. 30), and let H be any line *b* in β, and for H the plane α perpendicular to *b*; let *r* be the $\widehat{\alpha\beta}$. For H condu- in the plane α the perpendicular *a* to *r*; by the three-normal theorem, *a is* perpendicular to β. The uniqueness of the perpendicular to β for H is shown by absurdity.
If then the given point were P outside the plane β, with a straight line *b* whatever and for P the plane α perpendicular to *b*, it intersects *b* and therefore the plane β according to a straight line *r*. From P in α let PH' be perpendicular to *r* and by the theorem of the three normals PH' results perpendicular to β. By absurdity its uniqueness is immediately demonstrated.

i) Planes passing through a line perpendicular to a plane are perpendicular to it.

k) If the planes α and β are perpendicular to each other, the perpendicular PH' at the intersection we have seen is perpendicular to β. Vice versa, because of the uniqueness of the perpendicular to a plane, if two planes α and β are perpendicular to each other, and from a point P of α the perpendicular to β is led, it lies in α.

l) Normal section of any dihedral. For two points A and B (fig. 31) of the rib *r* of a dihedral $\hat{\alpha\beta}$ we conduct in the α face the perpendiculars *a*, *a'* to the *r*, and in the β face the perpendiculars *b*, *b'* at *r*. We will call dihedral normal sections $\hat{\alpha\beta}$ angles $\widehat{ab}$, $\widehat{a'b'}$. They are equal.

Taken in fact on α AC = BD and on β AE = BF the quadrilaterals ACDB, ABFE are rectangles and therefore CD = AB = EF. The *r is* perpendicular to the planes *ab* and *a'b'*; therefore the plane α is perpendicular to the planes *ab* and *a'b'*, the CD which is perpendicular to the intersection *a* of the two planes α and *ab* is perpendicular to the plane *ab* and therefore also to the CE; similarly it is perpendicular to the DF; and similarly the EF is perpendicular to the CE and FD. Furthermore, since CD is perpendicular to the plane ACE, the plane CDE is perpendicular to the plane ACE, and EF, which is also perpendicular to the plane ACE, lies in the plane ACE.
plane CDE; therefore the quadrilateral CDEF is a plane quadrilateral with right angles, i.e. it is a rectangle. The triangles ACE and BDF are therefore equal.

for the third criterion, and the angles $\widehat{CAE}$ and $\widehat{DBF}$

are equal. The normal sections of any dihedral are therefore equal.

m) If two planes α and β are perpendicular to a third γ their intersection is perpendicular to γ.

n) Two planes perpendicular to a straight line do not meet.

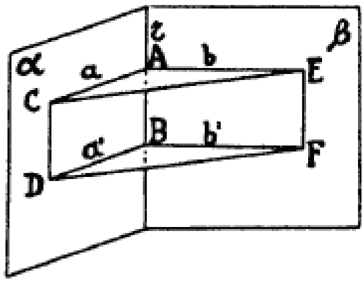

Fig. 81

o) Definition of axial plane of a segment.
It is shown that it is the geometric locus of the points equidistant from the extremities of the segment.

p) Distance a point from a plane; and geome- trical locus of points in the plane having an assigned distance from an external point.
Corollary: Given a regular polygon inscribed in a circumference, any point on the perpendicular to the plane of the polygon conducted through the centre is equidistant from the vertices of the polygon.

q) Bisector plane of a dihedral and its properties.
For a point P in the plane γ bisector of the dihedral $\hat{\alpha\beta}$ conduct the plane δ perpendicular to the spigolo *r*. The three *planes* α, β, γ are perpendicular to δ;

conducted from P, the perpendiculars PH and PK to α and β lie in δ; and joining the point M of the inter-section of *r* and δ with H, P, K, the right triangles PHM, PKM are equal in order to have the hypotenuse

Common PM and angles $\widehat{HMP}$, K $\widehat{MP}$ equals because γ is bisector of $\hat{\alpha\beta}$ and by rotating at-

back to *r*, when γ goes on β, α goes on γ and the two an- gles overlap.

Conversely, it is shown that if a point P inside $\hat{\alpha\beta}$ is equidistant from α and β, it belongs to the plane γ dihedral bisector $\widehat{\alpha\beta}$.

r) Definition of a trihedron and convex angular.

s) THEOREM: *In a trihedron, one face is less than the sum of the other two.*

It is demonstrated in the usual way, and extends to the ango-loid.

t) THEOREM: *The sum of the faces of a trihedron is less than four right angles.*

It is demonstrated in the usual way and extends to the convex ango- loid.

(*v*) Definition of regular angles.

All faces are equal, and the dihedrons formed by two consecutive faces are equal.

(*x*) Definition of polyhedron. A polyhedron is said to be regular when all faces are equal regular polygons and the angles are equal regular angles.

z) There can be a maximum of five regular polyhedra, one with three, one with four and one with five.

que faces congruent in a vertex equal to equilateral triangles; one with three squares congruent in a vertex, and one with three regular pentagons congruent in a vertex.

This possibility is demonstrated in the usual way.

2. *Construction of the regular tetrahedron.*

Having demonstrated the possibility of the existence of the five regular po- lihedra, we move on to their actual construction.

The property of the centre of gravity of any triangle can also be recognised as valid in our Pythagorean geometry, independently of Euclid's postulate; in the case of the equilateral triangle, it is then very easy to recognise that the centre of gravity is also the centre of the two circumscribed and inscribed circles, and that the radius of the former is double that of the latter.

For the centre H of an equilateral triangle ABC (fig. 32), the perpendicular *h is* taken to the plane ABC, and since AH is less than AB, the intersection of *h* with the circumference of centre A and ray AB is determined in the plane *Ah.* We join this point D with A, B, C; and we have DA = DB = DC = AB. The tetrahedron DABC has four equal equilateral triangles on its faces; the angular angles are trihedra with equal faces; and the dihedrons are also equal, since the dihedral edge AC has the normal section of the annular section of the dihedral angle AC.

gol DKB of the isosceles triangle KDB having side the height of the face and the base the edge, and is thus the same for all dihedra. There is therefore a king tetrahedron of given edge AB.

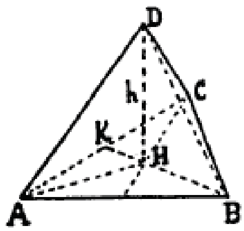

Fig. 82

Calling l_4 the edge, by the Pythagorean theorem we have:

$$(BK)^2 = \frac{3}{4} l_4^2 \quad \text{and thus} \quad (BH)^2 = \frac{4}{9} \cdot \frac{3}{4} l_4^2$$

$$(BH)^2 = \frac{1}{3} l_4^2 \quad \text{and} \quad (DH)^2 = \frac{2}{3} l_4^2$$

The centre of the circumscribed sphere lies on the *h*, which is the locus of the points equidistant from A, B, C; therefore if D' is the other extreme of the diameter OD, the plane ADD' is di- mondial, the triangle ADD' is right-angled because the midpoint of DD' is equidistant from the vertices, AH is the height of this right-angled triangle, and therefore we have

$$(AD)^2 = 2r \cdot DH \quad \text{and} \quad \frac{3}{2} (DH)^2 = 2r \cdot DH ;$$

$$3 (DH)^2 = 4r \cdot DH ;$$

$$3\,DH = 4r ; \quad DH = \frac{4}{3} r \quad \text{e} \quad OH = \frac{1}{3} r$$

The rule for the

Inscription of the regular tetrahedron in the sphere of rag- gio r.

Taken OD= *r* and on the opposite side OH= $\frac{1}{3} r$ let DH be the height. A circumference of diameter DD' = *2r*, and for H the perpendicular to the diameter DD' = 2r, is conducted; let its intersection with the circumference be vertex B of the tetrahedron. Finally, taking the plane passing through HB and perpendicular to the diameter DD', the circumference of radius HB is described and the equilateral triangle ABC is inscribed in it. The tetrahedron ABCD is the inscribed regular tetrahedron.

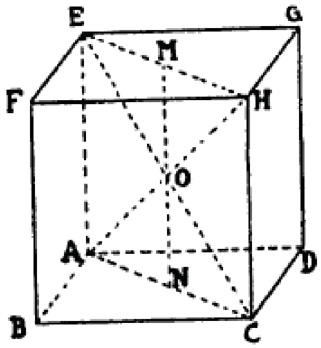

Fig. 33

3. *Existence and construction of the regular hexahedron.*

Let ABCD (fig. 33) be a square. Let us take the perpendiculars to the plane of the square ABCD from the vertices, and take the perpendiculars AE, BF, CH, DG equal to the side AB. The planes EAB, EAD are perpendicular to the plane α of the square ABCD; and the perpendiculars BF and DG to the plane ABCD lie in the planes EAB, EAD, respectively, so that ABFE and ADGE are two squares equal to the given.

Similarly, CH coincides with the intersection of the planes FBC and GDC perpendicular to α, and therefore FBCH and CDGH are also squares. Therefore CH is perpendicular to the plane FHG; CD is perpendicular to CB and CH, therefore also to the plane BCHF; the plane CDGH is perpendicular to the plane BCHF and the GH perpendicular to the intersection CH is also perpendicular to the plane BCHF,

and thus to HF. S$\hat{O}$ FHG = a rectum. The FH is thus perpendicular to the CDGH plane.

On the other hand DG is perpendicular to the plane HGE, planes HGD, HGE are perpendicular to each other and therefore FH perpendicular to the first of them belongs to the second. The quadrilateral FHGE is therefore a plane quadrilateral with sides all equal and a right angle and is therefore a square. The six faces of the hexahedron ABCDEFGH are squares; the three congruent faces at each vertex are squares and the dihedrons are all right angles; the regular hexahedron is constructed.

EA and HC are perpendicular to AC and EH, and the plane EAC is perpendicular to ABCD, CH is also perpendicular to AEC, so that EACH is a plane quadrilateral with right angles, i.e. it is a rectangle, and therefore the two diagonals of the cube CE, AH are equal and are cut in half. In the same way, EF and CD are perpendicular to FC and ED, EFCD is a rectangle, and the diagonal FD is equal to the other two and is cut in half by their midpoints; the same for BG. The four diagonals are equal, and they meet at the same point O which bisects them, so O is equidistant from vertices and is the centre of the circumscribed sphere.

We then have $(EC)^2=(EA)^2+(AB)^2+(BC)^2$ and then $4R^2=3l_6^2$ ed $l_6^2=\frac{4}{3}R^2$.

Conducted OM perpendicular to EH and thus to the EFHG facet, the segment OM, which is half of the edge

is equal to the apothem of the cube, and $a_{(6)}^2=\frac{R^2}{3}$.

On the other hand, it is easily recognised that the square constructed over the side of the equilateral triangle inscribed in a circumference of radius R is three times the square of the radius (i.e. the side of the equilateral triangle is $R\sqrt{3}$ and we therefore have the

THEOREM: The apothem of the cube inscribed in the sphere of radius R is $\frac{1}{3}$ of the side of the equilateral triangle inscribed in the circumference of radius R; and the edge of the cube is the $\frac{2}{3}$ of that side $\left(l_6=\frac{2}{3}R\sqrt{3}\right)$

After this, to solve the problem of inscribing the cube in the sphere of given radius, it is necessary to know how to divide an assigned segment into *n* (in our case 3) equal parts. The problem, regardless of the theory of parallels, can always be solved with the following

LEMMA: If the hypotenuse of a right-angled triangle is divided into *n* equal parts and the points of division are conducing the perpendiculars to one of the cathexes, they divide it into *n* equal parts.

Let ABC be a right-angled triangle (Fig. 34), and let the hypothetical BC be divided into *n* (5) equal parts; for the division points D, E, F, G, let us take the perpendiculars to the cathexes AC and AB. It is easily recognised that DMAL, ENAK, EPLK etc. are of the rectangles and that being

$\widehat{EDM}D = \widehat{MB}D + \widehat{BM}$ is also $\widehat{EDP}D = \widehat{BM}$; hence i

Rectangular triangles EDP, DBM are equal, and EP = DM and therefore AL = LK. Similarly LK = KI = HI = HC.

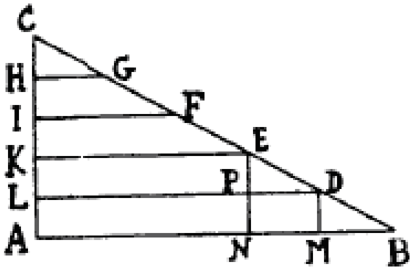

Fig. 34

On the other hand, due to the uniqueness of the submultiple of a given segment, if the hypotenuse and cathetus are divided into a minimum number of equal parts, the conjunctions of the corresponding division points LD, KE, etc. are perpendicular to the cathetus. are perpendicular to the cathetus.

We will see in the last chapter how one can always, independently of the theory of parallel lines, solve the problem of dividing a segment into an assigned number of equal parts. Meanwhile, for the case of *n*
= 5 the problem can be solved as follows: Given a segment such that its quintuple is greater than the given segment

(The point of intersection of the two circumferences is the vertex of a right-angled triangle whose hypotenuse is the diameter of the first circumference, and by taking the perpendiculars to the cathetus through the points of division of the diameter, it is divided into five equal parts.

The problem of dividing a segment into three equal parts is solved in a similar way. Let us now solve the problem of *the inscription of the cube in the sphere of radius* R: construct the equilateral triangle inscribed in the circle of radius R, and divide its side into three equal parts. A plane is created for a diameter CE of the sphere (Fig. 33), and the triangle is constructed in this plane.

tangent of hypotenuse CE and cathetus CH 2= 3 of the side of the constructed equilateral triangle. For the midpoint O of CE (centre of the sphere) we conduct the perpendicular plane MN to the cathetus EH; OM = ON is the apothem. For M and N the planes perpendicular to MN are conducted, and in the first of them the square with EH as its diagonal is constructed. It is a face of the cube; the symmetrics of the four vertices with respect to O give the other four vertices of the cube.

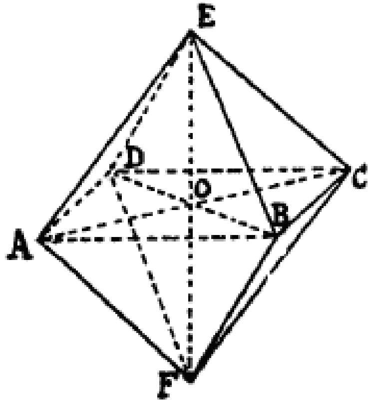

Fig. 35

4. *Inscription of the regular octahedron in the sphere of given radius.*

Let the plane perpendicular to the diameter EF be the centre of the sphere, and let ABCD (Fig. 35) be a square inscribed in the section circle. If we join the extremities of the diameter EF with A, B, C, D, we have inscribed regular octahedron. In fact, the eight faces are equilateral triangles, the angles of the angles are equal and the dihedrons are equal, since they are angles at the vertex of isosceles triangles with the side equal to the height of the face and the base equal to the diameter of the sphere.

It is easy to demonstrate that the octahedron having the centres of the six faces of the cube as vertices is regular, and that the thehedron having one vertex of the cube and the three opposite vertices of the three faces congruent with it as vertices is regular.

5. *The regular icosahedron*.

Divide a circumference (fig. 36) of centre V and any radius into 10 equal parts and inscribe in it the regular de- cagon $A_1B_1A_{(2)}B_2A_3B_3A_4B_4A_5B_5$ and the two regular penta- gons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B(_5)$.

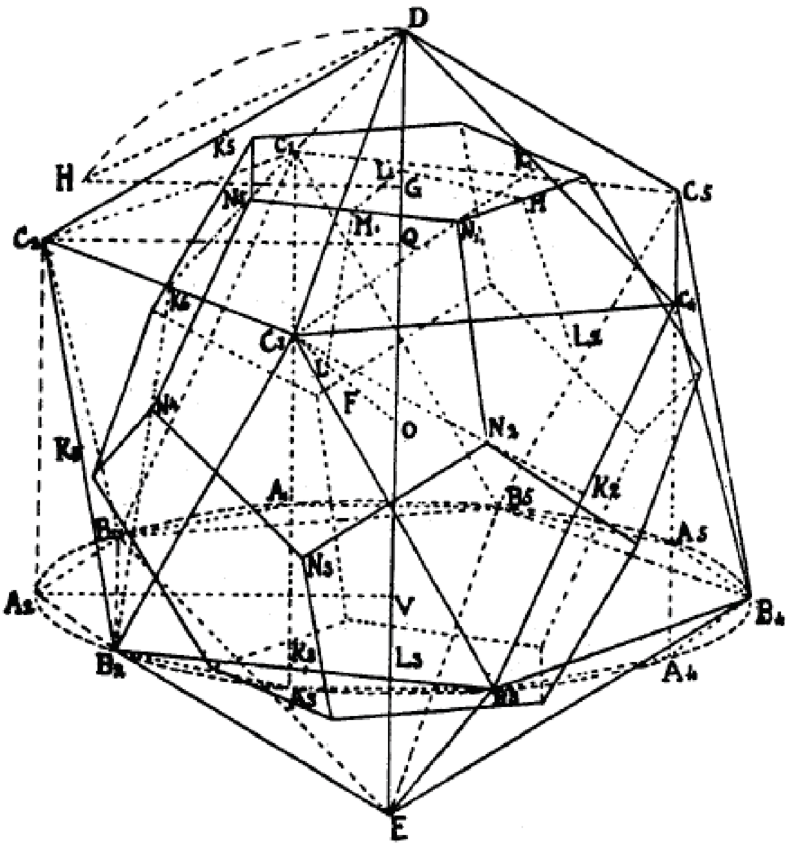

Fig. 36

For the vertices A of the first pentagon, take the perpendiculars to the plane α of the circumference, and take the perpendiculars to the plane α of the circle.

give the segments A_1C_1= A_2C_2= A_3C_3= $A_{4C(4)}$ = A_5C_5 = VA_1. The plane $C_{2)A(2}A_3$ is perpendicular to the plane α, so $A_3C_{(3}$) lies in it, the plane quadrilateral $C_2A_2A_{(3)}C_3$ is a rectangle and C_2C_3= A_2A_3. Similarly $A_4C(_4)$ lies in the plane $C_{(3)}A_3A_4$, the plane quadrilateral $C_{(3)}A_3A_{(4)}C_{(}4)$ is a rectangle and C_3C_4 = A_3A_4. And so the sides of the pentagon $C_1C_2C_{3C(4)}C_5$ are all equal to A_1A_2.

It is also a plane polygon. In fact, C_2A_2 is perpendicular to the plane α and the plane $C_1C_2C_3$; the plane $C_{(2)}A_2A_4$ is perpendicular to the plane α and therefore the perpendicular $A_{(4)}C(_4)$ to the plane α lies in the plane $C_2A_2A_4$; therefore $C_{(2)}A_{(2)}A_4C(_4)$ is a rectangle, and C_2C_4 is perpendicular to C_2A_2 and therefore C_4 lies in the plane $C_1C_2C_3$; similarly, C_5 lies in the plane $C_2C_{3C(4)}$; hence the polygon $C_1C_2C_{3C(4)}$- C_5 is a plane pentagon with all equal sides. Its ango-

th C_1C_2 C_3 is equal to the angl $_{A(1)}$ A_2 A_3 because

they are

both normal sections of the same dihedral, similarly for the other angles; and therefore $C_1C_2C_3C_4C_5$ is a regular plane pen- tagon equal to the two pentagons inscribed in the circumference of the plane α.

If we take the perpendicular to the plane α through the centre V, it lies in the plane C_2A_2V, and if we take the segment VQ = $VA(_{2)}$ = A_2C_2 on the side of C_2, the C_2Q lies in the plane of the pentagon $C_1C_2C_3C_4C_5$, and is QC_2= $VA(_2)$, and C_2A_2- VQ is a square. Similarly QC_1 = VA_2, etc., and thus Q is the centre of the circumcircle circumscribed to the regular pentagon $C_1C_2C_3C_4C_5$ and equal to the circumference of the plane α.

Being then C_1A_1 perpendicular to A_1B_5 we have:

$$(C_1B_5)^2=(C_1A_1)^2+(A_1B_5)^2$$

and since C_1A_1 is equal to the radius of the circumference V and A_1B_5 is the side of the regular decagon inscribed in it, C_1B_5 will be the side of the regular pentagon, i.e. $CB_5 = B_{(1)}B(_{5)}= C_1C_5 =$ ...

Similarly from the right triangles $C_1A_1B(_{1)}$, C_5A_5- B_5... we get $C_1B_{(1}=B_1B_5$, $C_5B_{(5}=B_5B_4$... then the trian- gles $C_{(1)}B_1C_{(5}$), $C_1B(_{5)}C(_5)$ are equilateral, and so continuing we recognise that the ten triangles $C_1C_2B_{(4}$), $C_2B_4B(_{2)}$, C_2C_3- B_2, $C_3B_2B_3$... which are obtained by neatly joining the vertices of the pentagon $C_1C_2C_{3C(4)}C_5$ to those of the pentagon $B_1B_2B_3B_4B_5$ are equilateral.

Let O be the midpoint of VQ; it can be seen at once that it is equidistant from the vertices C and the vertices B. Let us then take the segments OD= CE= OC_1= OB_1 on VQ; comparing with Fig. 23, we recognise that the segments QD and VE are the golden part of QV, i.e. the radius of the two circles of centre V and centre Q. We join D with the vertices of the pentagon $C_1C_2C_{3C(4)}C_5$ and E with those of the pentagon $B_{1B(2)}B_3B_4B_5$. From the right-angled triangle DQC_2, the result is: $(DC_{(}^{2))(2)}$= $(QC_{(}^{2}$+$(QD)^2$, and therefore DC_2 is also equal to the side of the pentagon. Similarly for DC_1, DC_3, DC_4, DC_5; therefore also the triangles having the vertex in D and for opposite sides the sides of the pentagon $C_1C_2C_{3C(4)}C_5$ are equilateral. And the same of course for triangles with vertex E having for opposite sides the sides of the pentagon $B_1B_2B_3B_4B_5$. We have thus obtained an icosahedron having for vertices the points D and E and the ten vertices of the two pentagons $C_1C_2C_3$- C_4C_5 and $B_{1B(2)}B_3B_4B_5$; it has for faces equal-sided triangles and is inscribed in the sphere of centre O and radius OD.

Since O is equidistant from D, C_2, $B_{(2),}$ and so is C_3 equidistant from the same points, the axial planes of the edges $C_2DC_2B_2$ definitely cut each other, and their intersection OC_3 is perpendicular to the plane DC(2$_)B_2$ and intersects it at a point F equidistant from D, C_2, B_2. On the other hand, the triangles DC_2O, C_3C_2O have OC_2 in common, OD = OC_3, DC_2= C_2C_3 and are therefore equal; the height C_2Q of the one is equal to the height C_2F of the other, and F is internal to OC_3 and OF = OQ and FC_3 = QD.

The isosceles triangles $OC_{(3)}D$, OC_3C_4 have the side the circumscribed sphere and the base of the edge of the icohedron, and are therefore equal. And, since OQ = OF, the triangles OC_3Q, OC_4F are also equal according to the first criterion.

rio, and $\widehat{OQC_3}$= also being a rectum $\widehat{OFC_4}$= a

FC_4 is therefore perpendicular to OC_3 and thus lies in the plane DC_2B_2; that is, C_4 lies in this plane. Analytically, it is shown that B_3 also lies in this plane; and we have: FB_3 = FC_4 = FD = FC_2 = FB_2. Hence the pentagon $DC_2B_{(2)}B_3C_4$ is an equilateral plane pentagon inscribed in the circle of centre F and radius FD, i.e. it is a regular plane pentagon and is the base of the regular pen- tential pyramid of vertex $C_{(3).}$ In the same way, we can prove that each vertex of the icosahedron is vertex of an equal regular pentagonal pyramid.

The normal section of the dihedral angle DC_3 is obtained by joining its midpoint with points C_2 and C_4. This angle is thus the angle at the vertex of an isosceles triangle whose side is the height of the face and whose base is the diagonal of the base pentagon; hence the normal section is the same for each dihedral of each icosahedron angle.

The constructed icosahedron is therefore a regular icosahedron.

To construct the regular icosahedron of the given edge C_1C_2, one can proceed as follows: 1º - the segment C_1C_4 of which C_1C_2 is the golden part is determined (fig. 23). 2º - determine the centre Q of the circle circumscribed to the isosceles triangle of side $C_{1C(4)}$ and base C_1C_2, and describe the circle of centre Q and radius QC_1. 3º - inscribe in this circle the regular pentagon $C_1C_2C_{3C(4)}C_5$. 4º - we take the perpendicular to the plane of the pentagon through the centre Q and we take QV equal to the radius of the circumference, and we have at the midpoint O of QV the centre of the circumscribed sphere and at OC_1 the radius. 5º - we take on the diameter QV the segments OD = OE equal to OC_1. 6º - the plane perpendicular to the diameter DE is conducted through V. 7º - the perpendicular to the plane through V is lowered from vertex C_1 and its foot A_1 belongs to the circumference of centre V and radius equal VQ. 8º - the perpendicular to this plane is lowered from C_2 and its foot A_2 also belongs to the circumference of centre V. 9º - the point mean point B_1 of the arc A_1A_2 and inscribe in the circumference of centre V the regular pentagon that has this mean point for one of its vertices, that is, the pentagon $B_{1B(2)}$- $B_3B_4B_5$. 10º - one joins D to the points C_1, C_2, $C_{(3),}$ C_4, C_5 and E to the points B_1, $B_{(2),}$ B_3, B_4, B_5; one then joins B_1 to C($_2$), $C_{(2}$) to B_2 etc., and one has the icosahedron.

6. *Inscription of the regular icosahedron in the sphere of radius* R.

The triangle DC_2E in Fig. 36 is rectangular in C_2 because its vertices are equidistant from O, the centre of the sphere. In it the height C_2Q = *r*, radius of the pentagon $C_1C_2C_{3C(4)}$- C_5; DQ= l_{10}; C_2D= l_5; QE= QV+ VE= r+ $l_{(10}$=s_{10}, and
so C_2E= s_5; therefore by [8]

$$(C_2D)^2+ (C_2E_1)^2= 5r^2$$

but by the Pythagorean theorem we have:

$$(C_2D)^2+ (C_2E)^2= (DE)^2= 4R^2$$

and therefore $5r^2$= $4R^2$. i.e. we have the

THEOREM: *The quintuple of the square whose side is the side of the base pentagon is equal to four times the square of the radius of the circumscribed sphere.*

Having stated this theorem, let us take (fig. 36) DE = 2R, and divide DE into five equal parts. Taking DG equal to one-fifth of DE, let the perpendicular to DE be led through G until it meets the circumference of diameter DE in H. We have: $(DH)^2$ = DE - DG or

$$(DH^{)(2)}= \frac{2R\ -2}{5} R = \frac{4}{5} R^2$$

DH is therefore equal to the radius r of the circumcircle circumscribed by the pentagon.

Then determine the side of the regular decagon in the circumference of radius r, and remove it from OD and OE, so as to obtain the segments OQ and OV. The planes perpendicular to the decagon DE are drawn through Q and V, and with centres Q and V and radius r two circumferences are described in them. The regular pentagons of vertex A, vertex B and vertex C are suitably inscribed in these circumferences; and joining vertex D to vertex C, vertex E to vertex B, the five vertices C to each other consecu- tively, the five vertices B to each other and the vertices C suitably to the vertices B, we have the inscribed regular icosahedron.

Let us denote by R the radius of the circumscribed sphere, by a the apothem of the icosahedron, by l_5 the edge, by r the radius of the circumcircle circumscribed by the pentagon of side $l_{(5)}$, by l_{10} the golden part of r, by s_5 and s_{10} the sides of the pentahalfa and the decalfa inscribed in this circumference, with R' the radius of the sphere tangent to the edges of the icohedron at their midpoints, with a_5 the apothem of the penta- gon of side l_5 and with a_{10} the apothem of the decagon of side l_{10}, the following relations are obtained:

$$5 r^2 = 4R(2)$$

$$2R\ r = +\ 2\, l_{10} = s_{10} + l_{10}$$

and thus, from the right-angled triangle DC_2E is derived:

$$R' = \frac{1}{(2)\ 5}\, s\, a_{10} \cdot$$

i.e.: the radius of the sphere tangent to the edges of the icohedron is equal to half the side of the pentalfa inscribed in the circle of radius *r*, or it is equal to the apo- tem of the decagon inscribed in this circle.

The radius of the inscribed sphere or apothem *a* is the cathetus of a right-angled triangle ON_5K_6 whose hypotenuse is R' and whose other cathetus is the third part of the height of the face; therefore:

$$a^2 = R'^2 - \frac{l_5^2}{12} = \frac{1}{4} s_5^2 - \frac{l_5^2}{12} = \frac{1}{12}\left(3 s_5^2 - l_5^2\right)$$

and [2] and [6]:

$$a^2 = \frac{1}{12}\left(3 s_5^2 - 4 r^2 + s_{10}^2\right) = \frac{1}{12}\left(3 s_{10}^2 - r^2 + s_{10}^2\right) =$$

$$= \frac{1}{12}\left(4 s_{10}^2 - 4 r^2\right) = \frac{1}{12}\left(2 s_{10} + r\right) + \left(2 s_{10} - r\right) =$$

$$= \frac{1}{12}\left(s_{10} + l_{10} + r + r\right)\left(s_{10} + s_{10} - r\right) =$$

$$= \frac{1}{12}\left(2R + 2r\right)\left(s_{10} + l_{10}\right) = \frac{(R + r) \cdot R}{3}$$

i.e. the square whose side is the apothem of the icosae- dro is equal to the third part of the rectangle whose sides are the radius of the circumscribed sphere, and this radius R is increased by the radius *r of* the circumference circumscribed by the pentagon. The relation may also be written in the form Rr = $3a^{(2)}$-R^{2}.[53]

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53 From triangle ON_5D we have instead:

$$a^2 = R^{(2)} - \left(\frac{2 l_5}{3 \cdot 2}\sqrt{3}\right)^2 = R^2 - \frac{l_5^2}{3}$$

Finally, it can be recognised that the diametral plane passing through the vertices D, B_2, E cuts the icosahedron according to a hexagon that has two opposite sides equal to the edge of the icosahedron and the other four equal to the height of the face, and it can be demonstrated geometrically that this hexagon has the same extension as the rectangle sides s_{10} and R + a_5.

By cutting the icosahedron with a diametral plane perpendicular to the diameter DE, however, a regular decagon is obtained in section, the side of which is equal to half the edge of the icosahedron and is inscribed in a circumference of radius R', from which it follows that half of l_5 is the au-

rea of R'; which also results from the formula: $R' = \frac{1}{(2)} s_5$.

7. *Construction of the regular dodecahedron.*

e

$$3 a^2 = 2) - l_5^2$$

and thus

$$3 R^2 - l_5^2 = R r + R(2)$$

e

$$2 R^2 = l_5(2) + R r ; \qquad l_5(2) = R (2 R - r)$$

One has as well:

$$s_5^2 + l_5^2 = 4 R^2$$

or

$$a_{10}^2 + \left(\frac{l_5}{2} \right)^2 = R^2$$

It also follows geometrically from the figure:

$$l_5^2 = 2R - l_{10} \quad ; \quad s_5^2 = 2R - s_{10}$$

Consider in Fig. 36 the pentagonal pyramid of vertex C_3 and base $DC_2B_2B_3C_4$. The midpoints K_1, K_2, K_3, K_4, K_5 of the sides of the base are in turn vertices of a regular pentagon with centre F, which is the base of another pyramid with vertex $C_{(3)}$ and edges $C_3K_1 = C_{(3)}K_2 = C_3K_{(3)} = C_{(3)}K_{(4)} = C_3K_5$. The centres N_1, $N_{(2)}$, N_3, N_4, N_5 of the lateral faces of the first pyramid stand on the edges of the seventh pyramid, and we have:

$$C_3N_1 = C_3N_2 = C_3N_3 = C_3N_4 = C_3N_5 = \frac{2}{3} C_3K_1$$

Since $\widehat{K_{(1)}C_{(3)}K_2} = \widehat{K_{(2)}C_{(3)}K_3} = \dots$ isosceles triangles $N_1C_3N_{(2)}$), $N_2C_3N_{(3)}$... are equal by the first criterion and thus $N_1N_{(2)} = N_2N_3 = N_{(3)}N_{4)} = N_{(4}N_5 = N_5N_1$.

Since the triangle C_3FK_1 is rectangular in F and N_1K_1 is one-third of the hypotenuse, the perpendicular to the cathetus C_3F conducted by N_1 meets the cathetus C_3F at a point L such that FL is one-third of C_3F.

The same is true for the other points N_2, N_3, N_4, N_5; and thus $N_1N_2N_3N_4N_5$ is an equilateral plane pentagon in the circumference of centre L and radius LN_1; or it is a plane pentagon that has for vertices the centres of the faces of the icosahedron congruent in C_3.

Similarly, by taking the centres of the lateral faces of the pyramid with vertex D and base $C_1C_2C_{3C(4)}C_5$, they are the vertices of another regular plane pentagon equal to the previous one and having in common with it the side N_5N_1; and by taking the centres of the lateral faces of the pyramid with vertex C_4 and base $DC_3B_{(3)}B_4C_{(5)}$, we obtain a third pentagon.

a regular plane equal to the preceding ones and having one side in common with the first and one in common with the second so that the vertex N_1 is common to the three pentagons.

Working in a similar manner with each of the twelve vertices of the icosahedron, one obtains a dodecahedron that has for faces regular pentagons equal to $N_1N_2N_3N_4N_5$, and for angles equal-faced trihedra.

The vertex C_3 and the centre L of the base are equidistant from the vertices of the base $N_1N_2N_3N_4N_5$ and therefore the centre O of the sphere circumscribed to the icosahedron is also equidistant from all the vertices of the pentagons as $N_1N_2N_3N_4N_5$; therefore the dodecahedron we have constructed is inscribed in the sphere of radius ON_1.

Taking then the midpoint M of the edge of the dodehedron common to the adjacent faces of centres L_1 and L_2 and

united with them, angl ê $_{L(1)}ML_2$ is the normal section

of this dihedral; and is the angle at the vertex of an isosceles triangle whose sides are the apotymes of the faces L_1M and L_2M and whose base is the segment L_1L_2 joining the centres of the two faces. But OL_1 and OL_2 are equal because the cathexes of the right triangles $ON_{(1)}L_{(1)}$, ON_1L_2 having the hypotenuse ON_1 in common and the cathexes $L_{(1)}N_{(1)}$, L_2N_1 are equal; therefore the segment L_1L_2 is the base of an isosceles triangle having for sides $OL_1 = OL_2$ and the angle at the vertex in common with the isosceles triangle having for sides the radii OD, $OC_{(4)}$ of the sphere and for base the edge DC_4 of the icosahedron. These elements therefore remain the same if we take the normal section of another dihedral of the dodecahedron; therefore these dihedrons are all equal, and we can conclude that the constructed doyhedron is regular, is inscribed in the sphere of radius ON_1 and has apothem OL_1.

We will see later on the construction of the dodecahedron of given edge.

8. *Inscription of the regular dodecahedron in the sphere of radius* R.

Let ABCD... UV (fig. 37) a regular dodecahedron.

A cube can be inscribed in it, having for vertices the vertices of the dodecahedron and for edges the diagonals of the faces of the dodecahedron.

In fact, if we take the vertex A, and in the three faces congruent with A the vertices G, C, P; and if we take the four vertices U, M, S, K, of the dodecahedron diametrically opposed to them, these eight points are the vertices of a figure whose angles are all equal to the diagonals of the faces of the dodecahedron, or to the side of the pentahalfa inscribed in the face. We show that the trihedra whose vertices are the vertices and whose edges are the edges of this competing figure are trirectangles; it suffices to show that, for example, the trihedron of vertex A is trirectangled, and that, for example, AG is perpendicular to AC.

Returning for a moment to Figure 26, we observe that if from the vertices C and I of the regular pentagon ACEGI we lower the perpendiculars CP, IQ to the side EG, the right triangles CPE, IQG, having the hypotenuse and an acute angle are equal, and we have CP = IQ; therefore the quadrilateral PQIC is by construction a rectangle of base PQ and height CP = QI. It is also obtained by plotting from the midpoint M of EG the two segments

$MP = MQ = \frac{1}{2} CI$, and joining P with C and Q with I.

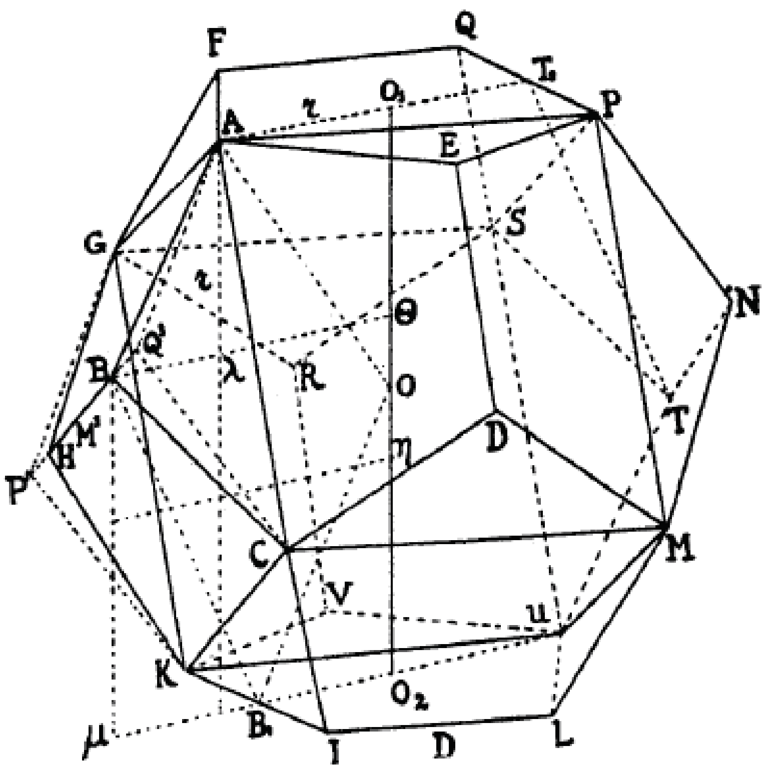

Fig. 37

Taking then (fig. 37) the midpoint M' of the edge HB of the dodecahedron e taken

$M'P' = M'Q' = \frac{1}{2} AG = \frac{1}{2} CK$, i quadrilaterals GP'Q'A, KP'Q'C are rectangles; and therefore P'Q' is perpendi-

perpendicular to Q'A and Q'C and their plane AQ'C, and likewise is perpendicular to P'G and P'K and their plane GP'K. The plane ABH that passes through P'Q' is perpendicular to the plane AQ'C and to the plane GP'K, and the line GA of this plane being perpendicular to the intersection AQ', as well as to GP', is also perpendicular to the plane AQ'C as well as to the plane GP'K; and therefore it is perpendicular to AC and GK. Therefore the quadrilateral AGKC, which has all its sides equal, has two right angles; and since the same is repeated for KC and KC is perpendicular to the plane Q'CA at a point C of its intersection AC with the plane GAC perpendicular to it, CK is in the plane GAC, and GACK is a square. Similarly, the other two faces ACMP and AGSP are shown to be squares.

Working in this way with the trihedrons of vertices G, S, P, K, U, M, C, the edges GK, SU, PM, AC are shown to be perpendicular to the plane of the square AGSP and equal to each other and to the side AP of this square; therefore AGSPCKUM is effectively a cube, inscribed in the doyhedron, and both are inscribed in the sphere whose diameter is the diagonal of the cube.

Fig. 36 shows that the centres of two opposite faces of the dodecahedron, such as L_1 and $L_{(3),}$ lie on the diameter DE and are equidistant from the centre O of the sphere circumscribed to the dodecahedron; therefore the conjunction of the centres of two opposite faces of the dodecahedron is perpendicular to them. Conjoining therefore in Fig. 37 the centres O_1 and $O_{(2})$, of two opposite faces, the $O_{1O(2)}$ passes through the centre O and is O_1O - O_2O

the apothem of the dodecahedron. It is the cathetus of the triangle OAO_1, having for its hypotenuse the radius OA = R and for its other cathetus the radius $O_1A = r$ of the circumcircle circumscribed to the pentagon AEPQF. This radius is merely the height of the right triangle that has for its cathexes $l(_{5)}$and s_5, that is AE and AP. But AP is the edge of the inscribed cube, and we know that three times the square of the edge is equal to the square of the diagonal, so we have:

$$3(AP)^2 = {}^{2R(2)}$$

or

[14] $$3\, s_5^2 = 4\, R^2$$

and since the square whose side is the side of the equilateral triangle inscribed in the circle of radius R is three times the square of the radius, while the square *of* s_5 is four thirds of this square, it follows that the square *of* $s_{(5})$ is the four nonsides of the square of the side of the inscribed equilateral triangle, and therefore the edge of the inscribed cube, which is also the side of the pentalfa inscribed in the face of the dodecahedron, is two-thirds of the side of the royal triangle inscribed in the circumference of radius R.

Therefore, to construct the regular dodecahedron inscribed in the sphere of radius OA = R, one can proceed as follows: 1º - Inscribe the equilateral triangle in the circumference of radius R, and take two-thirds of the side. Thus we have the edge of the inscribed cube and the side AP= $_{s(5)}$ of the pentalfa inscribed in the face. 2º - The golden part of this edge is determined and thus AE = l_5. 3º - The right-angled triangle of cathexes s_5 and l_5 is constructed; the height of this

Right triangle is the radius *r* of the circumcircle circumscribed to the face of the dodecahedron. 4º - The right triangle of hypotenuse R and cathetus *r* is constructed, the other cathetus being the apothem OO_1 of the dodecahedron. 5º - Taking a segment O_1O_2 equal to twice the apothegm, the planes perpendicular to it conducted through O_1 and $O_{(2)}$, the circumferences of radius *r* and centres O_1 and O_2 are described in these planes, and the regular pentagons AEPQF, UVKIL are inscribed in them, where U is symmetrical of A with respect to O, the midpoint of O_1O_2. The points A, P, K, U are four vertices of the inscribed cube. 6º - The planes perpendicular to AP are conducted through A and through P. 7º - In the first of these planes, one constructs the square whose diagonal is AK and in the second the square PSUM whose diagonal is PU; thus one has the other four vertices of the cube. 8º - In the plane AFG, the regular pentagon AFGHB is completed, and then in the plane EAB, the pentagon ABCDE is completed, and then HBCIK etc.

9. *Relations between the elements of the dodecahedron and solution of the problem of its inscription in the sphere of radius* R.

In Figure 26, the triangles AVO, CΘO, DOZ, EVO... are isosceles with the side equal to the radius OA of the circle and the base equal to the side of the inscribed regular decagon, so the circle with centre O and radius equal to the side AB of the decagon passes through Θ, V, Y, Z...; its radius is the golden part of the circle with radius OA. The isosceles triangles CΘY, OCA are equal because they have the equal side and the equal angle at the vertex, therefore the side ΘY of the pentagon inscribed in the minor is equal to the side of the pentagon inscribed in the major and is therefore the golden part of the side of the pentagon inscribed in the major: and therefore ΘV the side of the pentagon inscribed in the minor is the golden part of the side of the pentagon inscribed in the major. The isosceles triangles BCV and OYZ are equal because they have the side equal and the angle at the vertex equal, and therefore the side of the decagon inscribed in the minor is the golden part of the side of the decagon inscribed in the major; and the side of the decalfa inscribed in the minor, being equal to the radius of the minor increased by the side of the inscribed decagon, is equal to the radius of the major.

Vice versa, given the circumference of centre O and radius OV and described the concentric circumference whose radius is the side VZ of the decalfa, the circumference of radius OC is obtained and the relations now seen subsist, and in particular the side of the regular pentagon inscribed in the major is equal to the side of the pentalfa inscribed in the minor.

Let us now consider the opposite faces (fig. 37) AEPQF, KILUV of the dodecahedron, and let O_1 and O_2 be the centres of their respective circumscribed circumferences and *r* their radius O_1A = O_2K.

We know that O_1O_2 is perpendicular to the two faces and therefore the plane O_1AO_2 is also perpendicular to these two faces; it coincides with the plane DEN_5 of Figure 36, passes through the point K_6 of this figure and is perpendicular to the edge C_2C_3 because K_6Q is also perpendicular to the edge $C_2C_{(3).}$

and therefore cuts the plane of face $C(_{2)}C_3B_{(2})$ according to K_6B_2 perpendicular to the edge C_2C_3, and therefore passes through N_4, that is through vertex B of figure 37; and since this plane O_1AO_2 also passes through vertex U opposite vertex A, it intersects the lower face KILUV according to O_2U and thus the edge KI at its midpoint B_1; therefore the pentagon O_1AB- B_1O_2 is a plane pentagon. Similarly, the plane pentagon $O_1O_2UTT_{(1)}$ is a plane pentagon; and the plane O_1OA bisects the deodehedron according to the hexagon ABB_1UTT_1. Similarly, the pentagon $O_1O_2D_1DE$ is plane, and the two pentagons have sides that are neatly equal, vertex angles O_1 and O_2 right, vertex angles B_1 and D_1 equal because they are normal sections of the dodecahedron; and it is easy to recognise that the vertex angles A and B of the first pentagon are also equal to the vertex angles E and D of the second pentagon. Therefore, the two pentagons $O_1ABB_{(1)}O_{(2)}$), $O_1EDD_1O_2$ are equal; therefore, leading from B and D the perpendiculars to the common side O_1O_2 their feet coincide at a point Θ and ΘB = ΘD. In the same way ΘN, ΘS, ΘG are equal to ΘB and perpendicular to O_1O_2,; in short Θ is the centre of a circumference of radius ΘB placed in a plane perpendicular to O_1O_2, in which the regular plane pentagon BDNSG is inscribed.

Similarly, leading from C the perpendicular Cη to $O_{1O(2)}$ shows that η is the centre of a circumference (situated in a plane perpendicular to O_1O_2) in which the regular plane pentagon CMTRH is inscribed.

Since the edge AE of the dodecahedron is the golden part of AP and therefore of BD, we find that the side of the pentagon inscribed in the circle of radius *r* is the golden part of the side of the pentagon inscribed in the circle of centre Θ and radius ΘB; it follows that the radius *r* is the golden part of the radius ΘB, that is, that this radius is equal to the side s_{10} of the de- calphas inscribed in the face of the dodecahedron.

Taken now on BΘ, the segment Θλ, *equal to r* the segment Bλ, will be equal to l_{10}, and since O_1AλΘ is a rectangle by construction the triangle ABλ is rectangular. Its hypotenuse *is* l_5, its cathetus Bλ, is l_{10}, the other cathetus is therefore equal to *r*. The rectangle O_1AλΘ is therefore a square and the planes of the two circumferences with centres O_1 and Θ have a distance equal to *r*.

On the other hand, since the apothem O_2B_1 of the face is equal to half of BΘ= $s_{(10)}$, B_1 is the midpoint of the segment O_2μ taken equal to s_{10}, and therefore BΘO_2μ is a rectangle, and BμB_1 is a right-angled triangle whose hypo- tenuse *is equal to* $r + a_{(5)}$, the cathetus μB_1 is equal to a_5 and therefore:

$$(B\mu)^2 = (r + a_5)^2 - a_5^2 = r^2 + 2\,r a_5$$

but

$$r = s_{10} - l_{10} \quad \text{and} \quad a_5 = \frac{s_{10}}{2}$$

hence

$$(B\mu)^2 = r^2 + s_{10}(s_{10} - l_{10}) = r^2 + s_{10}^2 - l_{10}\,s_{10}$$

and since

$$r^2 = s_{10} - l_{10}$$

we then obtain

$$(B\mu)^{(2)} = s^{(2)}_{10}$$

$$B\mu = s_{(10)}$$

$$B\mu = O_{(2)}\Theta = B\Theta = s_{(10)}.$$

Thus $B\mu O_2\Theta$ is also a square; and the distance between the plane of vertices BDNSG and the lower face KILUV is equal to s_{10}.

Similarly taken the point η above $O_{1O(2)}$ such that $O_2\eta = O_1\Theta = r$ it is the centre of the circumference of radius s_{10} passing through CMTRH.

It follows that $\Theta\eta = \Theta O_2 - O_2\eta = s_{(10)} - r = l_{(10)}$. So the distance between the planes of the vertices BDNSG and CMTRH is equal to l_{10}, the side of the regular decagon inscribed in the face of the dodecahedron.

The distance between the two opposite faces of the dodecahedron AEPQF and KILUV is equal to *2a*; and we have:

[15] $$2a = 2r + l_{(10)} = s_{(10)} + r$$

and

$$a = \frac{2r + l_{(10)}}{2} = \frac{r + s_{10}}{2} = \frac{r}{2} + a_5 .$$

From the right-angled triangles $AO_1\eta$ and $B\Theta O_1$ which have r and s_{10} as cathexes, it is derived that the hypotenuses $A\eta$ and BO_1 are equal to s_5.

Since then r is the golden part of $s_{(10)}$), s_{10} in turn is the golden part of O_1O_2; thus the distance 2a between the two opposite faces of the dodecahedron is divided by the planes of the al-three vertices at two points Θ and η such that $\eta O_1 = O_2\Theta$ is the golden part of *2a*, the remaining part $O_1\Theta = O_2\eta$ is equal to the golden part *r* of s_{10} and the intermediate part is the golden part of *r*, i.e. it is the side of the decagon inscribed in the facia of the dodecahedron.

In summary, the two circumferences with centres Θ and η have a radius equal to twice the apothem of the face of the dodecahedron, have a distance from the two faces following them equal to the radius of the face, and have a distance from the other two faces equal to their radius, i.e. the side of the decalpha inscribed in the face of the dodecahedron.

In Figure 28, the section $ABB(_1)$ UTT_1 of the dodecahedron is drawn in its plane and consists of the hexagon PFQP'F'Q'. The points N and D correspond to the centres O_1 and O_2 of the faces in Figure 37.

The sides PF and P'F' are those equal to the edge l_5 of the dodecahedron. BD and PN are equal to the radius *r of* the facia; O midpoint of ND is the centre of the sphere and OB
= OF = OP is the radius R of the circumscribed sphere, DH is equal to s_{10}. By completing the square ADHF and the ret- tangle ADNV, AB is equal to l_{10}.

Taken above PB the point K such that PK= $s(10)$ it will be BK= r; taken for K the perpendicular to PD it cuts AV in C and DN in E such that AC = DE = *r* and BC = AK = l_5: then taken KL = BM = s_{10} the right triangles KBL, KPN are equal and so KN = BL = $s(5)$ and

$P\widehat{KN} = K\widehat{LB} = A\widehat{CB} = AKB$ then the points A, K, N are aligned, and the diagonal AN is divided by K into two parts, AK equal to l_5 and KN equal to s_5, so that AN is equal to $l_5 +$ s_5. AD is equal to s_{10}; then, taking the midpoint Q of AD, DQ will be the apothem a_5 of the face and OQ the radius R' of the sphere tangent to the edges of the doyhedron at their midpoints. And since OQ is half of AN, we have the simple relation:

[16] $$R' = \frac{l_{(5} + s_5}{2}$$

In Figure 28 FN and CD are equal to s_5. It follows from the figure that the rectangle BDNP is equal to the sum of the rectangle BDHG and the square GHNP, and so we have:

$$2a - r = r - s_{10} + r^{(2)\, r-s} = {}_{10} + {}_{s10} - l_{10} = s_{10} \ (r\ l + \quad) = s^2_{10} \quad {}_{10}$$

So

[17] $$2a - r = s^2_{10}$$

or even

[18] $$a - r = {}^2a^2_5$$

In Figure 28, the diagonal AN, and the axes of AD and DN meet at the midpoint of AN, and the rectangle with base AQ = a and height a is divided by BP and CE so that the rectangle with base AB = l_{10} and height a is equal in extent to the rectangle with base AQ = a_5 and height r.

One therefore has:

[19] $$a - l_{10} = r - a_5$$

or even

[19'] $$2a - l_{10} = r - s_{10}$$

From the OBD and OQD triangles in Fig. 28, we derive:

[20] $R^2 = a^{(}+r^2$

[21] $R^2 = a(+a^2_5$

and from these or even from the figure

[22] $R^2 = R(2) + r^2 - a^2_5 \quad R'(2 + 5))(2 \; (\;)2$

The hexagon $ABB_{(1)}UTT_1$ section of the dodecahedron is equal to the rectangle of sides $2s_{10}$ and $2a$, less the rectangles of sides r and l_{10} and a_5 and s_{10}. Thus we have:

$$2s_{10} - 2a - rl_{10} - a_5 s_{10} = 4a_5 - 2a - r(s_{10} - r) - 2a^2_5$$

$$= 4a_5(s_{10} + r) - r\text{-}s + r(2)_{10} - 2a^2 = 8a^2_5 + 4a_5 r - 2a_5 + r(2) - 2a^2_5$$

$$= 6a_5(+2a_5(s_{10} - l_{10}) + r^2 = 6a_5 + 4a^2_5 - s_{10} l_{10} + r^2) = 10a^2_5$$

Thus, the section made in the dodecahedron with the plane passing through the centres of two opposite faces and the vertex of one of these faces is the tenfold of the square whose side is the apothem of the face.

In the hexagon PFQP'F'Q' the diagonals PP' and FF' are equal to 2R and since they bisect each other in O it follows that PFP'F' is a rectangle; and therefore the isosceles triangles PQ'F' and FQP' which have equal sides also have equal bases PF' and FP' and are equal. These bases are equal to 2R'.

The angle $\widehat{Q'PF'}$ and $\widehat{QFP'}$ at the base of the two trian- previous isosceles goli are equal; and therefore are equal also the angle $\widehat{Q'PF}$ and $\widehat{PFQ}$; therefore i triangles

PFQ' and PFQ are equal by the first criterion and therefore the two diagonals of the hexagon PQ and FQ' are equal. The latter is the hypotenuse of the triangle FQ'T' and therefore the qua- drate constructed over it is given by 9 $a^2 + r_{(2)}$: and if we
other expressions can also be found. 5

After having found the expression of the three diagonals of the hexagon PFQP'F'Q' one can find that its area is also expressed by R'($2l_5$ + $s_{(5)}$) or even by R'(2R' + l_5), which si can prove identically equal to

10 a^2_5.

On the basis of the properties that we have found, we can give the following solution to the problem of inscribing the regular dodecahedron in the sphere of a given radius, a solution that is comparable to the first one and that we assume is consistent with that given by the Pythagoreans: 1º - Given R, determine, as in the other procedure, the edge AP of the inscribed cube, which is also equal to s_5, the side of the pentahalfa inscribed in the face of the dodecahedron. 2º - The golden part of this edge of the cube is determined, and the edge of the dodecahedron is found in it. 3º - The height of the right-angled triangle whose cathexes are s_5 and l_5, that is, the inscribed edges of the cube and of the dodecahedron, is equal to *r*, the radius of the circumference circumscribed to the face of the dodecahedron. 4º - The projections of the cathexes of this triangle are l_{10} and s_{10}, that is, the side of the regular decagon and the side of the decalpha inscribed in the circumcircle circumscribed to the face. 5º - We take a segment Θη= l_{10} side of the decagon and the golden part of the radius *r*, and we take its extensions ΘO_1= η$O_{(2}$=

r. The midpoint O of the segments Θη and $O_{1O(2)}$ is the centre of the inscribed sphere, and the segments $OO_1 = OO_2 =$ a are equal to the apothem of the dodecahedron. 6º - For the points O_1, Θ, η, and $O_{(2),}$ the planes perpendicular to O_1O_2 are conducted; in these planes we describe the circumferences of centres O_1 and O_2 and radius *r* and those of centres Θ and η and radius s_{10} = side of the decalfa, and we inscribe in them the regular pentagons AEPQF, KILUV, BDNSG, CMTRH in such a way that the vertices A and B are in the same plane OO_1AB and the vertices I, C in the same plane OO_2IC and that these two planes form an angle of 36°. All the vertices of the dodecahedron are thus obtained. 7º - We draw AB, ED, PN, QS, FG, IC, LM, UT, VR, KH; and then the points B, C, D, M, N, T, S, R, G, H, B are joined together successively and the dodecahedron is completed.

The problem of constructing the dodecahedron circumscribed by the sphere of radius *a* is solved immediately. It is sufficient to take the golden part of the diameter *2a*, and the remaining part is *r*; the difference between 2a and *r* is $s_{(10)}$); and the difference between s_{10} and *r* is l_{10}; and now proceed as in the previous case.

The problem of constructing the regular dodecahedron of given edge l_5, is solved by first constructing (fig. 23) the segment s_5 of which the assigned edge is the golden part; then constructing the right-angled triangle of sides s_5 and l_5, fig. 28 successively gives *r*, l_{10}, s_{10}, *a*, a_5, R, and R'.

Ipsicles and before him Aristaeus[54] demonstrated that the circles circumscribed by the pentagon of the dodecahedron and the face of the icosahedron inscribed in the same sphere have the same radius.

The demonstration can be done as follows: in Fig. 36 we have: ON_5 - R > OL_1. On the apothem OL, OL_1, OL_2 ... I take OL' = OL'_1 = OL'_2 = ... = R. These points are vertices of the icosahedron inscribed in the sphere of radius R. In fact, 1º - L'_1 = $L'_{(2)}$ = $L_{'(1)}L'_2$ = ... because bases of iso- chosen triangles of equal side and angle at the vertex; 2º - The equilateral triangle $L'L_{'(1)}L'_2$ has its centre on the axis ON_1 equidistant from them: this centre X is the foot of the heights of the vertices L', $L_{'(1)}$, L'_2 of the equal triangles ON_1L, $ON_{(1)}L'_{(1}$), $ON_1L'_2$; 3º - The right-angled triangle OXL'_1 = $ON_1L(_1)$ because the hypothetical OL'_1 = ON_1 and an acute angle is in common; then XL'_1 = L_1N_1; but XL'_1 is the radius of the circumcircle circumscribed to the face of the icosahedron, and L_1N_1 is the radius of that circumscribed to the pentagon of the dodecahedron; and therefore the property is proved geometrically.

54 Cf. G. Loria - *The Exact Sciences in Ancient Greece*, pp. 159 e 271.

CHAPTER V
THE SYMBOL OF THE UNIVERSE

1. In relation to regular polyhedra, and especially the regular dodecahedron, we must now take a moment to consider the three averages also considered by the Pythagoreans, namely the arithmetic mean, the geometric mean and the harmonic mean.

Nicomachus of Jerash, a writer from the first century of the vernacular, attests that Pythagoras knew the three proportions arithmetic, geometric and harmonic; and Jacobus attests that in his school, the three means were considered arithmetic, geometric and harmonic [55].

There is an arithmetic proportion between four numbers a, b, c, d when $a - b = c - d$; the proportion is continuous if $b = c$; and in that case b is the arithmetic mean or arithmetic average of a and d and we have

$$b = \frac{a + d}{2}.$$

If they are three segments in arithmetic proportion, the definition is the same and the segment b half-sum of the two segments a and d is their arithmetic mean.

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55 Cf. NICOMACO DI GERASA, ed. Teubner, p. 122; and JAMBLICHI, *Nicomachi Arith. introd.*, ed. Teubner, p. 100. See also G. LORIA, *Le scienze esatte*, p. 36.

There is a geometric proportion between four numbers *a*, *b*, *c*, *d* when *a* : *b*= c : *d*, and for segments when the rectangle of the medians is equal to the rectangle of the extremes. With this definition there is no need for the theory parallels and similarity, no consideration is given to the ratio of two segments, and the question of incommensurability does not arise. We have also seen that the Pythagoreans were able to solve the problem simple ap- plication, i.e. to construct the proportional fourth segment after three assigned segments *a*, *b*, *c*, if the first segment was greater than the other two by one, again without the need for parallels. If *b* is equal *c*, then the proportion is continuous and *b* is the geometric mean between *a* and *d*; the geometric mean of two segments is therefore the side of the square equal to the retentangle of the other two; and we have seen that the Pythagoreans were always able, as an application of the Pythagorean theorem, to construct such a geometric mean.

As for the harmonic proportion and the armo- nic mean, it will be said that four numbers *a*, *b*, *c*, *d* are in harmonic proportion when their inverses are in harmonic proportion.

arithmetic, i.e. when $\frac{1}{\ }=\frac{1 - 1}{\ }$; and conse-

a b c d

b is the harmonic mean between *a* and *d* when the inverse of *b* is equal to the arithmetic mean of the inverses of the other two.

Archita in one of his fragments handed down the Pythagorean definitions in the case of continuous proportion of three terms; the ancient definitions coincide with the modern ones in the case of the arithmetic mean and the geome- tric mean, the definition of the harmonic mean is different. We quote Archita's fragment,[56] inserting the numerical examples for clarity:

"The average is arithmetical when the three terms are in a similar ratio of surplus, i.e. such that the amount by which the first surpasses the second is precisely that by which the second surpasses the third; in this proportion, the ratio of the larger terms is smaller, and the ratio of the smaller terms is larger (e.g. 12, 9 and 6 are in arithmetical proportion because 12 - 9= 9 - 6; the ratio of the larger terms is larger).

large i.e. the ratio of 12 and 9 is equal to $1 + \frac{1}{3}$, the

ratio of the smallest, i.e. of 9 and of 6 is equal $1+\frac{1}{2}$, and $\frac{1}{3}$ is less than $\frac{1}{2}$)".

"There is a geometric mean, continues Archita, when the first term is to the second as the second is to the third, and in this case the ratio of the largest is equal to the ratio of the smallest (e.g. 6 is the geo- metric mean of 9 and 4 because 9 : =6 : 4); the sub-contra- mous mean that we [Archita] call harmonic exists when

56 Cf. H. Diels, *Die Fragmente der Vorsokratiker*, ed. Berlin 1912; fr. 2º. Archita's fragment is quoted in the Greek text by Mieli on p. 251 of the oft-quoted work. Chaignet (A. Ed. Chaignet - *Pythagore et la philosophie pythagoricienne*, 2ª ed., vol. I, p. 282-83) gives the translation.

the first term passes the second by a fraction of itself, identical to the fraction of the third by which the second passes the third; in this proportion, the ratio of the largest ter- mines is the largest and the ratio of the smallest the smallest (example: 8 is the arithmetic mean of

12 e of 6, because $12 = 8 + \frac{1}{3}$ of 12 ; ed $8 = 6 + \frac{1}{3}$ of 6; the ratio of 12 to 8 is equal to $1 + \frac{1}{2}$, that of 8 to 6 is equal to $1 + \frac{1}{3}$, e $\frac{1}{2}$ is maggiore of $\frac{1}{3}$)".

Before Archita (or the Pythagoreans?) this proportion was called ὑπεναντία also translated as sub-contrary by Loria, because according to the definition we have given, the opposite happens in this case than in the former [57]. From this definition the modern definition can be derived by simple arithmetical operations. In fact, if *a*, *b*, *c*, form harmonic proportions, this means that *a*, *b*, *c*, form harmonic proportions.

cunt according to Archita that $a = b + \frac{1}{n}a$ and $b = c + \frac{1}{n}c$;

from which it can easily be deduced:

$$n = a:(a - b) = c:(b - c)$$

and thus:

$$a(b - c) = c(a - b);\ ab - ac = ac - bc;\ 2ac = ab + bc;$$

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57 See JAMBLICHI, *Nicomachi Arith.*, ed Teubner, p. 100; and NICOMACO, ed Teubner, p. 135.

$$2ac = b(a+c);\ b = \frac{2ac}{a+c};\ \frac{1}{b} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}\right).$$

One can also write:

$$b\left(\frac{a+c}{2}\right) = a \cdot c$$

This results in the numerical proportion:

$$a : \frac{a+c}{2} = \frac{2ac}{a+c} : c$$

which, according to Nicomachus of Jerash, Pythagoras transported from Babylon to Greece.[58] In this very important geometrical proportion, the extremes are any two numbers (or magnitudes), the middles are ordinarily their arithmetic mean and their harmonic mean. In the case of segments, the geometric definition of the arithmetic mean results from the penultimate relation: the harmonic mean *b* of two segments *a* and *c* is the height of a rectangle with the arithmetic mean of the two segments as its base and equal to the rectangle with the two segments as its sides, i.e. equal also to the square with the geometric mean of the two segments as its side.

And since the arithmetic mean of two segments *a* and *c* is greater than the smallest of these segments, it follows that, given the two segments *a* and *c*, their arithmetic mean geometrically constructed, to also geometrically determine the harmonic mean, it was sufficient to solve the simple application problem, in this case solving the harmonic mean of two segments *a* and *c*.

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58 The testimony is by Giamblicus, cf. G. Loria, Le scienze esatte etc., p. 36.

certainly (even without the parallel theory); and we have thus also found the geometric relationship between the three averages.

The example of the harmonic mean we have given (8 as the harmonic mean between 12 and 6) makes it clear why Arita or the Pythagoreans called the sub-contrarian mean harmonic. These numbers in fact express the lengths of the first, third and fourth (and last) strings of the Greek tetrachord (Orpheus's lyre); that is, in modern terms, the respective lengths of the strings (which, given the same tension, diameter, etc.) give the fundamental note, the fifth and the octave [59].

This leads one to see the relationships that the Pythagoreans discovered (or established) between the chords of the tetrachord, and likewise of the octave (called harmony in Greek).

Philolaus tells us this, in part, in one of his fragments[60]. Philolaus says: "The extension of harmony is a fourth plus a fifth [let us use the modern terms of fourth and fifth for clarity]; the fifth is stronger than the fourth by nine octaves". Which means: having taken a chord and taken the chord that gives the first harmonic sound, i.e. the chord that gives the octave, and having in this way the two extreme chords of the tetrachord, the harmony, i.e. the octave, is

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59 The terms fourth, fifth and eighth can already be found in NICOMACO, ed. Teubner, p. 122.

60 Cf. CHAIGNET, *Pythagore etc.*, which quotes the fragment; vol. I, p. 230.

extends by the addition of two intermediate strings which are our fourth and fifth. We thus have the tetrachord composed of four strings which are (for us) neatly those of *C*, *F*, *G* and the upper *C* (the intermediate string in the double tetrachord)[61]. Considering the lengths of these strings, instead of the frequencies or pitches of the sounds emitted as they are used today, frequencies that are the inverse of the lengths, it is known how Pythagoras found the lengths of these strings experimentally. He found that the length of the last string was half that of the first, and that the length of the second, i.e. the *F* string was simply the arithmetic mean of the lengths of these two extreme strings. With regard to the *G* string, the sound of which gives the ear the sensation of an interval with respect to the lower *C*, equal to the length of the first.

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61 This tetrachord is none other than the lyre of Orpheus, an instrument with which acting and also singing were accompanied. Os- serva A. TACCHINARDI in his *Acustica musicale* (1912, Hoepli, p. 175), that it is "remarkable that the tetrachord contains the most characteristic intervals of the voice in declamation. In fact, by interrupting, the voice rises by a fourth; by reinforcing, it rises again by a degree; and finally, by concluding, it descends by a fifth'. It should also be borne in mind that "the Indo-European accent was a pitch accent; the tonic vowel was characterised, not by a raising of the voice, as in German and English, but by an elevation. The ancient Greek 'tone' consisted of an elevation of the voice, the tonic vowel was a higher vowel than the atonal vowels; the interval is given by Dionysius of Halicarnassus as an interval of a fifth' (A. MEILLET, *Aperçu d'une histoire de la langue grec- que*, Paris 1912, p. 22; see also p. 296).

the *C* interval above the *F* interval, had such a length that the four lengths in their order formed a geometric proportion. These lengths are in fact expressed respectively by $1, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}$; or in whole numbers, taking equal to 12 the length of the first chord, are expressed by the numbers 12, 9, 8, 6; and 9 being greater than 6, the length of the *G* string could always be determined by the method of simple application.

The length of the third chord is therefore 8, the sub-contrarian mean of 12 and 6; which is why Archita gives the name harmonic to this mean.

In conclusion, the four strings of the tetrachord have lengths that are simply established as follows: the last string is half the length of the first string, the second string has as its length the semi-sum of the lengths of the extreme strings, and the third string has as its length the arithmetic mean of the lengths of the extreme strings. All these lengths are constructed geometrically. If, instead of the lengths, the frequencies were taken, it would be found that the fifth has for frequency the arithmetic mean of the frequencies of the extreme strings, and the fourth the harmonic mean [62].

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62 In many physics and mathematics texts, it is said that the harmonic mean owes its name to the fact that the three notes of the major chord *C, E and G* form a harmonic progression in which the length of the *E* string is the harmonic mean of the lengths of the other two. This statement is incorrect, although

2. Let us now see what arithmetic, geometric and harmonic averages occur when considering the elements of regular polyhedra.

For the cube the thing is immediate. The cube has 12 edges, 8 vertices and 6 faces; it is precisely the numbers that give the lengths of first, third and last chord of the

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it is true that in the natural scale, the length of the *E* string is the harmonic mean of the lengths of *C* and *G*. But this is not the case in the Pythagorean scale.

In the natural scale, intervals are based on the law of simple ratios, and the harmonic mean of the lengths 1, $\frac{2}{3}$ of *C* and *G* is $\frac{4}{5}$ = length of *E*; like that of *D*= $\frac{8}{9}$ is the harmonic mean of those of *C* and *E*. The Pythagorean scale of Philolaus, on the other hand, hinges on the tetrachord; in it, the length of the third chord (*G*) is the harmonic mean of the lengths of the extreme chords; its elevation with respect to the first chord is the same as that of the last chord with respect to the second, and it is the same elevation that in spoken Greek occurred according to Dioges of Halicarnassus for the vowel on which the tonic accent fell. And the name of harmonic mean introduced by Archita derives from the property of the *G* chord in the Greek tetrachord, and not from the property of the *E* chord in the major chord of the natural scale, which does not exist at the time.

Philolaus tells us how intervals were set in the scale Pythagorean. One took the interval $\frac{2}{3}:\frac{3}{4}=\frac{8}{9}$ between the two strings middle of the tetrachord (*G* and *F*); and with it, starting from *C* and *sol* the lengths of the other strings were determined. This resulted in thus the lengths: *C*= 1, *D*= $\frac{8}{9}$, *E*= $\frac{64}{81}$, *F*= $\frac{3}{4}$, =

tetrachord. Furthermore, 8 is the first cube, it is the cube of the first number after unity. For this reason Philolaus calls the cube geometric harmony.[63] The numbers of its elements present the same relationship as the three chords first, third and fourth of the tetrachord.

The same, of course, could be said for the regular o- tahedron, which has 12 edges, 8 faces and 6 vertices.

In the regular icosahedron, denoting by R the radius of the circumscribed sphere, by *r* that of the circumference circumscribed to the pentagonal base of each angleid, and by l_{10} and s_{10} the sides of the regular decagon and the decalpha in

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$\frac{2}{3}$, *the*= $\frac{16}{27}$. In the natural scale, however, the length of the (mi) è $\frac{4}{5}=\frac{64}{80}$ with a difference of approximately $\frac{1}{100}$ by length of the Pythagorean *E*. In the Pythagorean scale, therefore, *E* is not the harmonic mean between *C* and *G*. And it is instead the third chord of the tetrachord (the *fifth* of our octave) that, due to its properties, suggested to Archita the term harmonic mean to designate the arithmetic mean of the inverses. In this way, and only in this way, can we understand the importance that the Pythagoreans must have attributed to this harmonic mean, which, with an identical mathematical law, was felt in music, in language, and in the dodecahedron, symbol of the universe. Naturally, this error recurs in philosophical texts. Robin, e.g., (LÉON ROBIN, *La pensée grecque*, Paris 1923, p. 75) takes for the four strings of the lyre *the low*, the *third*, the *middle* and the *high* represented (he says) by the integers 6, 8, 9, 12; and thus commits the double error of substituting the *third* for the fourth, and of inverting the order of the string lengths.

63 See NICOMACO, ed. Teubner, p. 125.

inscribed in it, we have found that: $s_{10} + l_{10} = 2R$. The arithmetic mean between s_{10} and l_{10} is therefore R, while for [9] the geometric mean is r. The harmonic mean can then be constructed; denoting it by M will be:

$$(s_{10} + l_{(10)}) \cdot M = 2 s_{10} l_{10}$$

and substituting

$$2R \cdot M = 2r^2$$

and since

$$r = \frac{4}{5} R^2$$

one has:

$$M \cdot R = \frac{4}{5} R^2$$

and finally

$$M = \frac{4}{5} R$$

Likewise, considering the radius R and the sum R + r of the two radii, we have found that their geome- tric mean is (R + r) - r = $3a^2$, where a denotes the apothem of the icohedron. And so, indicating by M the harmonic mean we have:

$$(2R + r) \cdot M = 6a^2$$

and since

$$2R = s_{(10)} + l_{(10)}$$

one will

have:

$$2 s_{10} \cdot M = 6a^2; \qquad s_{10} \cdot M = 3a^2$$

i.e. the harmonic mean between the sum of the radius of sphere circumscribed by the icosahedron and the radius of the circum-

circumscribed by the base pentagon and the radius of the sphere, is the height of a rectangle whose base is the side of the decalpha inscribed in this circumference and is equal to three times the square whose side is the apothem of the icohedron.

Coming to consider the elements of the regular dodecahedron and its face, we observe first of all the presence of two quaterns: the first constituted by the di- stances $2a$, s_{10}, r, l_{10} between the planes of two opposite faces, between the planes containing the other vertices of the two faces, and between them; the second by the side of the pentalpha and the segments de- terminated above it by the two sides of the pentalpha that intersect it, i.e. by the segments AE = $s(_{5)}$, AN_1 = EN = l_5, AN = EN_1, NN, of Fig. 26. In both of these quaterns of segments, each of them is the golden part of the one preceding it.

Now, if we denote by a, b, c, d four consecutive segments of the succession that is obtained by taking its golden part as the consecutive segment of a segment, we have

$$a=b+c \qquad b=c+d$$

and thus $a+d=2b$; thus: the second term of the succession is the arithmetic mean of the extremes.

One then has:

$$b^2=ac; \qquad c^2=bd$$

so

$$bc=(a-c)\,c=ac-c^2=b(2)-c^2=(+c)(b-c)=ad$$

On the other hand, denoting by M the harmonic mean of the extremes a, d, it is such that:

$$ad = \frac{a + d}{} - M\,2$$

i.e. replacing, that:

$$bc = b - M$$

therefore it is none other than the third segment c. It is therefore possible to state the property that, if four segments are consecutive segments of a sequence such that each segment is followed by its golden part, it follows that the second segment and the third segment are the arithmetic mean and the harmonic mean of the extremes respectively.

Exactly the same thing happens for the lengths of the second and third strings of the tetrachord with respect to the lengths of the extreme strings.

Considering then the quatern $2a$, s_{10}, r, l_{10} of the segments determined above the conjunction of the vertices of two opposite faces of the dodecahedron by the planes of the faces and the planes containing the other vertices, we have 1º - the distance s_{10}, (i.e. the side of the dec-alpha inscribed in the face) is the golden part of the double of the apothem and is the arithmetic mean between the double of the apothem and the side l_{10} of the decagon inscribed in the face (i.e. the distance between the planes containing the intermediate vertices); 2º - the distance between one of these planes and the nearest face, i.e. the radius *r of* the circumference circumscribed to the face, is the harmonious mean between 2a and $l_{(10)}$.

Similarly, the side l_5 of the regular pentagon in- written is the golden mean of the side s_5 of the pentalfa, and is the arithmetic mean between the side of the pentalfa and the side of the pentagon $NN_1N_2N_3N_4$ in Fig. 26; while the side AN of the pentalfa point is the harmonic mean between the side of the pentalfa and the side of the pentagon $NN_1N_2N_3N_4$.

In the dodecahedron, the distance 2a of the opposite faces, and in the face the side of the pentalpha, are thus subdivided so as to constitute two quaterns of segments, such that the middle segments are obtained from the extremities by taking their arithmetic mean and their harmonic mean, just as the two middle chords of the tetrachord are obtained from the extremities.

Taking s_{10} and r as extreme segments, we find by arithmetic mean a [15]; and by harmonic mean M we have:

$$a - M = r s_{10} = (s_{10} - l_{10}) s_{10} = s_{10}^{2} - s_{10} l_{10}$$

and for [9] $$a - M = s_{10}^2 - r^2 = (s_{10} + r)(s_{10} - r) = 2 a l_{10}$$

and finally $$M = 2 l_{10}$$

Similarly, the arithmetic mean between s_5 and l_5 is R' [16], and the harmonic mean is given by 2 ($s_5 - l_5$), which equals 4 (s_5 - R') and 4 (R' - l_5), and is twice the AN side of the pentalfa tip.

In these two quaterns, the fourth segment is the golden part of the first, and the two intermediate segments the arithmetic mean and harmonic mean of the extremes.

Finally, we have M as the harmonic mean of 2a and s_{10}:

$$(2a+s_{10}) - M = 4a - s_{10} = 2(s_{10}+r) - s_{10} = 2s^2_{10} + 2s_{10} - r$$

and for the [17]

$$(2a+s_{10}) - M = 4ar + 2s_{10} - r = 2r - (2a+s_{10})$$

and thus the harmonic mean between 2a and s_{10} is equal to the diameter of the circumcircle circumscribed by the face.

The existence of these harmonic averages, and of these species of tetrachords constituted by the elements of the dodecahedron and its face, must not have escaped the notice of the Pythagoreans (at least the later ones), and especially the tetrachord formed by the elements $2a$, s_{10}, r and l_{10} must have constituted in their eyes a significant confirmation of the symbolic reasons which made the dodecahedron the governing geometrical symbol of the universe; we say confirmation insofar as this correspondence between the dodecahedron and the universe is based on still other reasons.

3. The five regular polyhedra were called *co-semic figures* because they were regarded as symbols of the four elements and the universe. The dodecahedron was the symbol of the universe. If we want to see why, we only have to read a few pages of Plato's '*Timaeus*'. Let us summarise by using the Acre version[64]. Timaeus observes that 'every species of body has depth every depth must have a plane, and a plane right is made of triangles', in other words, every polygonal plane surface is made of triangles and correspondingly every polygonal plane surface is made of triangles.

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64 PLATO, *I Dialoghi*, volgarizzati da FRANCESCO ACRI, Milano 1915, vol. III, p. 142-45.

every polyhedron decomposes into tetrahedra: so that the plane corresponds to the number three of the vertices determining the triangle and the four to the number of vertices determining the tetrahedron. Two, as is known, corresponds to a straight line that is identified by two points. The point, the line, the plane or triangle and the tetrahedron are the elements of geometry, like the numbers: one, two, three and four are the numbers whose combination gives the whole decade. Due to the fact that every polygon is composed of triangles, the Pythagoreans said that the triangle is the principle of generation [65].

"Triangles" Timaeus continues, "arise from two species of triangles, the isosceles right triangle and the scalene right triangle. These are placed as the principles of fire and of the other bodies [elements]; and with them the four bodies [the four elements, i.e. the surfaces of the polyhedra, symbols of the four elements] are composed·

Since scalene right triangles are in-numerous (distinguished by their shape), Timaeus chooses the one
"beautiful" having the following properties: 1st - with two of them an equilateral triangle is formed; 2nd - the hypothetical double of the lesser cathetus; 3rd - the square of the greater cathex is triple that of the lesser. With six of these triangles, an equilateral triangle is formed (or vice-versa).

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65 See PROCLO, ed. Teubner, p. 166, 15. For other sources see CHAIGNET, vol. II, p. III.

66 What is within the brackets has been added by us for clarification.

verse, taken an equilateral triangle, the diameters of the circumscribed circle passing through its vertices de- compose it into six such triangles), and with four of these equilateral triangles, one obtains the regular tetrahedron,

"by which a sphere can be divided into twenty-four parts similar [in shape] and equal [in volume]". With eight of these equilateral triangles, one obtains the octahedron (composed, therefore, of 48 of these triangles); the third body, the icosahedron, has twenty triangular and equilateral faces, and thus twice sixty of these elementary triangles. Other regular polyhedra with triangular faces do not exist[67]. With the isosceles right triangle the cube is formed; for four isosceles triangles form a square (or also, the square is divided by the diameters passing through the vertices into four iso- chosen right triangles), and with six squares the cube is formed, which thus consists of twenty-four isosceles right triangles. Thus, says Timaeus, there remains a fifth form of composition, "which God used for the design of the universe".

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67 Timaeus seems quite sure of the fact. Mieli (p. 262 of his work) absolutely excludes that the Pythagoreans were able to recognise the impossibility of the existence of six regular polyhedra, and he quotes in a footnote, he does not say whether in support of this exclusion, but it seems so, Euclid's demonstration (XIII, 18) in his Greek text. It seems to us that the Pythagoreans could well have arrived at this; in any case it is certain that they knew of the five polyhedra that do exist.

At this point Plato silences Timaeus, perhaps out of reverence,[68] perhaps because in the case of the dodecahedron there is some difference. But applying the same method of decomposition into triangles to the faces of the dodecahedron, the pentagon with its diagonals gives the pentalpha, and the figure is divided into thirty right triangles with diameters passing through the ten vertices of the pentalpha. The surface of the diodehedron is therefore decomposed into 30× 12= 360 triangles.

''beautiful' dear to Timaeus. Now the number twelve (which also appears in the other polyhedra) already had a sacred and universal character of its own; twelve was the number of the zodiacal divisions and twelve in Greece, Etruria and Rome, was the number of the gods, twelve was the number of the Etruscan and Roman bundle rods, and an Etruscan dodecahedron and many surviving Celtic dodecahedrons indicate the importance of the number twelve and the dodecahedron [69]. The number 360 was then the number of divisions of the Chaldean zodiac, and the number of days in the Egyptian year, facts presumably known to Pythagoras. For these reasons, the dodecahedron naturally presented itself as the symbol of the universe.

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68 Plato's silence in this regard also gave Robin the eye, who says (LÉON ROBIN, *La pensée grecque*, Paris, 1923, p. 273) that '*au sujet du cinquième polyèdre regulier, le dodécaedre... Platon est très mysterieux*'. The Robin does not envisage any reason for such mystery.

69 See ARTURO REGHINI, *Il fascio littorio*, in the magazine 'DOCENS' 1934-XIII, numbers 10-11.

This is fully confirmed by what two ancient writers say. Alcinous[70], after explaining the nature of the first four polyhedra, says that the fifth has twelve faces, just as the zodiac has twelve signs, and adds that each face is composed of five triangles (with the centre of the face as a common vertex), each of which is composed of six others. A total of 360 triangles. Plutarch[71], after noting that each of the dodecahedron's two pentagonal faces consists of thirty scalene right-angled triangles, adds that this shows that the dodecahedron represents both the zodiac and the year, since it is divided into the same number of parts.

And just as the universe contains within itself and consists of the four elements, fire, air, water, and earth, so the dodecahedron, inscribed in the sphere like the cosmos in the fascia (the περιέχον), contains the four regular polyhedra that represent them. We have seen, in fact, how the regular hexahedron can be inscribed in it and in the sphere; it can easily be shown that the icosahedron having as vertices the centres of the faces of the dodecahedron is regular; so, too, a regular octahedron can be obtained by taking as vertices the centres of the faces of the cube; and by uniting a vertex of the cube with the opposite vertices of the faces congruent with it

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70 ALCINOUS, *De doctrina Platonis*, Paris 1567, ch. II. Cf. also the work of H. MARTIN - *Études sur le Timée de Platon*, Paris 1841, II, 246.

71 PLUTARCUS, *Platonic Matters*, v. I. Of course it is the Egyptian year, although Plutarch forgets to specify this.

and these three among them, it is shown that a regular tetrae- dro is obtained.

The tetrad of the four elements is contained in the uni- verse, the κόσμος, and this in the band, as the four polyhedra in the fifth and in the circumscribed sphere. Thus the te- trade of points, straight lines, planes and bodies is contained in space and constitutes it; and four points identify the polyhedron with the least number of faces and identify a sphere; Thus the sum of the first four integers gives the unity and totality of the decade (a number that belongs as much to the linear numbers of the natu- ral series as to the triangular numbers, as to the pyramidal numbers, and this regardless of the fact that ten is taken as the basis of the numbering system); thus the four notes of the tetrachord constitute harmony. The tetrahedron, the tetrad of the four elements, the *tetractis* of the four numbers, and the tetrachord are thus intimately related to each other, and to the four elements of the dodecahedron $2a$, s_{10}, r, $l_{(10)}$, each of which has the golden part that follows it, and of which the middle chords have exactly the same relation to the extremes as the middle chords to the extremes of the tetrachord, and which identify the four planes containing the vertices of the dodecahedron. And one can understand why the catechism of the Acusmatics identifies the oracle of Delphi (the navel of the world) with the *tetractis* and the harmony.[72]

The golden part is of great importance in the struc-
ture of the pentalfa and in that of the dodecahedron symbol

72 See LÉON ROBIN, *La pensée grecque*, Paris 1923, p. 78.

of the universe. It is therefore also understandable why the golden part is of such importance in pre-peri- clean architecture[73]; and many other things could be said about the influence and relations between Pythagorean geometry, co- smology, architecture and the various arts.

The digression would be too long, however. We will merely observe that the development of Pythagorean geometry thus ends (in both directions of the word) in the inscription of the dodecahedron in the sphere and the recognition of its properties, as we know it actually did.

Euclid too, according to Proclus' attestation[75], set as the final aim of his elements the construction of Platonic figures (regular polyhedra); and perhaps from Pythagoras' time to that of Euclid this final aim has traditionally remained the same; But while in Euclid the intent was purely geometrical, in Pythagoras the properties of the dodecahedron showed, if not proved, the existence in the cosmos of that same ar- mony that the ear and experience discovered in the notes of the tetrachord.

This was, we believe, the profound link that united geometry to cosmology, and provided the basis and impetus for the

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73 M. CANTOR, *Vorlesungen über Geschichte der Mathematik,* 2ª ed. I, 178.

74 The consideration of the harmonic mean connected to Polycletus' canon of statuary; cf. L. ROBIN, *La pensée grecque*, p. 74.

75 Cf. LORIA, *Le scienze esatte* etc., p. 189.

We can now understand with some precision, and no longer vaguely, how Plato could write that "geometry is a method for directing the soul towards the eternal being, a preparatory school for a scientific mind, capable of directing the activities of the soul towards superhuman things", and that "it is even impossible to arrive at a true faith in God if one does not know mathematics and its intimate bond with music"[76]. (76) For the Pythagoreans and Plato, geometry was therefore a *sacred* science, i.e. an exotic science, while Euclidean geometry, by breaking all conventions and becoming an end in itself, degenerated into a *profane* science.

No trace remains of this particular link between cosmology and music, perceptible in the tetrachord formed by the constituent elements of the dodecahedron, but in this case we believe that the absence of any material trace is not accidental, for this must have been one of the secret teachings of our school; and a clue to the fact is provided by Timaeus' immediate reserve in the Platonic dialogue of the same name as soon as he comes to speak of the dodecahedron.

Thus we can assume that we have taken an important step towards the restitution of Pythagorean geometry, not only from the modern point of view of restoring pure geometric construction, but from the Pythagorean point of view of studying the cosmos in order to discover

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76 Cf. LORIA, *Le scienze esatte* etc., p. 110.

connections between geometry and other sciences and disciplines.

Other things could be added on the subject, but we too must Pythagoreanly bear in mind: μὴ εἶναι πρὸς πάντας πάντα ῥητά.

CHAPTER VI DEMONSTRATION OF THE 'POSTULATE' OF EUCLID

1. Starting from the two-right theorem, and with the help of the consequent Pythagorean theorem, but without resorting to the theory of parallels, similarity and proportion, it is therefore possible to arrive at all the Pythagorean discoveries mentioned by Proclus, with the only restriction that the problem of the simple (*parabola*) application cannot be solved in all cases, but only in a special case, however important and sufficient to permit the full development of Pythagorean pious and solid geometry as we have been able to see it so far. And we have noted the eloquent fact that for the problems application, the testimony adduced by Proclus is not the authoritative testimony of Eudemus, but only that of the co-authors who were around Eudemus.

It will be objected that this is not enough to prove with absolute certainty that what we have reconstructed is indeed Pythagorean geometry. We know this perfectly well, but we also know that, given absolute lack of any direct document, which we should have taken into account as an eloand not as a pro- vided document, it was not possible to do more; and we know that in this circumstance even indirect evidence, which we have collected by the road, has its value in favour of our thesis.

In the development of Pythagorean geometry, we have limited ourselves to what is necessary to achieve the results mentioned by Proclus, but there are still other results that can be achieved.

The problem of simple application, which corresponds to solving the equation $ax = bc$ or $ax = b^2$, can be solved if a is greater than b or c. If this is not the case, the certainty of the existence of the solution can only be achieved if the property postulated by Euclid with his V postulates is available. A similar difficulty is encountered in other important questions. Thus, given three points on a circumference, it is proved that the axes of the three chords pass through the centre; but it cannot be proved in general that a circumference always passes through three unaligned points.

Now, faced with this obstacle that bars the way to the further development of geometry, how could the Pythagoreans behave? We have seen what important reasons lead us to believe that they did not admit the postulate of parallels, nor the concept of paralel as defined by Euclid; we now propose to show how they could, equally, overcome the dif- ficulty.

Let us observe how it is known how, knowing in any case the two-right theorem (Saccheri's proposition), it is possible, admitting Archimedes' postulate, to demonstrate with Legendre [77] the uniqueness of the non-secant of a given straight line passing through an assigned point (property equivalent to the postulate of the parallels); and let us also observe how Severi, admitting his postulate of the parallels [78], demonstrates, again with the help of Archimedes' postulate, the uniqueness of the non-secant. The thing is then possible by using the postulate of Archimedes, except that we cannot think of using this postulate because Archimedes is posterior even to Euclius, and it is not probable that the Pythagoreans would have admitted a postulate like that of Archimedes.

On the other hand, it is true that Archimedes' postulate is sufficient to allow result to be reached, but is it also necessary to have recourse to it? And if it is not necessary, could the Pythagoreans, without it and in a simpler way, achieve the result, i.e. demonstrate the uniqueness of the non-secant of a given line passing through an assigned point?

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77 Cr. R. BONOLA in ENRIQUEZ, *Matters Concerning etc.*, p. 323.

78 Cf. SEVERI, *Elements of Geometry*, Florence 1926, vol. I, p. 119.

We shall see that we do, and we shall see how; but is necessary for us to premise other propositions deduced from those already seen.

2. THEOREM: *If two lines* a *and* b *are both perpendicular to the same line AB, every other perpendicular to one of them also meets the other and is perpendicular to it.*

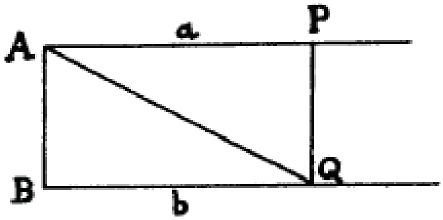

Fig. 38

Let the two straight lines *a* and *b* (fig. 38) be perpendicular to AB; and from a point P of *a* we lead the perpendicular to *b*. Its foot Q is necessarily distinct from B, because otherwise two perpendiculars to *b* would emerge from B. And since AB and PQ perpendicular at different points to the same line cannot meet, the points P and Q must be on the same side with respect to AB. By joining A with Q, the triangle ABQ is right-angled, and

then $\widehat{AQB}$ is smaller than the right angle $\widehat{PQB}$; the QA thus divides this right angle into two parts, and since we know that the two acute angles of the right triangle are complementary, the two angles $\widehat{AQP}$ and $\widehat{QAB}$ result in the following are equal because they are complementary of the same angle $\widehat{AQB}$. The two triangles ABQ, QPA, also having equal angles $\widehat{AQB}$ and $\widehat{QAP}$ because both complementaries of the same angle $\widehat{BAQ}$, are equal by the second criterion; and thus the angle $\widehat{APQ}$ is right, c.o.d.

On the other hand, since the perpendicular for P to a is unique, it coincides with PQ, i.e. the perpendicular PQ to *a* meets *b* and is perpendicular to it.

Remark: Any point P or Q of either line *a* or *b* has a constant distance from the other. In fact, since ABPQ is a rectangle, the side PQ is equal to the opposite side AB. Therefore two lines perpendicular to a third are equidistant from each other.

Conversely, if a point P in the plane on the side of A in relation to *b* has a distance PQ = AB from *b*, then we say that this point P belongs to the perpendicular to AB perpendicular to A, i.e. it lies on *a*.

Suppose in fact that the two points A and P situated on the same side of *b* have equal distances from *b* AB, PQ. The point P cannot naturally belong to AB, otherwise Q would coincide with B and therefore P with A; then Q and B are also distinct. Let us join A with Q; the angle $\widehat{AQB}$ of right-angled triangle AQB is acute e complementary of $\widehat{BAQ}$; the QA divides so $\widehat{BQP}$, and $\widehat{AQB}$ is the complement of $\widehat{AQP}$; hence i two triangles ABQ, QPA have AQ in common, = PQ and included angle are therefore equal; the angle $\widehat{PAQ}$ is therefore equal to the complement $\widehat{AQB}$ of $\widehat{BAQ}$ and therefore the angle $\widehat{BAP} = \widehat{BAQ} + \widehat{QAP}$

is equal to a straight line. The point P is therefore on the perpendicular to AB per A.

It follows that any other line passing through *a* cannot be such that its points have a constant distance from *b*; hence the *uniqueness* of the equidistant line; i.e. the

THEOREM: *A point is passed by one and only one line equidistant from a given line.*

The problem of leading to a point A the line equi-distant from a given line *b*, is solved immediately. It suffices from A to lower the perpendicular to *b*; and then from A to lower the perpendicular to this.

We have seen that all points of *a* and only they have constant distance AB from *b*.

This expressed by the

THEOREM: *The geometric locus of points in the plane located on the same side with respect to a given line and having an assigned constant distance from it is a line.*

This proposition is what Severi takes as his postulate, calling it the postulate of the parallels. For us, it is a theorem that is a consequence of the two-right theorem and therefore of the Pythagorean postulate of rotation. These three propositions are such that each of them carries the consequence of the other two; for we shall shortly see that from the proposition now stated, the two-right theorem can be deduced.

We finally observe that having proved the uniqueness of the equidistant from a line *b* passing through a point

It does not say that every other line passing through A must secant *b*; we can only say that *if* there are other lines passing through A that do not secant *b*, they are not equidistant from *b*: i.e. we have so far shown the *uniqueness* of the equidistant line, and we know nothing about the *uniqueness* of the non-secant line.

3. Certain theorems apply to equidistant lines that are analogous to those for parallel Eucli- dean lines.

THEOREM: *If a line meets two other lines and forms alternate equal internal angles with them, they are equidistant.*

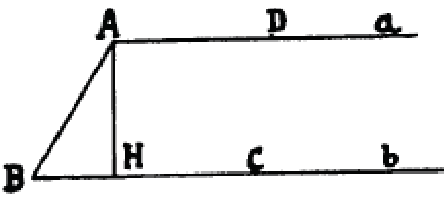

Fig. 39

Let *a* and *b* (fig. 39) be the two lines met by the traverse AB, and let the internal alternate angles be equal. It follows that the conjugate interior angles are supplementary. If these angles are also equal, i.e. if they are right angles, then *a* and *b* are both perpendicular to AB, and by the preceding theorem are equidistant. If the two angles are unequal and is for example $\widehat{DAB}$ A > $\widehat{BC}$, will be $\widehat{DAB}$ an obtuse angle and $\widehat{ABC}$ acute. Lowering from A the perpendicular AH to *b*, the foot H is located reto B on the acute angle side because a trian- gle cannot have more than one right or obtuse angle, and,

since the other angle $\widehat{BAH}$ of the triangle ret-
tangle ABH is acute, it follows that the AH divides the obtuse ango-
lo $\widehat{BAD}$ into two parts.

We have by hypothesis: $\widehat{ABH} + \widehat{BAD} =$ 2 straight and thus:

$\widehat{ABH}$ B + $\widehat{AH}$ + $\widehat{HAD}$ 2 = right ma

$\widehat{ABH} + \widehat{BAH} =$ a straight for the theorem of two straight:
then $\widehat{HAD} =$ a straight line; and the *a* and *b* perpendicular to the AH are two equidistant straight lines.

Note: The same applies if AB forms with *a* and *b* equal corresponding angles, equal alternate exterior angles, etc.

REVERSE THEOREM: *If a transversal follows two equidistant straight lines, it forms with them equal internal alternate angles, equal external alternate angles, etc., and so on.*

Suppose that AB (fig. 39) cuts the two equidistant straight lines *a* and *b*. If it were perpendicular to one of them we know that it would also be perpendicular to the other and the theorem would hold. If it is not, it will form angles with *a*

adjacent unequal; e.g. $\widehat{BAD}$ obtuse. Conducted by
A the perpendicular common to the two straight lines *a, b* it divides-
de BAD, and in triangle rectangle BAH the angle

$\widehat{ABH}$ is complementary to B $\widehat{AH}$;

$$\widehat{HBA} + \widehat{BAH} = \text{a rectum}$$

and

$$\widehat{HBA} + \widehat{BAH} + \widehat{HAD} = 2 \text{ rectified}$$

therefore

and

$$\widehat{HBA} + \widehat{BAD} = 2 \text{ recti}$$

The two internal conjugate angles are therefore supple-
mentary; and therefore the internal alternations are equal, etc.

Note: However, it is not proven that if two lines are equidistant, every secant of the first *must* also secant the second; therefore the simple application problem in the general case cannot yet be solved.

4. The demonstration of the two-rectified theorem attributed by Eudemus to the Pythagoreans, to which the passage in Aristotle's *Metaphysics* refers, now becomes possible.

Having taken vertex A of a triangle ABC (fig. 1) to be equidistant from the opposite side BC, by the equality of the interior alternate angles of vertices A and B, and A and C the theorem is proved in the well-known way.

Of course, this simple demonstration is a workhorse for us. Was it also for the Pythagoreans to whom Eudemus attributes the demonstration? Was it also for Aristotle? If it was not, i.e. if it was not based on the theorem of equidistant lines, derived from the two-right theorem, it must necessarily be based on this property of equidistant lines, which is either admitted by rule or deduced from an equivalent postulate; but the existence of the ancient proof of the two-right theorem mentioned by Eudemus would remain unexplained. In any case, this demonstration is based on the properties of equidistant lines, and is therefore valid whether or not one accepts or uses Euclius' postulate.

de. The equidistant is a non-secant, which unlike the other possible non-secants (or parallels according to Euclid's definition) enjoys the properties seen, and therefore allows the demonstration of the two-right theorem.

The ancient Pythagoreans, for the reasons we have seen, admitted neither Euclid's postulate nor a postulate on equidistant straight lines like that of Se- veri. If, as we believe, they did arrive at the concept of equidistant straight lines, it was as a consequence of the two-rectified theorem they proved with the unknown three-step proof, and not vice versa. Unless we want to suppose that at a certain moment a part of the Pythagoreans believed that they could take the concept of equidistant straight lines as a starting point, and derive from it the demonstration of the two-right theorem instead of the anecdotal demonstration.

After Euclid, Poseidonius and Geminus resorted to the concept of equidistant straight lines with the aim of eliminating Euclid's postulate; and other attempts were made, as is known later, but always in a non rigorous way, because, as Saccheri has shown, the admission that equidistant lines actually exist is to be considered as a new postulate [79]. This is Severi's postulate, equivalent to Saccheri's proposition, and to our Pythagorean postulate of rotation.

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79 GIOVANNI VAILATI, *Di un opera dimenticata del P. Girolamo Saccheri*, in *Scritti*, 1911, p. 481.

For us, it is a theorem because it is a consequence of the two-rights theorem, itself a consequence of the rotation postulate.

For the reasons we have seen, it is certain that the ancient Pythagoreans did not admit, but proved, the Saccheri proposition, and proved it in a way that is not true - such a way derives from a postulate of equidistant lines or from the very concept of equidistant lines; whereas it is at least possible that the demonstration was based on a postulate such as that of rotation.

If they accepted this postulate, not only could they deduce the two-right theorem and the Pythagorean theorem, but also all the discoveries attributed to them by Proclus-Eu-Demos, as well as the theory of equidistants and, in addition, the proof of the two-right theorem attributed to them by Eudemus.

5. THEOREM: *If a transversal meets two equidistant lines and from a point on one of them the line equidistant from the transversal is conducted, it also meets the other.*

Let *m* be the transversal of the two equidistant straight lines *a* and *b* (fig. 40), and let P be the point assigned above *a*. Let us join B with P, and take the segment BQ = AP on *b*, located with respect to *m* on the side of P. BP forms with *a* and *b* equal interior alternate angles; therefore the trian- gles APB, QBP are equal by the first criterion; therefore

also $\widehat{APB} = BPQ$ and the *m* and PQ are equidistan-

ti. And since we know that only one line passes through P equidistant from *m*, it coincides with PQ; thus the equidistant from *m* conducted through P point of *a* also encounters *b* at point Q.

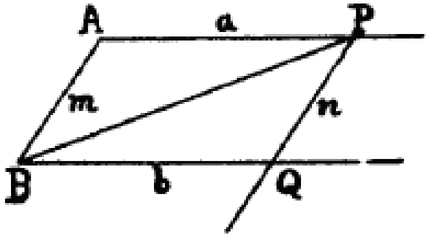

Fig. 40

Remark: the quadrilateral ABQP is a rhomboid. On the other hand, if ABPQ is a rhomboid, since a diagonal makes with the opposite sides alternate equal interior angles, they are equidistant. Therefore in the rhomboid and the rhombus, the opposite sides are equidistant.

This constant distance is called the *height of* the rom- boid.

THEOREM: *If a line equidistant from one of the other two sides of a triangle is taken to the midpoint of one side, it meets the third side at its midpoint.*

For the midpoint M of the side AB (fig. 41) of the trian- gle ABC, we take the line equidistant from BC. All the points of BC are on the same side with respect to it; the points A and B are on the opposite side with respect to it, and therefore the points A and C are also on the opposite side, and therefore the segment AC is cut at one of its points N by this line. We complete the rhomboid that has for consecutive sides MN, MB; side NP of this rhomboid is equidistant from AB and leaves point C and AB on opposite sides; hence vertex P between B and C.

Since PN= BM= AM, and it is $\widehat{MAN} = \widehat{PNC}$ because corresponding from to the equidistant AB, PN, and $\widehat{AMN} =$ NPC for similar reason, the AMN triangles, NPC are equal and thus AN= NC, i.e. N is the midpoint of AC.

Of course for the same reason P is the midpoint of BC and we have

$$MN = BP = PC = \frac{1}{2}BC$$

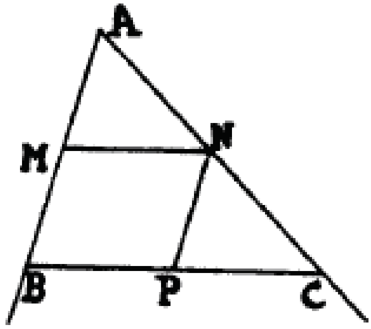

Fig. 41

REVERSE THEOREM: *The conjunction of the midpoints of two sides of a triangle is equidistant from the third side and equal to half of it.*

It is proved by absurdity, as a consequence of the uniqueness of the equidistant BC passing through M, and the uniqueness of the midpoint M.

As a consequence of these theorems, other theorems be demonstrated on the bundle of equidistant lines, the trapezium, etc.; the problem of dividing a segment into an assigned number of equal parts can be solved; it can be demonstrated that the three medians of a trianogram meet at a single point, etc. [(80).]

We will limit ourselves to the following theorem.

THEOREM: *If a segment equal to the side is taken on the prolongation of one side of a triangle, and the line equidistant from one of the other two sides is taken to the end of the segment, it meets the prolongation of the third side.*

Let AMN be the given triangle; take the segment MB = AM on the extension of AM, and the segment NC = AN on the extension of AN. Let us join B with C. By the preceding theorem, MN and BC are equidistant. Therefore the equidistant MN through B meets the prolongation of AN at point C.

6. We now want to demonstrate the fundamental property that for an assigned point A outside a given line *b*, only one line can be led that does not follow it.

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80 In a similar way, the theory of equidistant planes and the theory of equidistant planes can be developed. We could have premised these developments and then obtained with their aid many simplifications in various issues we have dealt with, but with a little patience we could do without them as well.

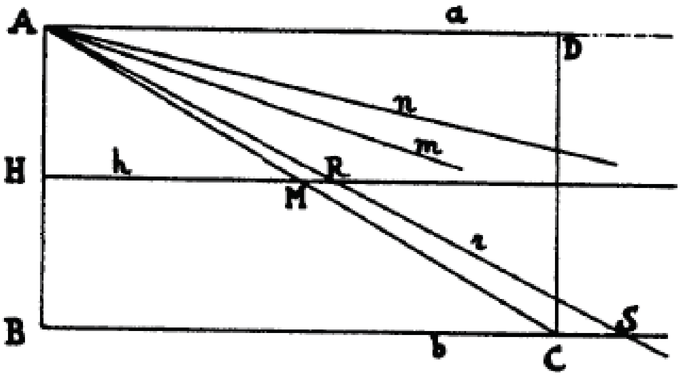

Fig. 42

From point A (fig. 42) let us take the perpendicular to b and let B be the foot; and from point A let us take *a* perpendicular to AB. We know that *a* and *b* both perpendicular to AB cannot meet. It must be shown that every other line passing through A and distinct from *a* is a secant of *b*.

Suppose it is possible that this does not happen.

There will then be, in addition to *a*, at least another straight line *m* that passes through A and does not meet *b*. The point A divides *m* into two rays situated on opposite sides of *a*; let us consider the ray *m* that is situated with respect to *a on* the side of the point *B*, i.e. of the strip of sides *a* and *b*. And let us consider the ray *a* and *b* situated with respect to AB on the same side of the ray *m*. The ray *m* is one of the rays of origin A and included in the ango-

lo $\widehat{BAa}$ of the half-lines AB and *a*, which by hypothesis does not *meet b*. In addition to this ray, there may be others of origin A that do not meet the ray.

b; indeed, there are certainly some, and they are all the half-lines of origin A and included in the angle $\widehat{ma}$, because if one of

if e.g. *n* were to meet *b* at a point N, since the ray *m* would be inside the angl $\widehat{BAN}$ of the trian-

ABN and would therefore leave the points B and N on opposite sides would have to saw off the segment BN contrary to the assumption made on *m*. Therefore, any line *n*, internal to the

the angle $\widehat{mAa}$, , is therefore a non-secant if *m* is

a non-secant.

On the other hand, more than

to the AB of the half lines included in $\widehat{BAa}$ and secant to the *b*. One of these is, for example, the one that forms the angle of 60° with AB and the angle of 30° with *a*; in fact, taking the segment AC = 2AB from A on this ray, and joining B with C and the midpoint M of BC, the isosceles triangle BAM has angle at the vertex of the triangle BAM.

c $\widehat{BAM}$ of 60° is equilateral; therefore the triangle MBC is isosceles and the angl $\widehat{ABC}$ is right, which means that the point C of the AM lies on *b*, i.e. that the AM is a bisector of *b*. Naturally, all half lines for A in-

triads at $\widehat{BAC}$ are secants of the half-line *b*.

On the other hand, the half lines of the bundle of centre A com-taken between ray AB and ray *a are* either secants of ray *b* or are non-secants of *b*. To the class secants belong AB, AC and all se-mirette within the angle $\widehat{BAC}$; and belong-

certainly also a part of the half lines

of origin A and internal to angle $\widehat{CAa}$; in fact, it is enough to take any point S on the prolongation of the segment BC on the side of C, and the ray of origin

A, passing through S, is included in the angle $\widehat{CAa}$ and is a secant of the ray *b*. To the class of non-secants belongs *a* for sure, *m* for hypothesis, and as we have seen, also all the ray semicircles of origin A and b.

internal ad $\widehat{mAa}$.

The class of ray segments with origin A and *secant to* ray *b* constitutes an ordered set, because it is in bi-univocal correspondence with the set of points of ray *b*. If we actually order them in correspondence, AB will be the first secant ray, followed by the others; and since is no last point of ray *b*, so there is no last secant ray of origin A; that is, after any secant of *b* in the ordered bundle of the centre rays A there are others.

With these considerations in mind, let us take the perpendicular line common to the straight lines *a* and *b* from point C. The semi-lines of origin A that follow AB and precede AC are in bi-univocal correspondence with the points of the segment BC; the semi-lines following AC are similarly in bi-univocal correspondence with the points of the segment CD, so that the semi-lines of the bundle with centre A between AB and *a are* in bi-univocal correspondence with the points of the orthogonal segment ABC, including the extremities. AB is the first of the secant rays, *a* the last of the non-secant rays *b*.

Let us make an observation at this point: The bi-univocal correlation between the points of segment BC and the half-line of the convex angle BAC projecting the segment from a point A off line BC, allows to order the set of half lines of the angle $\widehat{BAC}$.

In order to deduce from the ordinability of the line the possibility of ordering the semi-routes of a bundle, Severi[81] notes that it is necessary first to introduce the postulate of the parallels, and then in the correspondence to exclude one of the semi-routes from the bundle. This double necessity disappears if, instead of ordering the semi-routines in correspondence with the points of a straight line, one can order the semi-routines in correspondence with the points of the perimeter of a rectangle whose diagonals pass through A, and the correspondence is complete, no semi-route excluded.

Of course, in order to do this, one must know the tangles independently of the parallel postulate, which is precisely what occurs in the development of our Pythagorean geometry.

Having thus established the orderability of the set of the semi-routes of the bundle with centre A between AB and AD, and having established the direction of this order; and having observed that these semi-routes are necessarily secants or non-secants of the semi-route *b*, that every semi-route that precedes a secant is also a secant and every semi-route that follows a non-secant is also a non-secant, we observe that just as there is no last of the secants, there is no last of the secants.

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81 SEVERI, *Elements of Geometry*, vol. I, p. 177.

Thus, from a purely logical point of view, one might think that the first of the non-secundant rays does not exist or may not exist; that is, that given any ray that does not secant *b*, one can always find other pure non-secundant rays that precede it.

Intuition, however, observes that starting from the initial position AB, or even AC, and circling around A until arriving at the final position *a*, the ray that was a secant has eventually become a non-secant. If the metamorphosis did not occur at the final moment for the ray *a*, it must have occurred at a certain moment for an intermediate position, before which the ray had always been a secant and after which it had always been a non-secant. In short, it is intuitively evident that there is one and only one ray that is the first of the non-secant; and everything is reduced to showing that this first non-secant is none other than *a*.

From a logical point of view, it is correspondingly necessary to resort to a postulate; and it was natural and predictable that this should happen, otherwise the Pythagorean rotation postulate (or the equi- valent Saccheri proposition) would have been equivalent to Euclid's postulate; only it is not the Archimedes postulate but a much simpler case of the continuity postulate. The existence of a separating ray of the two classes of secant and non-secant rays *of b* must be admitted as a postulate.

so obvious to intuition that in the eyes of the ancients it must have been a given, a primordial truth so axiomatic that they did not even feel the need to postulate it explicitly. In fact, if Euclid did not feel the need to postulate the continuity principle in the two cases that we have expressly noted at the time, it would be strange to believe or presume that this is or must have been the case in a perfectly analogous case, and this two centuries before Euclid when Pythagoras was the first to make geometry a liberal science.

7. Let us therefore explicitly admit the *postulate* that there is at least one ray of origin A that separates the rays of origin A and secant to *b* from those not secant to *b*.

We know that it cannot be a secant so it will necessarily be a non-secant. Moreover, we immediately recognise, by absurdity, its uniqueness. It is therefore the *first* non-secant. We intend to show that no ray of the bundle A that is distinct from *a* can be the *first* non-secant, so that *a* is as we know non-secant, and is the first and only one.

Let us make an observation first:

If the perpendicular *h* to AB (axis of AB and equidistant from *a* and *b*) is taken to the midpoint H of AB (fig. 42), any ray to A that cuts *h* also cuts *b*. If in fact *r* bisects *h* in R, since HB is equal to AH and *b* is equidistant from HR bisects *b* as we know.

r, so a ray for A that does not cut *b* cannot cut *h* either; in particular, the first ray that does not cut *b* cannot cut *h* and is therefore contained in the strip *ah*.

Let us now demonstrate the

FUNDAMENTAL THEOREM: *A point that does not belong to a given line is passed by one and only one line that does not follow it.*

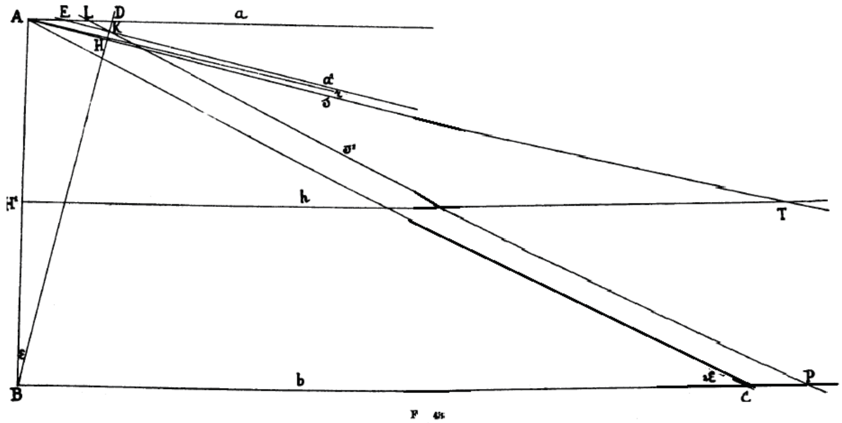
Let (fig. 43) A be the given point and *b* the given line. Let A be the perpendicular AB to the given line, and B be the foot. Then from A the ray *a* perpendicular to AB on the same side as the ray *b*, and from the midpoint H of AB the ray *h* perpendicular to AB on the same side as *a* and *b*.

Suppose it is possible that the ray *r* that forms with the ray at a certain angle δ (with $\delta \neq 0$) is *any* non-secant of *b* (possibly also the first). Then the first non-secant, i.e. the semi-line of separation of the secants from the non-secants we have admitted the existence of, cannot follow *r*, and therefore either coincides with *r* or precedes *r*, i.e. the semi-line of separation must form with *a* an angle $\varepsilon \geq \delta$ where $\varepsilon < 30°$ is certainly. Let it be *s*.

Then, taking the ray that forms the angle 2ε with the ray a, it cuts the *b* at a point C. Let us conduct for B the perpendicular to *s* and let H be the pie-

de. Since the anglê HAB of the triangle must be acute

rectangle AHB, the foot H stands on the ray *s*, and the an-
gl ABH= ε.



Since BH makes with BA an angle ε 30° and thus also less than 60°, it certainly meets the semi- straight line *a* at a point D. This also results from the fact that the s is all within the strip *ha*, because the *s does* not meet the *b* does not meet the *h* either, therefore B and H are on opposite sides of the h, BH meets the *h*, and therefore also the *a*.

One immediately has: BD > BA BH.>

Taking therefore BK equal to BA, it will be the point K taken between H and D. Rotating the figure around B by angle ε so that A goes on K, BA goes on BK and *a*, perpendicular to BA in A, goes on *a*' perpendicular to BK in K.

The *a*' and the *s*, both perpendicular to the BD are equidistant, and since K lies between H and D, D and the *s* lie on opposite sides of the *a*', and thus also D and A; therefore the segment AD is cut at one of its points E by *a*'.

With the rotation, *s* goes *on s*', which passes through K and forms with *a*' the angle ε, thus penetrating into the right angle

$\widehat{EKD}$ and meeting the segment ED at a point L.

The DA forms with the equidistant straight lines a' and *s* angles corresponding $\widehat{DEK}$, $\widehat{DAH}$ equals; thus

$\widehat{DEK} = \varepsilon$, the triangle LEK is isosceles and therefore the angle external $\widehat{DLK} = 2\varepsilon$.

Let us now take the segment CP= AL on the prolongation of BC, and join P with L. The triangles ALC and PCL have LC in common, AL = CP, and the angle is equal because the transversal CL forms with the two equidistant straight lines a and balternate angles internal equal; therefore

$\widehat{ALP}$ A= $\widehat{CP}$, and then $\widehat{PLD}$ A= $\widehat{CB} = 2\varepsilon$. Thus Both PL and KL form an angle with AD equal to 2ε; therefore the half lines LK and LP coincide, i.e. the three points L, K, P are aligned, i.e. *s*' meets *b*.

The triangle PBK is isosceles having the angles at the base complementary to ε, its vertex P is therefore on the axis of BK. Rotating this triangle around B by E in such a way that the base BK returns to BA, its axis goes on the *h*, the *s*' returns above the s, and the point P of the *s*' goes above the *h*. The *s* then meets the *h* at a point T. By now taking a segment TV= AT the point V of the *s* belongs to *b*.

So s is a secant of *b*.

Conclusion: the first non-secant *s* cannot form with *a* an angle $\varepsilon \geq \delta$; but we have seen that cannot form with *a* an angle less than δ either; therefore, if there existed a first non-secant *b* distinct from *a*, it would have to satisfy the condition of forming with *a* an angle which should be greater than, equal to or less than the angle S formed with *a* by any non-secant *r*. It follows that, since it is impossible to satisfy these conditions, such a first non-secant distinct from *a* does not exist; and therefore *a* a non-secant of *b*, it is the first and the only one among all the half-lines of origin A between AB and *a*, which does not secant *b*.

This demonstration can easily be transformed so as to dispense with the rotation movement at point B.

We conclude that, by admitting the Pythagorean postulate of rotation, or the equivalent two-right theorem (Saccheri's proposition), or the equivalent postulate of Se- veri over equidistant straight lines, Euclid's postulate can be demonstrated, either by resorting to Archimedes' postulate, or without resorting to Archimedes' postulate, and admitting only the existence of that half-line separating secants from non-secants, which must have appeared undeniable to the intuition of the ancients.

8. Having proved Euclid's postulate, we are naturally back in the realm of non-archaic Euclidean geometry; and our task is finished.

We are interested in the restitution of the Pythagorean geometry, not insofar as it agrees with the Euclidean geometry, but insofar as it differs from it. That it differed substantially is proved by the existence of that archaic demonstration of the two-right theorem, which could not be based on the properties of alternate internal angles.

In order to obtain this demonstration, we have resorted to the supposition that the Pythagoreans admitted the Pythagorean postulate of rotation, which we have enunciated, and we have seen that the theorem of the two right angles immediately follows in the first particular case mentioned by Eutocius, then in the other cases, and we have seen that from there we can certainly derive the theorem of Pythagoras, and we can arrive at all the discoveries attributed to the Pythagoreans by successive developments. Having done this, and always without introducing the concept of parallels and its postulate, we have been able to arrive at the theory of equidistant lines, which alone allows the most recent demonstration of the two-right theorem reported by Aristotle and attributed by Eudemus to the Pythagoreans.

We know what objections can be raised to the adoption of the Pythagorean postulate of rotation, which presupposes the concept of *rigid motion of* the plane, and the ability to recognise the equality of figures by superposition. But this is a theoretical problem in which we are not interested; we are interested instead in seeing whether or not the Pythagoreans may have explicitly adopted this postulate of rotation.

As proof of the fact that they did not admit the postulate of the parallels, defined as in Euclid, we put forward the reason that for the Pythagoreans the concept of infinity was identified with that of the imperfect. Now, for a similar reason, from a Pythagorean point of view, one could object that they could not accept or rely on the concept of motion either. For in the series of Pythagorean oppositions, just as the concept of finite and perfect is opposed to the concept of infinite and imperfect, so, correspondingly, the concept of im- mobility is opposed to that of movement. This is for us a much more serious objection than the other.

Following a pure rule of schematic consistency, both the concept of infinity and that of motion should have been banned. But we must bear in mind the links that bind the geometrical conceptions of the Pythagoreans to the cosmological ones; and if "no one has ever seen two parallel lines in the sense mentioned above, that is, two lines that, prolonged indefinitely, never meet"[82], on the other hand, anyone sees and knows from experience that movement is an essential character of human and universal life. The stars, i.e. the gods, moved eternally in their celestial dances. And according to the Pythagoreans, the circular movement was the perfect one, perhaps not only because of its regularity and simplicity, but also because centre and axis of rotation remained im-

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82 GIUSEPPE VERONESE, *Appendice agli elementi di geometria*, Padua, 1898, p. 23.

mobile and participating in perfection. Admitting, therefore, that a straight line situated at any finite distance from the centre of rotation also rotated, was admitting what seemed to occur in the universe with the rotation around the earth or the central fire or the sun (Aristarchus of Samos), and admitting that the angle of the initial vector ray with its final position was equal to the angle of the initial and final positions of the straight line, was admitting a fact that conformed to intuition and was verified by experience in the field our observation could reach.

Veronese[83] says "that it is truly a credit to Euclid that he did without movement where he could, since in his *elements* the tendency to avoid it as much as possible is clear". So if Euclid, although reluctant, makes use of motion, before him he had to make even greater use of it, and we thus have a proof that the Pythagoreans made use of it without so many scruples and that they could therefore also very well make use of a postulate relating to the motion of rotation such as the one we have stated. In time, the Pythagorean point of view, which intimately bound the various sciences together, was less and less taken into account, accentuating the tendency to make geometry a separate, purely logical science; and Euclid, by admitting his postulate, achieved the double aim of freeing himself more and more from the concept of motion and of obtaining a

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83 G. VERONESE, *Appendix to the Elements* etc., p. 38.

convenient and rapid means of solving difficulties that can only be overcome with much more patience and work. On the other hand, he introduced his postulate that has never satisfied anyone and that D'Alembert called 'the stumbling block and scandal of geometry'.

9. To recapitulate, let us consider two half-lines *a* and *b* perpendicular at two points A and B to the same line AB. They do not meet; and this results from the mere fact that only one perpendicular to a given line can be led from any point in the plane.

Secondly, if one admits the Pythagorean postulate of rotation or the Saccheri proposition, one has that these lines are also equidistant[84].

Thirdly, if one also admits the Ar- chimedean postulate or the special case of the con-

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84 Previously, assuming was known that two perpendicular lines at distinct points to the same line cannot meet, we deduced that a line *r* with a rotation of half a revolution around a point O outside it takes a position such that *r* and *r*' do not meet. This fact, however, is but a consequence of the Pythagorean postulate of rotation. In fact, with this rotation, a point A of *r* goes on the symmetrical A' of A with respect to O; and A' does not belong to *r*, because otherwise O should also belong to *r*. On the other hand, if *r* and *r*' had a point P in common, they should, according to the Pythagorean postulate, form an angle of 180°, that is coincide, and this cannot happen because A' of *r*' does not belong to *r*: therefore they do not meet.

tinuity that we have used, we have that the half-line
a is the only ray of origin A that does not secede from *b*.

Let us return after this to the question of the second Pythagorean demonstration of the two-right theorem. According to Proclus, Eudemus would say verbatim like this: 'Let there be a triangle αβγ and let the parallel to βγ be conducted by α... (καὶ ἤθω διὰ τοῦ ᾶ τῇ βγ παράλληλος ἡ)'.

Here the term parallel appears and the determining article ἡ implies its acknowledged uniqueness; but even if Proclus may have quoted the term used by Eudemus, it remains to be seen whether Eudemus used the term parallel in the sense attributed to it by Euclid's later definition, and whether the notion of the uniqueness of this line also originated in Eudemus from the acceptance of a postulate such as that admitted by Euclid.

Aristotle in the passage of the Metaphysics in which he refers to this same demonstration also conducts by the vertex α the straight line that serves the demonstration, but he does not call it either parallel, equidistant, or non-secant; he simply says: εἰ οὖν ἀνῆκτω ἡ παρὰ τὴν πλευράν, i.e.: if one conducts the [straight line] alongside [or opposite] the side...

In this passage, too, ἡ shows that this ret- ter is considered unique, but again, it is not defined in any way and it is not known where this uniqueness derives from.

The obvious etymology of the word parallel does not shed any light on this; the term is used in astro- nomy for the parallels of the celestial sphere; and it is used in the ordinary language by Aristotle, as later, for example, by Plutarch in '*Parallel Lives*'.

From ordinary language it then switched to geometrical language, but when and with what precision is not known. Aristotle uses it three times in the *Analytics*, as a geometrical term, and judges that those who endeavour to describe parallels are committing a petition of principle.

As things stand, the passage of Eudemus and that of his master Aristotle do not prove at all that the later description of the Pythagoreans was based on a definition of the parallel and a postulate equal to the definition and postulate of Euclid. And it cannot be ruled out that this line was the equidistant, and was called the *parallel*, and was considered the only non-secant simply because the doubt had not yet arisen that there could be other non-secant lines besides the equidistant. In that case the doubt would have arisen later, and Euclid would have eliminated it of his own accord by introducing his postulate. In such a case, Aristotle's proof would be correct if such a line con- ducted to the vertex of the triangle is understood to be equidistant, and would be incorrect if conceived as a parallel, its uniqueness was assumed without basis; whereas Eudemus's proof would be correct if the term parallel is understood to mean equidistant (whose uniqueness and properties the Pythagoreans could deduce from the two-right theorem) and would be incorrect if it designated a parallel in the Euclidean sense and did not admit or prove Euclid's postulate.

However, the two passages, by Aristotle and Eudemus, do not prove that the later Pythagoreans gave a demonstration of the two-right theorem identical to that of Eu- clid. If, as it seems to us, this later Pythagorean demonstration was based on the properties of equidistant lines, even if we call them parallels, this demonstration was also independent of the concept of lines that when extended to infinity never meet, and of Euclid's postulate, which are so little in agreement with the Pythagorean conception.

Finally, we note that in the demonstration we have given of the uniqueness of the non-secant, there is no need to extend the line to infinity, and therefore it too squares with the Pythagorean conception. And we also note that, even if we do not want to agree that the Pythagorean geomatics were based on our Pythagorean postulate of rotation, the demonstration of Euclid's "postulate" that we have exposed can be done in the same way, if we admit the Saccheri proposition or the Severi postulate. And since the Pythagoreans certainly knew the two-rectified theorem independently of parallel theorem, it is thus clear that they could, from the two-rectified theorem and without Euclid's postulate, demonstrate the uniqueness of the non-secundity. The question did not transcend their means, nor certainly the intelligence of those so-called 'primitives'.

10. The transformation of Euclid's postulate into theory is a secondary result of our study. And it is beyond the scope of this study, nor do we presume to be so, to judge whether the Euclidean set-up of geo- metry is, from a modern theoretical point of view, preferable to the ancient set-up we have tried to reconstitute. Of course, all postulates are convenient; and by cutting the Gordian knot of parallels with the sword of Euclid's postulate, things are simplified. But having to choose between the fifth postulate and the Pythagorean postulate of rotation, which one is less difficult? Which one is less restrictive? Appreciation in these matters is also somewhat personal, and we let everyone choose according to their own taste.

We are interested in observing that the Pythagorean postulate of rotation allows us to prove the two-right theorem and the Pythagorean theorem independently of the postulate and the theory of parallels in a way that seems to be the ancient way, and by itself allows us to obtain the entire development of Pythagorean geometry; and we do not know that until now we have found a way, not only more satisfactory, but any way at all, to achieve the same result. The postulate of continuity to which we have resorted has only served to resolve the last question, that of proving Euclid's "postulate" in a way that does not transcend the possibilities of the Pythagoreans.

11. Once Euclid's fifth postulate has been introduced as a postulate, the property enunciated by the postulate pita-

gorical rotation loses all importance. It is therefore not surprising that no trace of it can be found on the surface. It would be strange if it had been otherwise when all traces of Pythagorean demonstration have been lost except for the late demonstration of the two-right theorem.

If our reconstruction is correct, the introduction of Euclid's postulate must have profoundly upset the structure of geometry; and this is also in accordance with the information we have on the subject, for we know that Euclid changed the order and the dimensions and in general altered the whole structure of geo- metry, so that, for example, the Pythagorean theorem became the last and was given another demonstration.

The almost uncontested favour enjoyed by the '*Elements of Euclid*' for more than twenty centuries, added to these unfavourable conditions for the transmission of Pythagorean geometry, led to the exaltation of the Greek-Alexandrian school, to the detriment of the 'Italic School'.

Of the Greek school, everything, or almost everything, has come down to us; of our school, of the school he created from the ground up, nothing has been saved. An adverse fate seems to have raged against the great philosopher's vast and daring work. The Pythagorean regime in Cotron was overthrown by democracy; the order and school were dispersed; discoveries and knowledge were fought against, mocked and forgotten. Aristotle, with his self-righteousness then put at the service of prejudices of another nature, prevented the acceptance of the Pythagorean cosmological theories, ensuring the triumph of the erroneous geocentric theory for twenty centuries; philosophy, understood in the etymological and Pythagorean sense of the word, was obscured in the thicket of speculations, systems, beliefs, moralism and fetishism; and even geometric work, which must have had a firm foundation, was lost to the benefit of the later Greek school.

However arduous the task, it was, after twenty-five centuries, time to do something in favour of our school, repairing as far as possible the disastrous action of time and contingencies. Trying to restore the geometric work of the 'Scuola Italica' was not only an important topic of study for us, but also a welcome task of vindication.

In closing, we wish to explicitly state that we are fully aware of how much our modest forces have fallen short of the undertaking and the daring. So let others come, do more and better, and we will be the first to rejoice. And so, of course, we are well aware of the relationship between us and Pythagoras. Therefore it is natural to impute to us, and only to us, the errors and shortcomings of these pages; but, if there are merits, we beg the readers to ascribe them, *not to us*, but to the immortal founder of our school. Αὑτὸς ἔφα. Our only merit, if any, having been able to take inspiration directly from him.

ΤΕΛΟΣ

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